

The category of uniform spaces as a completion of the category of metric spaces

JIŘÍ ADÁMEK, JAN REITERMAN

Abstract. A criterion for the existence of an initial completion of a concrete category \mathbf{K} universal w.r.t. finite products and subobjects is presented. For \mathbf{K} = metric spaces and uniformly continuous maps this completion is the category of uniform spaces.

Keywords: universal completion, metric space, uniform space

Classification: 18A35, 54B30, 54E15

Introduction.

We investigate initial completions of concrete categories, i.e. initially complete categories \mathbf{K}^* such that \mathbf{K} is a full, concrete subcategory of \mathbf{K}^* . Recall that \mathbf{K}^* is said to be a universal initial completion of \mathbf{K} , see [H], provided that (a) \mathbf{K} is closed under initial sources in \mathbf{K}^* and (b) for each initially complete \mathbf{L} and each concrete functor $F : \mathbf{K} \rightarrow \mathbf{L}$ preserving initial sources there exists an extension to an initial-sources preserving functor $F^* : \mathbf{K}^* \rightarrow \mathbf{L}$, unique up-to natural isomorphism. More in general, let Δ be a collection of sources in the base-category, then a Δ -universal initial completion of \mathbf{K} , see [E], is an initial completion \mathbf{K}^* such that

- (a) \mathbf{K} is closed in \mathbf{K}^* under initial sources carried by Δ -sources (shortly: initial Δ -sources)

and

- (b) for each initially complete category \mathbf{L} and each concrete functor $F : \mathbf{K} \rightarrow \mathbf{L}$ preserving initial Δ -sources there exists an extension to an initial-sources preserving functor $F^* : \mathbf{K}^* \rightarrow \mathbf{L}$, unique up-to a natural isomorphism.

A description of the Δ -universal initial completion (as a category of “ Δ -complete sources”) has been presented in [E]. Unfortunately, the conclusion made in that paper that \mathbf{K}^* is always legitimate (i.e. lives in the universe U of classes), formulated in Theorem 4, is false: For example, if \mathbf{K} is a discrete, large category, then the Δ -universal completion is illegitimate, being codable by the collection of all subclasses of \mathbf{K} . In the case $\Delta =$ all sources or $\Delta = \emptyset$, the legitimacy of the Δ -universal completion is characterized in [AHS].

In the present paper we concentrate on the case

$$\Delta_{fm} = \text{all finite products and monomorphisms.}$$

(Or, equivalently, all finite monosources.) We prove that each fibre-small, hereditary concrete category has a fibre-small (thus, legitimate) Δ_{fm} -universal initial completion. For

$\mathbf{K} = \mathbf{Met}$, metric spaces and uniformly continuous maps

this completion is

$\mathbf{K}^* = \mathbf{Unif}$, uniform spaces and uniformly continuous maps.

In this special case the same result also holds for $\Delta =$ all finite sources, or $\Delta =$ all countable sources, but it does not hold for $\Delta =$ all sources. In fact, a concrete description of the universal initial completion of \mathbf{Met} is not known.

The paper has been inspired by Zdeněk Frolík who asked us about a categorical motivation of uniform spaces from the point of view of metric spaces. This paper is, most unfortunately, the end of a long, fruitful, and happy collaboration of the authors: Jan Reiterman died precisely when it was completed.

I. Δ -universal completion.

Recall from [A] that a construct (i.e. a concrete category over \mathbf{Set}) is called hereditary provided that every subset of the underlying set of any object K gives rise to an initial subobject of K . This can be generalized to concrete categories over a base category \mathbf{X} (i.e. pairs consisting of a category \mathbf{K} and a faithful, amnesic functor $|\cdot| : \mathbf{K} \rightarrow \mathbf{X}$) provided that a fixed factorization structure (\mathbf{E}, \mathbf{M}) for \mathbf{X} -morphisms is given:

Definition. A concrete category \mathbf{K} over an (\mathbf{E}, \mathbf{M}) -base category is called hereditary provided that given an object K of \mathbf{K} , every \mathbf{M} -morphism $m : X \rightarrow |K|$ has an initial lift.

Recall that a concrete category is fibre-small provided that for each object X of the underlying category the fibre $\{K \in \mathbf{K} \mid |K| = X\}$ is a set.

