On binary coproducts of frames

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Abstract. The structure of binary coproducts in the category of frames is analyzed, and the results are then applied widely in the study of compactness, local compactness (continuous frames), separatedness, pushouts and closed frame homomorphisms.

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Coproducts in the category of frames are usually viewed as counterparts of products in the category of topological spaces. The discrepancy between them has produced well-known remarkable properties of frames, for instance, the localic Tychonoff Theorem is constructively valid, and paracompactness is preserved under coproducts of frames. It is certainly worthwhile to study in particular the simplest type of coproducts — binary coproducts.

Let π be the nucleus, defined on the downset-frame of $L_1 \times L_2$, determining the coproduct $L_1 \oplus L_2$. When dealing with $L_1 \oplus L_2$, we often meet the following problem: Given a downset U of $L_1 \times L_2$ and $(a, b) \in \pi(U)$, what are the internal relations between (a, b) and U? Through the analysis of the (pre)nuclei and their combinations involved in constructing binary coproducts, we obtain a useful result, Proposition 2.2, which is a generalization of the technique introduced by Banaschewski [1] and Vermeulen [13] to show that strongly Hausdorff compact frame is regular, from which we gain substantial insight. The great power of Proposition 2.2 is then illustrated by its wide applications in the study of frame counterparts of classical topological facts related to (local)compactness, Hausdorff space and closed continuous maps. All discussions, except the last part of Section 4, are constructively valid.

1. Preliminaries.

For general facts concerning frames we refer to Johnstone [10].

Let L be a frame. The *top* (*bottom*) element of L will be denoted by e (0). For any subset $A \subseteq L$, let $\downarrow A = \{x \in L \mid x \leq a \text{ for some } a \in A\}$. For $a \in L$, its *pseudocomplement* is denoted by a^*. For a frame homomorphism $h : L \longrightarrow M$, its *right adjoint* is denoted by $h^r : M \longrightarrow L$ and is given by $h^r(b) = \bigvee\{x \in L \mid h(x) \leq b\}$. A frame homomorphism $h : L \longrightarrow M$ is called *dense* if $h(x) = 0$ implies $x = 0$; it is called *codense* if $h(x) = e$ implies $x = e$.

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A frame L is called *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, where $x \prec a$ means $x^* \vee a = e$.

Given $a, b \in L$, we say $a \ll b$ (*way below*) if $b \leq \bigvee S$ for some $S \subseteq L$ implies that $a \leq \bigvee E$ for some finite subset E of S. A frame L is called *continuous* if $a = \bigvee\{x \in L \mid x \ll a\}$ for each $a \in L$. In a continuous frame, the $\ll$-relation interpolates, that is, $a \ll b$ implies there exists a c with $a \ll c \ll b$.

Concerning the construction of coproducts in the category **Frm**, we adopt the approach introduced by Banaschewski [1], [4] as follows.

Recall that a *nucleus* on a frame L is a closure operator on L which preserves binary meets. A *prenucleus* on L is a map $k_0 : L \longrightarrow L$ such that, for all $x, y \in L$: (1) $x \leq k_0(x)$, (2) if $x \leq y$ then $k_0(x) \leq k_0(y)$, (3) $k_0(x) \wedge y \leq k_0(x \wedge y)$. For each prenucleus k_0 on L, there is a unique nucleus k which has the same fixed points as k_0 and is given by $k(x) = \bigwedge\{t \mid x \leq t, t \in K\}$. We will call k as the *associated nucleus* of k_0.

Consider a family $(L_i)_{i\in I}$ of frames with a decidable index set I. Let $\mathbf{L} \subseteq \prod L_i$ consist of all those $a = (a_i)_{i\in I}$ whose support $spt(a) = \{i \in I \mid a_i < e_i\}$, e_i the unit of L_i, is finite. Then $\mathbf{L}$ is a sublattice of $\prod L_i$. The maps $k_i : L_i \longrightarrow \mathbf{L}$ defined by

$$k_i(x)_j = \begin{cases} x & (j = i) \\ e_j & (j \neq i) \end{cases}$$

preserve arbitrary joins and arbitrary meets.

Let $\mathfrak{D}$ be the frame of all down-sets in $\mathbf{L}$, and define $\pi_0 : \mathfrak{D} \longrightarrow \mathfrak{D}$ by

$$\pi_0(U) = \{a \wedge k_i(\bigvee T) \mid a \in \mathbf{L}, i \in I, T \subseteq L_i, a \wedge k_i(t) \in U \text{ for all } t \in T\}.$$

Then π_0 is a prenucleus on $\mathfrak{D}$. We use π to denote the associated nucleus.

