Order continuous linear functionals on non-locally convex Orlicz spaces

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Abstract. The space of all order continuous linear functionals on an Orlicz space L^φ defined by an arbitrary (not necessarily convex) Orlicz function φ is described.

Keywords: Orlicz spaces, modular spaces, locally solid Riesz spaces, Köthe dual, order continuous linear functionals

Classification: Primary 46E30, 46B10, 46B30

1. Introduction and preliminaries.

In the theory of duality of function spaces an investigation of the order continuous dual is of importance. In this paper we examine order continuous dual $(L^\varphi)^\sim_n$ of an Orlicz space L^φ defined by an arbitrary (not necessarily convex) finite valued Orlicz function φ over a σ-finite measure space. By making use of Kalton's and Drewnowski's results concerning the Mackey topology of E^φ (= the ideal of L^φ of all elements with order continuous F-norm) we describe the Köthe dual $(L^\varphi)^x$ of L^φ. Thus in view of the Riesz isomorphism between $(L^\varphi)^x$ and $(L^\varphi)^\sim_n$ we can establish the general form of order continuous linear functionals on L^φ. Moreover, considering L^φ (equipped with its usual integral modular m_φ) from the viewpoint of Nakano's theory of modular spaces [16] one can define on the topological dual $(L^\varphi)^*$ of L^φ the conjugate convex semimodular $\overline{m}_\varphi$, and next by means of $\overline{m}_\varphi$, we can define two modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $|||\cdot|||_{\overline{m}_\varphi}$. In this paper we obtain a description of the semimodular $\overline{m}_\varphi$ and the modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $|||\cdot|||_{\overline{m}_\varphi}$ restricted to $(L^\varphi)^\sim_n$.

We generalize the well-known results concerning the duality of Orlicz spaces obtained by W.A. Luxemburg and A.C. Zaanen [10], J. Musielak and W. Orlicz [13], W. Orlicz [20] and B. Gramsch [5] (see Remarks 3.2, 3.3 and Remark 4.1).

For the terminology concerning Riesz spaces we refer to [1], [24].

Let (Ω, Σ, μ) be a σ-finite and atomless measure space, and let L^0 denote the set of equivalence classes of all real valued measurable functions defined and finite a.e. on Ω. Then L^0 is a super Dedekind complete Riesz space under the ordering $x \leq y$, whenever $x(t) \leq y(t)$ a.e. on Ω. For a subset A of Ω, $\mathcal{X}_A$ stands for the characteristic function of A.

Now we recall some notation and terminology concerning Orlicz spaces (see [8], [9], [23], [24] for more details).

By an Orlicz function we mean a function $\varphi : [0, \infty) \to [0, \infty]$ that is non-decreasing, left continuous at 0 with $\varphi(0) = 0$, non identically equal to 0.

A convex Orlicz function is usually called a Young function.

An Orlicz function φ determines a functional $m_\varphi : L^0 \to [0,\infty]$ by the formula

$$m_\varphi(x) = \int_\Omega \varphi(|x(t)|)\, d\mu.$$

The Orlicz space determined by φ is the ideal of L^0 defined as follows:

$$L^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \ \text{ for some } \ \lambda > 0\}.$$

The functional m_φ restricted to L^φ is an orthogonally additive semimodular (see [13], [15], [16]). The space L^φ can be equipped with the complete metrizable topology $\mathcal{T}_\varphi$ of the Riesz F-norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \le \lambda\}.$$

Moreover, if φ is a Young function, then two norms (equivalent to $\|\cdot\|_\varphi$) on L^φ can be defined by

$$\|x\|_\varphi = \inf_{\lambda>0}\{\frac{1}{\lambda}(1 + m_\varphi(\lambda x))\},$$
$$|||x|||_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \le 1\}$$

and $|||x|||_\varphi \le \|x\|_\varphi \le 2|||x|||_\varphi$.

Let

$$E^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \ \text{ for all } \ \lambda > 0\}.$$

It is well known that E^φ coincides with the ideal of L^φ of all elements with order continuous F-norm $\|\cdot\|_\varphi$, that is,

$$E^\varphi = \{x \in L^\varphi : |x| \ge u_n \downarrow 0 \ \text{ in } \ L^\varphi \ \text{ implies } \ \|u_n\|_\varphi \downarrow 0\}.$$

Since $\operatorname{supp} E^\varphi = \Omega$, there exists a sequence (Ω_n) of μ-measurable subsets of Ω, such that $\Omega_n \uparrow$, $\bigcup_{n=1}^\infty \Omega_n = \Omega$ and $\mathcal{X}_{\Omega_n} \in E^\varphi$ (see [24, Theorem 86.2]).