Theorem. *Let \mathbf{X} be an \mathbf{E} -co-wellpowered (\mathbf{E}, \mathbf{M}) -category, and let Δ be a collection of sources containing all \mathbf{M} -maps (considered as singleton sources). Then each hereditary, fibre-small concrete category over \mathbf{X} has a fibre-small Δ -universal initial completion.*

PROOF: The Δ -universal initial completion has been described in the proof of Theorem 4 of [E] (for the case of $\Gamma =$ all sources and $P =$ forgetful functor of \mathbf{K}) as the category of all Δ -complete sources in \mathbf{K} . It is our task to show that this category is fibre-small (thus, legitimate).

For each Δ -complete source $\sigma = (X \xrightarrow{f_i} |S_i|)_{i \in I}$ and each i we factorize $f_i = m_i \cdot e_i$ ($m_i \in \mathbf{M}$, $e_i \in \mathbf{E}$) and denote by $m_i : S'_i \rightarrow S_i$ the initial morphism induced by m_i (which exists since \mathbf{K} is hereditary). From the fact that Δ contains $\{m_i\}$ it follows that $X \xrightarrow{e_i} |S'_i|$ is a member of the (Δ -complete) source σ . Thus,

$$\sigma' = (X \xrightarrow{e_i} |S'_i|)_{i \in I}$$

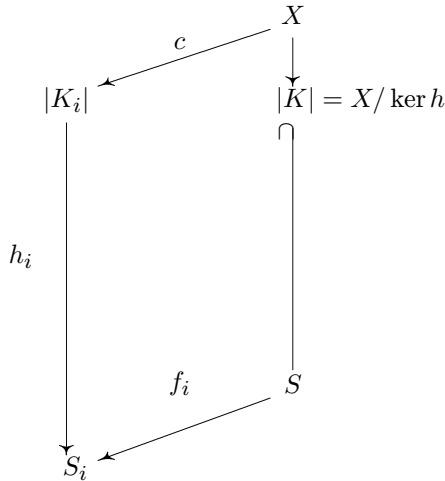
is a subsource of σ . This subsource fully determines σ (in other words, given two Δ -complete sources $\sigma_1 \neq \sigma_2$, it follows that $\sigma'_1 \neq \sigma'_2$). In fact, from the Δ -completeness

it follows that for each $i \in I$ and each morphism $h : S'_i \rightarrow S$ in \mathbf{K} we have $X \xrightarrow{he_i} |S|$ in σ ; consequently, σ is precisely the source of all composites of the members of σ' with morphisms of \mathbf{K} . Thus, to prove the fibre-smallness, it is sufficient to observe that for each object X of \mathbf{X} , all the possible sources σ' form a set. In fact, since \mathbf{X} is \mathbf{E} -co-wellpowered, we have a set of representatives for all e_i 's, and for each such a representative $e_i : X \rightarrow Y$ we have (since \mathbf{K} is fibre-small) only a set of possible objects S'_i with $|S'_i| = Y$. Thus, there exists a set A of representative structured maps $e_i : X \rightarrow |S'_i|$, and the collection of all Δ -complete sources with the domain X can, obviously, be coded by the set of all subsets of A . \square

Remark. For constructs \mathbf{K} , i.e. concrete categories over \mathbf{Set} , the Δ -universal initial completion \mathbf{K}^* can be described as follows.

Objects on the underlying set X are all collections σ of \mathbf{K} -objects K with the underlying sets $|K| = X / \sim$ (where \sim is an equivalence relation on X) such that:

- (1) If $K \in \sigma$ then $K' \in \sigma$ whenever $|K'| = X / \sim$ and the canonical map $c : X \rightarrow |K'|$ of $|K'|$ (assigning to each $x \in X$ the equivalence class of x) factorizes as the canonical map of $|K|$ followed by a \mathbf{K} -morphism $K \rightarrow K'$;
- (2) For each initial Δ -source $(S \xrightarrow{f_i} S_i)_{i \in I}$ in \mathbf{K}



and each map $h : X \rightarrow |S|$ such that every $f_i h$ factorizes as the canonical map of $|K_i|$ for some $K_i \in \sigma$ followed by a \mathbf{K} -morphism $h_i : K_i \rightarrow S_i$ it follows that σ contains the initial lift of the inclusion $X / \ker h \hookrightarrow |S|$.

Morphisms from σ to σ' are maps $f : |\sigma| \rightarrow |\sigma'|$ such that for each $K' \in \sigma'$ (with the canonical map $c : |\sigma'| \rightarrow |K'|$) σ contains the initial lift of the inclusion $|\sigma| / \sim \hookrightarrow |K|$, where \sim is the kernel equivalence of c.f.