Proposition 1.1. $\mathrm{Fix}\,(\pi_0)$ *is the coproduct of* $(L_i)_{i\in I}$ *in* **Frm**, *with coproduct maps* $q_i = \pi \circ \downarrow \circ k_i : L_i \longrightarrow \mathrm{Fix}\,(\pi_0)$.

Furthermore, Banaschewski [4] introduced prenuclei σ_0 and μ_0 on $\mathfrak{D}$, which are defined respectively by, for any $U \in \mathfrak{D}$,

$$\sigma_0(U) = \{\bigvee D \mid \text{ updirected } D \subseteq U\},$$

and

$$\mu_0(U) = \{a \wedge k_i(\bigvee T) \mid a \in \mathbf{L}, i \in I, \text{ finite } T \subseteq L_i, a \wedge k_i(t) \in U \text{ for all } t \in T\}.$$

Let σ and μ denote the associated nuclei. One of the benefits of having σ_0 and μ_0 is shown by

Proposition 1.2. $\pi = \sigma \circ \mu$.

A constructive proof of this was provided recently by Banaschewski [1]. Based on it, we are able to study the construction of binary coproducts later on.

We use $\bigoplus L_i$ to denote the coproduct of $(L_i)_{i\in I}$. The coproduct map $q_i : L_i \longrightarrow \bigoplus L_i$ is given by $q_i(x) = \{a \in \mathbf{L} \mid a_i \leq x\} \cup Z$ for each $x \in L_i$, where $Z = \{a \in \mathbf{L} \mid a_i = 0 \text{ for some } i \in I\}$ is the bottom of $\bigoplus L_i$. Let $a_{i_1} \oplus \cdots \oplus a_{i_n}$ denote the element $q_{i_1}(a_{i_1}) \cap \cdots \cap q_{i_n}(a_{i_n})$ of $\bigoplus L_i$.

2. Binary coproducts.

For binary coproducts, we have a more detailed description of the above general construction. Consider two frames L_1 and L_2 and let $p_i : L_1 \times L_2 \longrightarrow L_i$ $(i = 1, 2)$ be the projection maps.

Let $\pi_1, \hat{\pi}_1, \pi_2, \hat{\pi}_2 : \mathfrak{D} \longrightarrow \mathfrak{D}$ be defined by

$$\pi_1(U) = \{(\bigvee X, y) \mid X \times \{y\} \subseteq U\},$$

$$\hat{\pi}_1(U) = \{(\bigvee X, y) \mid X \text{ is finite and } X \times \{y\} \subseteq U\},$$

$$\pi_2(U) = \{(x, \bigvee Y) \mid \{x\} \times Y \subseteq U\},$$

$$\hat{\pi}_2(U) = \{(x, \bigvee Y) \mid Y \text{ is finite and } \{x\} \times Y \subseteq U\}.$$

Lemma 2.1. *Let $i, j \in \{1,\ 2\}$.*

- **(1)** *For any $U \in \mathfrak{D}$, $U \subseteq \hat{\pi}_i(U) \subseteq \pi_i(U) \subseteq \pi(U)$.*
- **(2)** *$\hat{\pi}_i$ and π_i are nuclei on $\mathfrak{D}$.*
- **(3)** *$\hat{\pi}_i \circ \pi_j \circ \hat{\pi}_i = \pi_j \circ \hat{\pi}_i$ for $i \neq j$.*
- **(4)** *$\hat{\pi}_i \circ \hat{\pi}_j \circ \hat{\pi}_i = \hat{\pi}_j \circ \hat{\pi}_i$ for $i \neq j$.*
- **(5)** *$\hat{\pi}_1 \circ \hat{\pi}_2 = \hat{\pi}_2 \circ \hat{\pi}_1$.*

Proof: We only provide proofs for (2) and (3), other parts are obvious.

(2) We show that π_1 is a nucleus. Obviously, π_1 preserves the partial order. To prove that π_1 is idempotent, we consider $U \in \mathfrak{D}$ and $Z \times \{y\} \subseteq \pi_1(U)$. For each $z \in Z$, take $X_z = \{x \in L_1 \mid (x, y) \in U \text{ and } x \leq z\}$, which satisfies $X_z \times \{y\} = ((\downarrow z) \times \{y\}) \cap U$ and $z = \bigvee X_z$. Then,

$$\bigvee Z = \bigvee \bigcup_{z \in Z} X_z, \quad \text{and} \quad (\bigcup_{z \in Z} X_z) \times \{y\} \subseteq U,$$

hence $(\bigvee Z, y) \in \pi_1(U)$. It follows that $\pi_1 \circ \pi_1(U) = \pi_1(U)$, therefore π_1 is idempotent. Finally, if $(x, y) \in \pi_1(U) \cap V$ for some $U, V \in \mathfrak{D}$ then $x = \bigvee X$ and $X \times \{y\} \subseteq U$ with some $X \subseteq L_1$. One has also $X \times \{y\} \subseteq V$ since V is a downset, hence $(x, y) \in \pi_1(U \cap V)$. This shows that $\pi_1(U) \cap V \leq \pi_1(U \cap V)$, hence π_1 preserves binary meets.