Throughout the paper, for a given $x \in L^\varphi$, we will denote by $x^{(n)}$ $(n = 1, 2, \dots)$ the functions defined on Ω as follows:

$$x^{(n)}(t) = \begin{cases} x(t) & \text{if } \ |x(t)| \le n \ \text{ and } \ t \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

By $(L^\varphi)^*$ we will denote the dual of L^φ with respect to $\mathcal{T}_\varphi$. Since $\mathcal{T}_\varphi$ is a complete, metrizable locally solid topology, we have (see [1, Theorem 16.9]):

$$(L^\varphi)^* = (L^\varphi)^\sim,$$

where $(L^\varphi)^\sim$ stands, as usual, for the space of all order bounded linear functionals on L^φ.

A linear functional f on L^φ is said to be order continuous whenever $x_\sigma \xrightarrow{0} 0$ in L^φ implies $f(x_\sigma) \to 0$ for a net (x_σ) in L^φ. Since the measure space is σ-finite, a linear functional f is order continuous iff f is σ-order continuous (i.e., $x_n \xrightarrow{0} 0$ in L^φ implies $f(x_n) \to 0$ for a sequence (x_n)). As usual, let $(L^\varphi)_n^\sim$ stand for the space of all order continuous linear functionals on L^φ. It is known that $(L^\varphi)_n^\sim \subset (L^\varphi)^\sim$ and $(L^\varphi)_n^\sim$ is a band of $(L^\varphi)^\sim$ (see [22, Proposition 5.22]). Moreover, it is known that $(L^\varphi)_n^\sim = (L^\varphi)^\sim$ whenever φ satisfies the so-called Δ_2-condition, that is

$$\limsup \frac{\varphi(2u)}{\varphi(u)} < \infty \quad \text{as} \quad u \to 0 \quad \text{and} \quad u \to \infty.$$

In view of [20] the dual $(L^\varphi)^*$ is a Banach space under the norm

$$P_{m_\varphi}(f) = \sup\{|f(x)| : x \in L^\varphi,\ m_\varphi(x) \le 1\},$$

which is called, due to Nakano [16], a polar of the semimodular m_φ.

In general, given a linear topological space (X, ξ), by $(X, \xi)^*$ we will denote its topological dual.

2. The convex minorant of an Orlicz function.

Throughout the remainder of the paper we will assume that an Orlicz function φ takes only finite values.

For an Orlicz function φ satisfying the condition

$$(+) \qquad \liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$$

let

$$\varphi^*(v) = \sup\{uv - \varphi(u) : u \ge 0\} \quad \text{for} \quad v \ge 0.$$

Then φ^* is a Young function, complementary to φ in the sense of Young. The function

$$\overline{\varphi}(u) = (\varphi^*)^*(u) \quad \text{for} \quad u \ge 0$$

is called a convex minorant of φ, because it is the largest Young function that is smaller than φ on $[0, \infty)$.

In this section we give more details about φ^* and $\overline{\varphi}$ that will be useful in the sections 3 and 4. To the end of this section we will assume that the condition $(+)$ is satisfied.

We start with the following

Lemma 2.1. (i) *If* $\liminf_{u\to 0} \frac{\varphi(u)}{u} = 0$, *then* φ^* *vanishes only at zero.*

(ii) *If* $\liminf_{u\to 0} \frac{\varphi(u)}{u} > 0$, *then* φ^* *vanishes in some neighbourhood of zero.*

Proof: (i) For every $u > 0$ and $v > 0$, there exists $0 < u_1 < u$ such that $\varphi(u_1) < u_1 v$. Hence $\varphi^*(v) \ge u_1 v - \varphi(u_1) > 0$.

(ii) There exist $u_1 > 0$ and $v_1 > 0$ such that $\varphi(u) > uv_1$ for all $u \geq u_1$, and there exist $u_2 > 0$ and $v_2 > 0$ such that $\varphi(u) \geq uv_2$ for all $0 < u \leq u_2$. We can assume that $u_2 < u_1$ and let us take $v_3 > 0$ such that $1/v_3 = \sup\{u/\varphi(u) : u_2 \leq u \leq u_1\}$. Then putting $v' = \max(v_1, v_2, v_3)$ we have $uv' \leq \varphi(u)$ for all $u \geq 0$. Hence $\varphi^*(v') = 0$. Since the function φ^* is convex and left continuous, there exists a number $v_0 > 0$ such that $\varphi^*(v_0) = 0$ for $0 \leq v \leq v_0$ and $\varphi^*(v) > 0$ for $v \geq v_0$. □

In case of φ satisfying the condition $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = \infty$, the functions φ^* and $\overline{\varphi}$ were examined by Z. Birnbaum and W. Orlicz [2], W. Orlicz [20] and W. Matuszewska, W. Orlicz [12], and the main properties of φ^* and $\overline{\varphi}$ can be summarized in the following

Lemma 2.2. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. *Then the following hold:*

(i) *For every $v > 0$ there exists the least number $u_v > 0$ such that*

$$\varphi^*(v) + \varphi(u_v) = u_v v.$$

(ii) *The set $\{u_v : v \in A \subset [0,\infty)\}$ is bounded if the set A is bounded.*
(iii) *$u_v \to 0$ as $v \to 0$.*
(iv) *$\overline{\varphi}(u_v) = \varphi(u_v)$ for $v > 0$.*
(v) $\lim_{v\to 0} \frac{\varphi^*(v)}{v} = 0$.
(vi) *φ^* takes only finite values and* $\lim_{v\to\infty} \frac{\varphi^*(v)}{v} = \infty$.