II. A completion of the category of metric spaces.

Proposition. *The category \mathbf{Unif} is a Δ_{fm} -universal initial completion of the category \mathbf{Met} .*

PROOF: We apply the general result above to the special case of $\Delta = \Delta_{fm}$ and $\mathbf{K} = \mathbf{Met}$. The objects of Δ_{fm} -universal initial completion \mathbf{K}^* with the underlying set X are collections σ of metric spaces $K = (X/\sim, d_K)$, where \sim is an equivalence on X , such that (1) and (2) are satisfied. Each K induces a pseudometric d_K^* on the set X by $d_K^*(x, y) = d_K([x], [y])$ (where $x \mapsto [x]$ denotes the canonical map). The conditions (1) and (2) guarantee that the resulting set of pseudometrics on X forms a uniformity (defined as a collection of pseudometrics). Conversely, for each uniformity on X and each pseudometric d of that uniformity we have a corresponding metric space on the set X/\sim where $x \sim y$ means that $d(x, y) = 0$. The axioms of a uniformity guarantee that the resulting set of metric spaces is an object of \mathbf{K}^* . The morphisms of \mathbf{K}^* correspond precisely to the uniformly continuous maps. Thus, \mathbf{Unif} is a Δ_{fm} -universal initial completion of \mathbf{Met} . \square

Remark. Since \mathbf{Met} is, obviously, closed under countable initial sources in \mathbf{Met} , we can also say that for $\Delta =$ all countable sources, \mathbf{Unif} is a Δ -universal initial completion of \mathbf{Met} . However, the universal completion ($\Delta =$ all sources) is different:

Example of an initial source in \mathbf{Met} which is not initial in \mathbf{Unif} .

Let δ be the discrete metric (with value 1) on the set N of natural numbers. Let \mathbf{F} be a free ultrafilter on N containing the set E of even numbers. For each $F \in \mathbf{F}$ let δ_F be the following metric on N :

$$\delta_F(x, y) = \begin{cases} \frac{1}{n} & \text{if } \{x, y\} = \{2n - 1, 2n\} \text{ with } 2n \in F \\ 0 & \text{if } x = y \\ 1 & \text{else.} \end{cases}$$

As proved in [PRRS], the set of all $\delta_F, F \in \mathbf{F}$, is a base of a uniformity σ on N which is an atom in the fibre of N (i.e. the only strictly finer uniformity is that induced by δ). Consequently, the following source

$$((N, \sigma) \xrightarrow{id} (N, \delta_F))_{F \in \mathbf{F}}$$

is initial in \mathbf{Unif} . We will show that, nevertheless,

$$((N, \delta) \xrightarrow{id} (N, \delta_F))_{F \in \mathbf{F}}$$

is initial in \mathbf{Met} . In fact, let (X, δ_0) be a metric space, and let $f : X \rightarrow N$ be a map such that $f : (X, \delta_0) \rightarrow (N, \delta_F)$ is uniformly continuous for each $F \in \mathbf{F}$. We will prove that the image of f , considered as a uniform subspace of (N, σ) , is discrete — thus, $f : (X, \delta_0) \rightarrow (N, \delta)$ is uniformly continuous.

Suppose that image of f is not discrete. Then it is isomorphic to (N, σ) . [In fact, every uniform subspace A of (N, σ) is isomorphic to (N, σ) or is discrete, according to whether the set $\{2n \mid n \in N, 2n \in A, 2n + 1 \in A\}$ is a member of \mathbf{F} or not.] Thus, we can assume that f is surjective. Since σ is an atom, it follows that f is a final morphism. However, (N, σ) is not a quotient of a metric space, since it is not generated by a single pseudometric — this is a contradiction.

REFERENCES

- [A] Adámek J., *Theory of Mathematical Structures*, Reidel Publ. Comp., Dordrecht, 1983.
- [AHS] Adámek J., Herrlich H., Strecker G.E., *Least and largest initial completion*, Comment. Math. Univ. Carolinae **20** (1979), 43–75.
- [C] Čech E., *Topological Spaces*, Academia Prague, 1966.
- [E] Ehlersmann A.E., *Partial completions of concrete functors*, Cahiers Topo. Géom. Diff. **22** (1981), 315–328.
- [H] Herrlich H., *Initial completions*, Math. Z. **150** (1976), 101–110.
- [PRRS] Pelant J., Reiterman J., Rödl V., Simon P., *Ultrafilters on ω and atoms in the lattice of uniformities I*, Topology and Appl. **30** (1988), 1–17.

FACULTY OF ELECTRICAL ENGINEERING, TECHNICAL UNIVERSITY PRAGUE, TECHNICKÁ 2,
PRAHA 6, CZECHOSLOVAKIA

(Received September 25, 1992)