On the rhomboidal heredity in ideal lattices

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Abstract. We show that the class of principal ideals and the class of semiprime ideals are rhomboidal hereditary in the class of modular lattices. Similar results are presented for the class of ideals with forbidden exterior quotients and for the class of prime ideals.

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Let C be a class of lattices and let D denote a class of ideals. We say that D is *rhomboidal hereditary* in C, if any ideals P, Q of a lattice $L \in C$ belong to D whenever $P \vee Q$ and $P \wedge Q$ belong to D.

In this terminology, a result of G. Grätzer and E.T. Schmidt [3, p. 83] can be restated as the assertion that the class of principal ideals is rhomboidal hereditary in the class of distributive lattices.

Our first theorem generalizes the mentioned result:

Theorem 1. *The class of principal ideals is rhomboidal hereditary in the class of modular lattices.*

Proof: Suppose P and Q are ideals of a modular lattice L such that $P \vee Q = (a]$ and $P \wedge Q = (b]$.

Since $P \vee Q = (a]$, there exist $p_0 \in P$ and $q_0 \in Q$ such that $a \leq p_0 \vee q_0$. However, $(p_0] \subset P \subset (a]$, and so $p_0 \leq a$. Similarly we get $q_0 \leq a$. This implies $p_0 \vee q_0 = a$. Let $p_1 = p_0 \vee b$ and $q_1 = q_0 \vee b$. From $b \leq a$ we can see that

$$p_1 \vee q_1 = a. \tag{1}$$

Let p be an arbitrary element in P and let $p_2 = p \vee p_1$. From $(p] \subset P \subset (a]$ and (1) we conclude that

$$p_2 \vee q_1 = a. \tag{2}$$

Clearly, $p_2 \geq p_1 = p_0 \vee b \geq b$ and $q_1 = q_0 \vee b \geq b$. This gives $p_2 \wedge q_1 \geq b$. But $p_2 \wedge q_1 \in P \wedge Q = (b]$, and so

$$p_2 \wedge q_1 = p_1 \wedge q_1 = b. \tag{3}$$

Using (2), (3), the modularity of L and the fact that $p_1 \leq p_2$, we get $p_1 = p_2$. It follows that $p_1 = p \vee p_1 \geq p$, and, consequently, $I = (p_1]$. $\square$

For any lattice L, let $\theta(L)$ denote a congruence of L and let θ be the class of all these congruences. We shall say that an ideal I of L is an *ideal with forbidden exterior quotients in* θ, if the following condition is satisfied for any $a, b \in L$: If $a \leq b$, $(a, b) \in \theta(L)$ and $a \in I$, then $b \in I$.

Theorem 2. *For any θ, the class of ideals with forbidden exterior quotients in θ is rhomboidal hereditary in the class of modular lattices.*

PROOF: Let P and Q be ideals of a lattice L. Suppose that $P \vee Q$ and $P \wedge Q$ are ideals with forbidden exterior quotients in θ and let $b \geq a \in P$ be such that $(a, b) \in \theta(L)$.

Since the ideal $P \vee Q$ is an ideal with forbidden exterior quotients in θ, it follows that $b \in P \vee Q$. Hence we have $b \leq p \vee q$ for some $p \in P$ and $q \in Q$.

Let $a_1 = p \vee a$, $b_1 = p \vee b$. Note that $(a_1, b_1) \in \theta(L)$ and $a_1 \leq b_1$. Similarly, $(a_1 \wedge q, b_1 \wedge q) \in \theta(L)$. But $a_1 \wedge q \in P \wedge Q$ and so finally $b_1 \wedge q \in P \wedge Q$. Now set $a_2 = a_1 \vee (b_1 \wedge q)$, so that $a_2 \in P$.

Observe that the relations $b_1 \wedge q \leq a_2 \leq b_1$ imply $b_1 \wedge q \leq a_2 \wedge q \leq b_1 \wedge q$. Hence

$$b_1 \wedge q = a_2 \wedge q. \tag{4}$$

On the other hand

$$a_2 \vee q = a_1 \vee q = p \vee q \vee a \geq p \vee b = b_1 \geq a_2$$

which implies $a_2 \vee q \leq b_1 \vee q \leq a_2 \vee q$. Thus

$$b_1 \vee q = a_2 \vee q. \tag{5}$$

A combination of (4), (5) and $a_2 \leq b_1$ with the modularity of L yields $b \leq b_1 = a_2 \in P$. Consequently, $b \in P$. □

Remark. If P and Q are ideals of a relatively complemented lattice L such that $P \wedge Q$ is an ideal with forbidden exterior quotients, then the same can be said about the ideals P and Q. Indeed, suppose $a \in P, a \leq b$ and $(a, b) \in \theta(L)$ and let d be an element of $P \wedge Q$. If a^+ denotes a relative complement of a in the interval $[e, b]$ where $e = d \wedge a$, then $(e, a^+) \in \theta(L)$. By assumption, $a^+ \in P \wedge Q$, and, therefore, $b = a \vee a^+ \in P \vee (P \wedge Q) = P$.