Now we are going to extend the results of the previous lemma to the case of Orlicz function φ satisfying the condition $\liminf_{u\to\infty} \frac{\varphi(u)}{u} < \infty$.

Lemma 2.3. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = a < \infty$. *Then the following statements hold:*

(i) *For every $0 < v < a$ there exists the least number $u_v > 0$ such that*

$$\varphi^*(v) + \varphi(u_v) = u_v v.$$

(ii) *The set $\{u_v : 0 < v < a\}$ is bounded.*
(iii) *$u_v \to 0$ as $v \to 0$ $(0 < v < a)$.*
(iv) *$\overline{\varphi}(u_v) = \varphi(u_v)$ for $0 < v < a$.*
(v) $\lim_{v\to 0} \frac{\varphi^*(v)}{v} = 0$.
(vi) *φ^* jumps to infinity, more precisely: $\varphi^*(v) < \infty$ for $v \leq a$, $\varphi^*(v) = \infty$ for $v > a$.*

Proof: (i) For every $0 < v < a$ there exists $c_v > 0$ such that $\varphi(u)/u > v$ for $u > c_v$. Hence $\varphi^*(v) = \sup\{uv - \varphi(u) : 0 \leq u \leq c_v\}$ and for every $0 < v < a$ there exists the least number $u_v > 0$ such that $\varphi^*(v) = u_v v - \varphi(u_v)$.

(ii) Assume that the set $\{u_v : 0 < v < a\}$ is not bounded. Then there would exist a sequence (v_n) such that $0 < v_n < a$ and $u_{v_n} > \max(n, c_{v_n})$. Hence $\varphi(u_{v_n}) > u_{v_n} v_n$, so $\varphi^*(v_n) = u_{v_n} v_n - \varphi(u_{v_n}) < 0$. This contradiction establishes the boundedness of our set.

(iii) Let $v_n \to 0$ $(0 < v_n < a)$ and assume by way of contradiction that $u_{v_n} \nrightarrow 0$. Then there would exist a number $\alpha > 0$ and an increasing sequence (k_n) of natural numbers such that $u_{v_{k_n}} > \alpha$. On the other hand, in view of (ii) there exists $\beta > 0$ such that $u_{v_{k_n}} \leq \beta$. Choose an index n_0 such that $v_{k_n} < \varphi(\alpha)/\beta$ for $n \geq n_0$. Then, by (i), for $n \geq n_0$ we have

$$\varphi^*(v_{k_n}) = u_{v_{k_n}}(v_{k_n} - (\varphi(u_{v_{k_n}})/u_{v_{k_n}})) \leq u_{v_{k_n}}(v_{k_n} - \frac{\varphi(\alpha)}{\beta}) < 0.$$

This contradiction establishes that $v_n \to 0$ implies $u_{v_n} \to 0$.

(iv) In view of (i), for $0 < v < a$, the equality $\varphi(u_v) + \varphi^*(v) = u_v v$ holds. On the other hand, from the definition of $\overline{\varphi}$ we get $u_v v \leq \overline{\varphi}(u_v) + \varphi^*(v)$. Hence $\varphi(u_v) \leq \overline{\varphi}(u_v)$, so $\overline{\varphi}(u_v) = \varphi(u_v)$ because $\overline{\varphi}(u) \leq \varphi(u)$ for $u \geq 0$.

(v) Let $v_n \to 0$. Without loss of generality we can assume that $v_n < a$. Thus, by (i) and (ii) we get

$$0 \leq \varphi^*(v_n)/v_n = u_{v_n} - (\varphi(u_{v_n})/v_n) \leq u_{v_n} \to 0.$$

(vi) Let $v > a$. Then $v > a + \varepsilon$ for some $\varepsilon > 0$, and since $\liminf_{u\to\infty} \varphi(u)/u < a + \varepsilon$, there exists a sequence (u_n) such that $0 < u_n \uparrow \infty$ and $\varphi(u_n) < (a+\varepsilon)u_n$. Hence

$$\begin{aligned}\varphi^*(v) &= \sup\{u(v - (a+\varepsilon)) + u(a+\varepsilon) - \varphi(u) : u \geq 0\}\\ &\geq u_n(v - (a+\varepsilon)) + u_n(a+\varepsilon) - \varphi(u_n) \to \infty,\end{aligned}$$

so $\varphi^*(v) = \infty$ for $v > a$. Moreover, it follows from (i) that $\varphi^*(v) < \infty$ for $v < a$, and, by the left-hand continuity of φ^* we get $\varphi^*(a) < \infty$. □

The following lemma will be of importance in the proof of Theorem 4.2.