(3) To see $\hat{\pi}_2 \circ \pi_1 \circ \hat{\pi}_2 = \pi_1 \circ \hat{\pi}_2$, it suffices to show that $\pi_1(U)$ is fixed by $\hat{\pi}_2$ whenever U is fixed by $\hat{\pi}_2$. Suppose U is fixed by $\hat{\pi}_2$. Then $(e, 0) \in U \subseteq \pi_1(U)$. Further, for any $(x, y_1), (x, y_2) \in \pi_1(U)$, take $A, B \subseteq L_1$ such that $x = \bigvee A = \bigvee B$, $A \times \{y_1\} \subseteq U$ and $B \times \{y_2\} \subseteq U$, then $(a \wedge b, y_1 \vee y_2) \in U$ for any $a \in A$, $b \in B$, which implies $(x, y_1 \vee y_2) = (\bigvee\{a \wedge b \mid a \in A, b \in B\}, y_1 \vee y_2) \in \pi_1(U)$. Hence $\pi_1(U)$ is fixed by $\hat{\pi}_2$. □

Now, we have three nuclei: $\mu = \hat{\pi}_1 \circ \hat{\pi}_2 = \hat{\pi}_2 \circ \hat{\pi}_1,\ \pi_2 \circ \hat{\pi}_1,\ \pi_1 \circ \hat{\pi}_2$. Combining with Proposition 1.2, it follows easily that

Proposition 2.1. $\pi = \sigma \circ \mu = \sigma \circ \pi_2 \circ \hat{\pi}_1 = \sigma \circ \pi_1 \circ \hat{\pi}_2$.

Since there are explicit expressions for μ, $\pi_2 \circ \hat{\pi}_1$ and $\pi_1 \circ \hat{\pi}_2$, the main problem in analyzing the values of π arises how to deal with σ. Recalling the nature of σ_0 (defined via updirected sets), we expect that certain conditions of compactness will be helpful.

For a frame L, $a \in L$ is called i-compact if, for any $b \in L$, $a \ll b$ implies $a \ll c \ll b$ for some $c \in L$. It is easy to see that any compact element is i-compact; and in a continuous frame L, $a \ll b$ implies that a is i-compact by the interpolation property of $\ll$. The purpose for this new concept is to unite the study of compact elements, on the one hand, and the relation $\ll$ in a continuous frame, on the other.

Lemma 2.2. *Let $U \in \mathfrak{D}$. If $(a,b) \in \pi(U)$, c is i-compact and $c \ll a$, then $(c,b) \in \pi_2 \circ \hat{\pi}_1(U)$.*

Proof: Let $S = \pi_2 \circ \hat{\pi}_1(U)$. In the interval $[S,\pi(S)]$ in $\mathfrak{D}$, let $\mathbf{W} = \{V \in [S,\pi(S)] \mid (a,b) \in V \text{ and } c \ll a \text{ implies } (c,b) \in S\}$.

(1) $\mathbf{W}$ is σ_0-stable: Take $V \in \mathbf{W}$. Consider any $(a,b) \in \sigma_0(V)$ and $c \ll a$. Suppose $(a,b) = \bigvee D$ for updirected $D \subseteq V$. We can find m such that $c \ll m \ll a$, and get $(x_0,y_0) \in D$ such that $m \leq x_0$. For any $(x,y) \in D$ with $(x,y) \geq (x_0,y_0)$, by $c \ll m \leq x_0 \leq x$ we get $(c,y) \in S$. Since $b = \bigvee\{y \mid (x,y) \in D, (x,y) \geq (x_0,y_0)\}$, we get $(c,b) \in S$. Hence $\sigma_0(V) \in \mathbf{W}$.

(2) It is trivial that S, $W = \bigcup \mathbf{W} \in \mathbf{W}$. Hence $\sigma(W) = W$, and $\sigma(S) \subseteq W$, which implies $\sigma(S) \in \mathbf{W}$.

Finally, by Proposition 2.1, $\pi(U) = \sigma(S) \in \mathbf{W}$. □

Applying this, we immediately get a key result as follows.

Proposition 2.2. *Consider $U \in \mathfrak{D}$.*

(1) *If $a \in L_1$ is compact and $a \oplus b \leq \pi(U)$, then $(a,b) \in \pi_2 \circ \hat{\pi}_1(U)$.*
(2) *If L_1 is continuous, $a \oplus b \leq \pi(U)$ and $c \ll a$, then $(c,b) \in \pi_2 \circ \hat{\pi}_1(U)$.*
(3) *If $a \in L_1$ is an atom and $a \oplus b \leq \pi(U)$, then $(a,b) \in \pi_2(U)$.*

For the convenience of further study, we make the following general observation on the values of π_1 and $\pi_2 \circ \hat{\pi}_1$.