However, simple examples show that such a simplification is not possible for modular lattices.

It is also easy to draw the following conclusion: *In any relatively complemented lattice, the ideal $P \wedge Q$ is an ideal with forbidden exterior quotients if and only if P and Q have the same property.* A similar assertion is of course true for any non-void intersection of an arbitrary system of ideals in a relatively complemented lattice.

We now recall some basic definitions and some related techniques.

A quotient q/p of a lattice L is *weakly perspective* into a quotient s/r of L (written $q/p \sim_w s/r$) if either

$$q \wedge r = p \ \ \& \ \ q \vee r \leq s$$

or

$$s \vee p = q \ \ \& \ \ s \wedge p \geq r.$$

A quotient u/t is *weakly projective* into a quotient w/v (written $u/t \approx_w w/v$) if there exist quotients u_i/t_i $(i = 0, 1, \dots, n)$ such that

$$u_0/t_0 = u/t \sim_w u_1/t_1 \sim_w \cdots \sim_w u_n/t_n = w/v.$$

An *allele* is a quotient y/x for which there exists a quotient b/a such that $y/x \approx_w b/a$ and either $y \leq a$ or $b \leq x$.

Let $\hat{C}(L)$ be the congruence defined on L in the following way [1]: the couple (a, b) belongs to $\hat{C}(L)$ if and only if there exists $n \in \mathbf{N}$ and a sequence of alleles c_{i+1}/c_i $(i = 0, 1, \dots, n-1)$ such that

$$a \wedge b = c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_n = a \vee b.$$

An ideal I of a lattice L is said to be *semiprime* [4] if the following implication holds for any a, b, c of L:

$$(a \wedge b \in I \ \ \& \ \ a \wedge c \in I) \Rightarrow a \wedge (b \vee c) \in I.$$

Finally, we shall make use of the following main result of [2]: *An ideal I of a lattice L is semiprime if and only if there exists no allele p/i with $i \in I$ and $p \notin I$.* As an immediate consequence of this we obtain the following proposition:

Proposition 3. *Let $\hat{C}$ be the class of all the congruences $\hat{C}(L)$. An ideal I of a lattice L is semiprime if and only if it is an ideal with forbidden exterior quotients in $\hat{C}$.*

We are now in a position to state the next theorem on the rhomboidal heredity:

Theorem 4. *The class of semiprime ideals is rhomboidal hereditary in the class of modular lattices.*

Proof: Apply Theorem 2 and Proposition 3. □

Corollary 5. *Let $P_i, i \in I \neq \emptyset$, be ideals of a relatively complemented lattice L. If $\bigcap(P_i,\ i \in I)$ is non-void, then it is a semiprime ideal of L if and only if all the ideals P_i are semiprime.*

Corollary 6. *Let $P_i, i \in I \neq \emptyset$, be ideals of an orthomodular lattice L. Then $\bigcap(P_i,\ i \in I)$ is a semiprime ideal of L if and only if all the ideals P_i are semiprime.*

Theorem 7. *The class of prime ideals is rhomboidal hereditary in the class of all lattices.*

Proof: Let P and Q be ideals of a lattice L such that $P \vee Q$ and $P \wedge Q$ are prime and let $a \wedge b \in P$ for $a, b \in L$. Without loss of generality, we can suppose that $a \in P \vee Q$. Moreover, we can suppose that Q is not contained in P. It follows that there exists q belonging to $Q \setminus P$. From $a \wedge b \in P$ and $q \in Q$ we get $a \wedge (b \wedge q) \in P \wedge Q$. If $a \in P \wedge Q$, then it is clear that $a \in P$. We may thus assume that $a \notin P \wedge Q$. However, in this case we have $b \wedge q \in P \wedge Q$ with $q \in Q \setminus P$. Since $P \wedge Q$ is prime, this gives $b \in P \wedge Q \subset P$, completing the proof. □

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