Lemma 2.4. (i) *If* $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$, *then for every measurable bounded function* $y \geq 0$ *there exists a measurable bounded function* $z \geq 0$ *such that*

$$\varphi(z(t)) + \varphi^*(y(t)) = z(t)y(t) \quad \textit{for all} \quad t \in \Omega.$$

(ii) *If* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = a < \infty$, *then for every measurable function* y *such that* $0 \leq y(t) \leq a$ *for* $t \in \Omega$, *there exists a measurable bounded function* $z \geq 0$ *such that*

$$\varphi(z(t)) + \varphi^*(y(t)) = z(t)y(t) \quad \textit{for all} \quad t \in \Omega.$$

Proof: (i) By Lemma 2.2 for every $v > 0$ there is $u_v > 0$ such that

$$\overline{\varphi}(u_v) + \varphi^*(v) = u_v v. \tag{+}$$

It is well known that φ^* is of the form $\varphi^*(v) = \int_0^v q(s)\,ds$, where the function $q : [0,\infty) \to [0,\infty)$ is non-decreasing with $q(0) = 0$ and left continuous (see [9, p. 37]). By Theorem 1 of [9, Ch. II, § 1], the equality (+) holds if $u_v = q(v)$ for $v > 0$. Putting $z(t) = q(y(t))$ for $t \in \Omega$, by Lemma 2.2 we get that $z \geq 0$ is a measurable bounded function on Ω and $\varphi(z(t)) + \varphi^*(y(t)) = z(t)y(t)$ for $t \in \Omega$.

(ii) Proceeding as in (i) and making use of Lemma 2.3 we get (ii). □

3. Order continuous linear functionals on L^φ.

In this section we will find the general form of order continuous linear functionals on an Orlicz space L^φ.

Let us recall that the Köthe dual X^x of a function space $X \subset L^0$ (with supp $X = \Omega$) is defined as follows:

$$X^x = \{y \in L^0 : \int_\Omega |x(t)y(t)|\, d\mu < \infty \;\text{ for all }\; x \in X\}.$$

It is well known that $(L^\varphi)^x = L^{\varphi^*}$, whenever φ is a Young function (see [9], [24]). It was originally proved by Z. Birnbaum and W. Orlicz [2].

Putting

$$f_y(x) = \int_\Omega x(t)y(t)\, d\mu \;\text{ for all }\; x \in X,$$

we have the following important equality

$$X_n^\sim = \{f_y : y \in X^x\} \tag{3.1}$$

where the mapping $X^x \ni y \to f_y \in X_n^\sim$ is a Riesz isomorphism (see [7, Ch. 6, § 1, Theorem 1]).

Next, let us recall that the Mackey topology of a linear topological space (X, ξ) is the finest locally convex topology τ on X that produces the same continuous linear functionals as the original topology ξ.

The following result due to N.J. Kalton [6] and L. Drewnowski [4, Corollary 1, Corollary 2] will be of importance in this section.

Theorem 3.1. (i) *There exists a nonzero continuous linear functional on* $(E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})$ *iff* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$.

(ii) *The Mackey topology* τ_{E^φ} *of* $(E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})$ *coincides with the seminormed topology* $\mathcal{T}_{\overline{\varphi}|_{E^\varphi}}$, *i.e.,* $\tau_{E^\varphi} = \mathcal{T}_{\overline{\varphi}|_{E^\varphi}}$.

Now we are ready to give a description of the Köthe dual $(L^\varphi)^x$.

Theorem 3.2. (i) *If* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = 0$, *then* $(L^\varphi)^x = (E^\varphi)^x = \{0\}$.

(ii) *If* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$, *then* $(L^\varphi)^x = (E^\varphi)^x = (E^{\overline{\varphi}})^x = L^{\varphi^*}$.

PROOF: (i) We have $(E^\varphi)_n^\sim \subset (E^\varphi)^\sim = (E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})^*$ (see [1, Proposition 16.9]), so by Theorem 3.1, $(E^\varphi)_n^\sim = \{0\}$. Hence, by (3.1), $(E^\varphi)^x = \{0\}$, and since $(L^\varphi)^x \subset (E^\varphi)^x$, the proof is completed.

(ii) First, we shall show that $(L^\varphi)^x = (E^\varphi)^x = (E^{\overline{\varphi}})^x$. It suffices to show that $(E^\varphi)^x \subset (L^\varphi)^x$ and $(E^\varphi)^x \subset (E^{\overline{\varphi}})^x$. Indeed, let $y \in (E^\varphi)^x$, i.e., $\int_\Omega |x(t)y(t)|\, d\mu < \infty$ for all $x \in E^\varphi$. Putting

$$g_y(z) = \int_\Omega z(t)y(t)\, d\mu \;\text{ for }\; z \in E^\varphi,$$

we get $g_y \in (E^\varphi)^\sim_n$ by (3.1). But according to Theorem 3.1 we have $(E^\varphi)^\sim_n \subset (E^\varphi)^\sim = (E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})^* = (E^\varphi, \mathcal{T}_{\overline{\varphi}|_{E^\varphi}})^*$, so we can put

$$\|g_y\|_{\overline{\varphi}} = \sup\Big\{\Big|\int_\Omega z(t)y(t)\,d\mu\Big| : z \in E^\varphi,\ |||z|||_{\overline{\varphi}} \le 1\Big\}.$$

To prove that $y \in (L^\varphi)^x$ (resp. $y \in (E^{\overline{\varphi}})^x$), let now $x \in L^\varphi$ (resp. $x \in E^{\overline{\varphi}}$), $x \neq 0$. Then $|x^{(n)}(t)y(t)| \uparrow_n |x(t)y(t)|$ on Ω, so by applying Fatou's lemma we get

$$\begin{aligned}\frac{1}{|||x|||_{\overline{\varphi}}}\int_\Omega |x(t)y(t)|\,d\mu &\le \frac{1}{|||x|||_{\overline{\varphi}}}\sup_n \int_\Omega (|x^{(n)}(t)|\operatorname{sign} y(t))y(t)\,d\mu \\ &\le \sup\Big\{\Big|\int_\Omega z(t)y(t)\,d\mu\Big| : z \in E^\varphi,\ |||z|||_{\overline{\varphi}} \le 1\Big\} = \|g_y\|_{\overline{\varphi}}.\end{aligned}$$

Hence $y \in (L^\varphi)^x$ (resp. $y \in (E^{\overline{\varphi}})^x$), so $(L^\varphi)^x = (E^\varphi)^x = (E^{\overline{\varphi}})^x$.

Since the topology $\mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}}$ is locally convex and satisfies the Lebesgue property, we have $(E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^* \subset (E^{\overline{\varphi}})^\sim_n \subset (E^{\overline{\varphi}})^\sim$ (see [1, Theorem 9.1]). On the other hand $(E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^* = (E^{\overline{\varphi}})^\sim$, because $\mathcal{T}_{\overline{\varphi}|_{E^\varphi}}$ is metrizable and complete (see [1, Theorem 16.9]). Thus $(E^{\overline{\varphi}})^\sim_n = (E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^*$. By the mapping $(y \mapsto g_y)$ the space $(E^{\overline{\varphi}})^x$ can be identified with $(E^{\overline{\varphi}})^\sim_n$ (see (3.1)) and the space $L^{\overline{\varphi}^*}$ with $(E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^*$ (see [10, Ch. II, § 3, Theorem 2]). Thus $(E^{\overline{\varphi}})^x = L^{\overline{\varphi}^*}$, so $(E^{\overline{\varphi}})^x = L^{\varphi^*}$, because $\overline{\varphi}^* = \varphi^*$. □

As an application of Theorem 3.2 and the equality (3.1) we obtain a condition for the existence of non-zero order continuous linear functionals on L^φ.

Theorem 3.3. *The following statements are equivalent:*

(i) $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$.
(ii) $(L^\varphi)^\sim_n \neq \{0\}$.

Finally, by making use of Theorem 3.2 and (3.1) we can establish the general form of order continuous linear functionals on L^φ.

Theorem 3.4. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$. *Then for a linear functional f on L^φ the following statements are equivalent:*

(i) *f is order continuous, i.e., $f \in (L^\varphi)^\sim_n$.*
(ii) *There exists a unique $y \in L^{\varphi^*}$ such that*

$$f(x) = f_y(x) = \int_\Omega x(t)y(t)\,d\mu \quad \text{for} \quad x \in L^\varphi.$$

Moreover, the map $L^{\varphi^*} \ni y \mapsto f_y \in (L^\varphi)^\sim_n$ is a Riesz isomorphism.

Remark 3.1. The equality $(L^\varphi)^x = L^{\varphi^*}$ from Theorem 3.2 has been recently proved in a different way by L. Maligranda and W. Wnuk ([11, Theorem 2]).

Remark 3.2. In the theory of Orlicz spaces the class of modular continuous linear functionals is considered. Let us recall that a linear functional f on L^φ is called modular continuous if $m_\varphi(x_n) \to 0$ implies $f(x_n) \to 0$ for a sequence (x_n) in L^φ [21]. In some special cases the class of all modular continuous linear functional on L^φ was investigated by J. Musielak and W. Orlicz [13], W. Orlicz [20] and J. Musielak and A. Waszak [14]. On the other hand, from [18, Theorem 5.4] it follows that the class of all modular continuous linear functionals on L^φ coincides with $(L^\varphi)_n^\sim$, and one can see that the well known results concerning modular continuous linear functionals on L^φ follow immediately from Theorems 3.3 and 3.4.

Remark 3.3. B. Gramsch [5] examined the topological dual of L^φ when φ is a concave Orlicz function. Gramsch's result contains the classical result of M.M. Day [3] on the triviality of the duals of L^p for $0 < p < 1$. But for φ being concave, the topological dual $(L^\varphi)^*$ coincides with $(L^\varphi)_n^\sim$ and the Gramsch's result follows easily from Theorems 3.3 and 3.4.

Remark 3.4. The order continuous dual of Orlicz sequence spaces l^φ (without local convexity) was described by the present author [19].

4. The conjugate semimodular and modular norms on $(L^\varphi)_n^\sim$.

In view of [16], the conjugate $\overline{m}_\varphi$ of the semimodular m_φ can be defined on the algebraic dual $\widetilde{L^\varphi}$ of L^φ as follows:

$$\overline{m}_\varphi(f) = \sup\{|f(x)| - m_\varphi(x) : x \in L^\varphi\}.$$

According to [17, Theorem 3.1] we have the following

Theorem 4.1. (i) $(L^\varphi)^* = \{f \in \widetilde{L^\varphi} : \overline{m}_\varphi(\lambda f) < \infty$ *for some* $\lambda > 0\}$.

(ii) *The conjugate* $\overline{m}_\varphi$ *restricted to* $(L^\varphi)^*$ *is a convex orthogonally additive semimodular. Moreover, if* $f \geq 0$, *then*

$$\overline{m}_\varphi(f) = \sup\{f(x) - m_\varphi(x) : 0 \leq x \in L^\varphi,\ m_\varphi(x) < \infty\}.$$

By means of the conjugate semimodular $\overline{m}_\varphi$, one can define on the dual $(L^\varphi)^*$ two Riesz norms (see [21]):

$$\|f\|_{\overline{m}_\varphi} = \inf_{\lambda>0}\Big\{\frac{1}{\lambda}(1 + \overline{m}_\varphi(\lambda f))\Big\},$$
$$|||f|||_{\overline{m}_\varphi} = \inf\Big\{\lambda > 0 : \overline{m}_\varphi\Big(\frac{f}{\lambda}\Big)\Big\}.$$

In view of the general fact (see [21, 1.51]), for any $f \in (L^\varphi)^*$,

$$|||f|||_{\overline{m}_\varphi} \leq \|f\|_{\overline{m}_\varphi} \leq 2|||f|||_{\overline{m}_\varphi} \text{ and } |||f|||_{\overline{m}_\varphi} \leq 1 \text{ iff } \overline{m}_\varphi(f) \leq 1.$$

Since $((L^\varphi)^*, P_{m_\varphi})$ is a Banach lattice, in view of the above inequalities and by applying the Open Mapping Theorem, the dual $(L^\varphi)^*$ endowed with the modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $|||\cdot|||_{\overline{m}_\varphi}$ is also a Banach lattice. Further, since $(L^\varphi)^\sim_n$ is a band of $(L^\varphi)^*$ (see [1, Theorem 3.7]), $(L^\varphi)^\sim_n$ is closed with respect to P_{m_φ} (resp. $\|\cdot\|_{\overline{m}_\varphi}$, $|||\cdot|||_{\overline{m}_\varphi}$) restricted to $(L^\varphi)^\sim_n$ by [1, Theorem 5.6]. Thus $(L^\varphi)^\sim_n$ is a Banach lattice with respect to the norms P_{m_φ} (resp. $\|\cdot\|_{\overline{m}_\varphi}, |||\cdot|||_{\overline{m}_\varphi}$) restricted to $(L^\varphi)^\sim_n$.

Remark 4.1. In 1956, W.A. Luxemburg and A.C. Zaanen [10, Theorem 1] showed that if φ is a Young function, then for any $y \in L^{\varphi^*}$,

$$m_{\varphi^*}(y) = \sup\Big\{\Big|\int_\Omega x(t)y(t)\,d\mu\Big| - m_\varphi(x) : x \in L^\varphi\Big\}.$$

Now we will extend the above equality over an arbitrary finite valued Orlicz function. Moreover, using this equality we will obtain a description of the modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $|||\cdot|||_{\overline{m}_\varphi}$ and the polar P_{m_φ} restricted to $(L^\varphi)^\sim_n$. The details follow.

Theorem 4.2. *Let* $\liminf_{u\to\infty}\frac{\varphi(u)}{u} > 0$. *Then for every* $y \in L^{\varphi^*}$ *the following equalities hold:*

(i) $\overline{m}_\varphi(f_y) = m_{\varphi^*}(y)$.
(ii) $\|f_y\|_{\overline{m}_\varphi} = \|y\|_{\varphi^*} = \sup\big\{\big|\int_\Omega x(t)y(t)\,d\mu\big| : x \in E^\varphi,\ m_{\overline{\varphi}}(x) \le 1\big\}$.
(iii) $|||f_y|||_{\overline{m}_\varphi} = |||y|||_{\varphi^*}$.
(iv) $P_{m_\varphi}(f_y) = \sup\big\{\big|\int_\Omega x(t)y(t)\,d\mu\big| : x \in E^\varphi,\ m_\varphi(x) \le 1\big\}$.

Proof: (i) From the definition of φ^* it easily follows that

$$\overline{m}_\varphi(f_y) \le m_{\varphi^*}(y).$$

Now we shall show that $\overline{m}_\varphi(f_y) \ge m_{\varphi^*}(y)$.

I. Assume first that $\liminf_{u\to\infty}\frac{\varphi(u)}{u} = a < \infty$. Then, by Theorem 2.2, $\varphi^*(v) < \infty$ for $0 \le v \le a$ and $\varphi^*(v) = \infty$ for $v > a$. Hence the inclusion $L^{\varphi^*} \subset L^\infty$ holds and we can consider two subcases:

1°. $\|y\|_\infty \le a$ ($\|\cdot\|_\infty$ — the norm in L^∞), i.e., $|y(t)| \le a$ a.e. on Ω. Let $y_n(t) = y^{(n)}(t)$ for $t \in \Omega$ ($n = 1, 2, \dots$). Then by Lemma 2.4, there exists a sequence (z_n) of bounded, measurable functions such that $z_n \ge 0$, supp $z_n \subset \Omega_n$ and

$$\varphi(z_n(t)) + \varphi^*(|y_n(t)|) = |z_n(t)y_n(t)|$$

for $n = 1, 2, \dots$ and $t \in \Omega$. Putting $x_n(t) = (\operatorname{sign} y_n(t))z_n(t)$ for $n = 1, 2, \dots$, we have $x_n \in L^\varphi$. Since $\varphi^*(|y_n(t)|) \uparrow_n \varphi^*(|y(t)|)$ for $t \in \Omega$, by applying Fatou's lemma we get

$$\begin{aligned} m_{\varphi^*}(y) &\le \sup_n \int_\Omega \varphi^*(|y_n(t)|)\,d\mu \\ &= \sup_n\Big\{\int_\Omega |z_n(t)y_n(t)|\,d\mu - \int_\Omega \varphi(z_n(t))\,d\mu\Big\} \\ &= \sup\Big\{\Big|\int_\Omega x_n(t)y(t)\,d\mu\Big| - m_\varphi(x_n)\Big\} \le \overline{m}_\varphi(f_y). \end{aligned}$$

Thus the equality $\overline{m}_\varphi(f_y) = m_{\varphi^*}(y)$ holds.

2°. $\|y\|_\infty > a$. Then $m_{\varphi^*}(y) = \infty$. Let us take $0 < \lambda < 1$ and $0 < \delta < a$ such that $\|\lambda y\|_\infty = a$ and $\lambda(a+\delta)/(a-\delta) < 1$. Let $F = \{t \in \Omega : |\lambda y(t)| > a - \delta\}$ and choose a measurable subset E of F such that $0 < \mu(E) < \infty$.

There exists a sequence (u_n) of positive numbers such that $u_n \uparrow \infty$ and $\varphi(u_n) < (a+\delta)u_n$.

Putting $x_n = u_n \cdot \mathcal{X}_E$ $(n = 1, 2, \ldots)$ we can easily show that

$$\int_\Omega \varphi(|x_n(t)|)\, d\mu \leq \frac{\lambda(a+\delta)}{a-\delta} \int_\Omega x_n(t)|y(t)|\, d\mu.$$

Hence

$$\begin{aligned}\overline{m}_\varphi(f_y) &\geq \int_\Omega x_n(t)|y(t)|\, d\mu - \int_\Omega \varphi(|x_n(t)|)\, d\mu \\ &\geq \Big(1 - \frac{\lambda(a+\delta)}{a-\delta}\Big) \int_\Omega x_n(t)|y(t)|\, d\mu \\ &\geq \Big(1 - \frac{\lambda(a+\delta)}{a-\delta}\Big) \frac{a-\delta}{\lambda} u_n \mu(E).\end{aligned}$$

Thus $\overline{m}_\varphi(f_y) = \infty$ and $\overline{m}_\varphi(f_y) = m_{\varphi^*}(y)$.

II. Next assume that $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. Then in view of Lemma 2.4 the same proof as in 1° works.

(ii) Since $\lambda f_y = f_{\lambda y}$, by making use of (i) and (1.1) we get

$$\begin{aligned}\|f\|_{\overline{m}_\varphi} &= \inf_{\lambda>0} \Big\{\frac{1}{\lambda}(1 + \overline{m}_\varphi(\lambda f_y))\Big\} \\ &= \inf \Big\{\frac{1}{\lambda}(1 + m_{\varphi^*}(\lambda y))\Big\} = \|y\|_{\varphi^*}.\end{aligned}$$

It is well known that (see [10]) that

$$\|y\|_{\varphi^*} = \sup\{|\int_\Omega z(t)y(t)\, d\mu| : z \in L^{\overline{\varphi}},\ m_{\overline{\varphi}}(z) \leq 1\}.$$

Let $z \in L^{\overline{\varphi}}$ with $m_{\overline{\varphi}}(z) \leq 1$. Putting $x_n(t) = |z^{(n)}(t)| \operatorname{sign} y(t)$ for $t \in \Omega$ $(n = 1, 2, \ldots)$, we have that $x_n \in E^\varphi$, $m_{\overline{\varphi}}(x_n) \leq 1$ and $|z^{(n)}(t)y(t)| \uparrow_n |z(t)y(t)|$ for $t \in \Omega$. Hence by applying Fatou's lemma we easily get

$$|\int_\Omega z(t)y(t)\, d\mu| \leq \sup_n |\int_\Omega x_n(t)y(t)\, d\mu|.$$

Thus $\|y\|_{\varphi^*} = \sup\{|\int_\Omega x(t)y(t)\, d\mu| : x \in E^\varphi,\ m_{\overline{\varphi}}(x) \leq 1\}$.

(iii) Using (i) and (1.2) we get

$$\begin{aligned}|||f_y|||_{\overline{m}_\varphi} &= \inf\{\lambda > 0 : \overline{m}_\varphi(f_y/\lambda) \leq 1\} \\ &= \inf\{\lambda > 0 : m_{\varphi^*}(y/\lambda) \leq 1\} = |||y|||_{\varphi^*}.\end{aligned}$$

(iv) Similarly as in (ii). □

References

[1] Aliprantis C.D., Burkinshaw O., *Locally Solid Riesz Spaces*, Academic Press, New York, 1978.
[2] Birnbaum Z., Orlicz W., *Über die Verallgemeinerung des Begriffes der zueinander Konjugierten Potenzen*, Studia Math. **3** (1931), 1–67.
[3] Day M.M., *The space L^p with $0 < p < 1$*, Bull. Amer. Math. Soc. **46** (1940), 816–823.
[4] Drewnowski L., *Compact operators on Musielak-Orlicz spaces*, Comment. Math. **27** (1988), 225–232.
[5] Gramsch B., *Die Klasse metrischer linearer Räume $\mathscr{L}_\Phi$*, Math. Ann. **176** (1967), 61–78.
[6] Kalton N.J., *Compact and strictly singular operators on Orlicz spaces*, Israel J. Math. **26** (1977), 126–136.
[7] Kantorovich L.V., Akilov G.P., *Functional Analysis* (in Russian), Moscow, 1984.
[8] Krasnoselskii M., Rutickii Ya.B., *Convex Functions and Orlicz spaces*, P. Noordhoff Ltd., Groningen, 1961.
[9] Luxemburg W.A., *Banach Function Spaces*, Delft, 1955.
[10] Luxemburg W.A., Zaanen A.C., *Conjugate spaces of Orlicz spaces*, Indagationes Math. **59** (1956), 217–228.
[11] Maligranda L., Wnuk W., *Landau type theorem for Orlicz spaces*, Math. Zeitschrift **208** (1991), 57–64.
[12] Matuszewska W., Orlicz W., *A note on the theory of s-normed spaces of φ-integrable functions*, Studia Math. **21** (1968), 801–808.
[13] Musielak J., Orlicz W., *Some remarks on modular spaces*, Bull. Acad. Polon. Sci. **7** (1959), 661–668.
[14] Musielak J., Waszak A., *Linear continuous functionals over some two-modular spaces*, Colloquia Math. Soc. János Bolyai **35** (1980), 877–890.
[15] Nakano H., *Modular semi-ordered linear spaces*, Maruzen Co. Ltd., Tokyo, 1950.
[16] ______, *On generalized modular spaces*, Studia Math. **31** (1968), 439–449.
[17] Nowak M., *Duality of non-locally convex Orlicz spaces*, to appear.
[18] ______, *Orlicz lattices with modular topology I*, Comment. Math. Univ. Carolinae **30** (1989), 261–270.
[19] ______, *Linear functionals on Orlicz sequence spaces without local convexity*, Inter. J. for Math. and Math. Sciences **15**, No. 2 (1992), 241–254.
[20] Orlicz W., *On integral representability of linear functionals over the space of φ-integrable functions*, Bull. Acad. Polon. Sci. **8** (1960), 567–569.
[21] ______, *A note on modular spaces I*, Bull. Acad. Polon. Sci. **9** (1961), 157–162.
[22] Peressini A., *Order topological vector spaces*, Harper and Row, New York, London, 1967.
[23] Turpin P., *Convexités dans les espaces vectoriels topologiques généraux*, Dissertationes Math. **131** (1976).
[24] Zaanen A.C., *Riesz spaces II*, North-Holland Publ. Co., Amsterdam, New York, Oxford, 1983.

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(*Received* February 18, 1992)