Lemma 2.3. *For any $A \subseteq L_1 \times L_2$, $\pi_1(\downarrow A) = \downarrow\{(\bigvee p_1[K], \bigwedge p_2[K]) \mid K \subseteq A\}$.*

Proof: Suppose $(x,y) \in \pi_1(\downarrow A)$. Then there is a $Z \subseteq L_1$ such that $Z \times \{y\} \subseteq \downarrow A$ and $x = \bigvee Z$. Put

$$K = \{(a,b) \in A \mid (z,y) \leq (a,b) \text{ for some } z \in Z\}.$$

Then

$$(x,y) = (\bigvee Z, y) \leq (\bigvee p_1[K], \bigwedge p_2[K]).$$

The other inclusion is trivial. □

Lemma 2.4. *For any $A \subseteq L_1 \times L_2$, $(x, y) \in \pi_2 \circ \hat{\pi}_1(\downarrow A)$ if and only if*

$$y \leq \bigvee\{\bigwedge p_2[K] \mid \text{ finite } K \subseteq A \text{ and } \bigvee p_1[K] \geq x\}.$$

Proof: Let $U = \downarrow A$. We have $\hat{\pi}_1(U) = \downarrow \{(\bigvee p_1[K], \bigwedge p_2[K])] \text{ finite } K \subseteq A\}$. $(x, y) \in \pi_2(\hat{\pi}_1(U))$ means that there exists a subset $Y \subseteq L_2$ such that $\{x\} \times Y \subseteq \hat{\pi}_1(U)$ and $\bigvee Y = y$. However, $\{x\} \times Y \subseteq \hat{\pi}_1(U)$ is equivalent to

$$Y \subseteq \downarrow\{\bigwedge p_2[K] \mid \text{ finite } K \subseteq A \text{ and } \bigvee p_1[K] \geq x\}.$$

Therefore, $(x, y) \in \pi_2 \circ \hat{\pi}_1(U)$ if and only if

$$y = \bigvee Y \leq \bigvee\{\bigwedge p_2[K] \mid \text{ finite } K \subseteq A \text{ and } \bigvee p_1[K] \geq x\}.$$

□

Due to Proposition 2.2 and Lemma 2.4, the following result becomes apparent. Kříž and Pultr [11] proved it in a different way, employing the Axiom of Choice.

Proposition 2.3. *Suppose L is compact and M is an arbitrary frame. If*

$$e_{L \oplus M} = \bigvee\{a \oplus b \mid (a, b) \in A\} \text{ for some } A \subseteq L \times M,$$

then

$$e = \bigvee\{\bigwedge p_2[K] \mid K \subseteq A \text{ is finite and } \bigvee p_1[K] = e\}.$$

Sometimes, it is convenient to use the one-one correspondence between elements of $L_1 \oplus L_2$ and Galois connections between L_1 and L_2. Recall that a pair $[g, g']$ of mappings $g : L_1 \longrightarrow L_2$, $g' : L_2 \longrightarrow L_1$ is a *Galois connection* between L_1 and L_2 if (i) g, g' are antitone; (ii) $g' \circ g \geq id_{L_1}$ and $g \circ g' \geq id_{L_2}$. Given an element $T \in L_1 \oplus L_2$, let $g_T : L_1 \longrightarrow L_2$ be defined by $g_T(x) = \bigvee\{y \in L_2 \mid (x, y) \in T\}$, and $g'_T : L_2 \longrightarrow L_1$ defined by $g'_T(y) = \bigvee\{x \in L_1 \mid (x, y) \in T\}$, then $[g_T, g'_T]$ is a Galois connection between L_1 and L_2. Conversely, for a Galois connection $[g, g']$ between L_1 and L_2, we make $T_{[g,g']} = \{(x, y) \mid g(x) \geq y\}$, which is a downset and closed under π_1 and π_2, hence $T_{[g,g']} \in L_1 \oplus L_2$. In particular, for the pseudocomplement operator $*$ of L, the Galois connection $[*, *]$ between L and itself corresponds to the element $S = \{(x, y) \in L \times L \mid x \wedge y = 0\}$ of $L \oplus L$.

In the following context, regarding the Galois connection $[g_T, g'_T]$ determined by some $T \in L_1 \oplus L_2$, we shall simply write x^T to denote $g_T(x)$ and also y^T for $g'_T(y)$. Hence

$$(\bigvee X)^T = \bigwedge\{x^T \mid x \in X\} \text{ for any } X \subseteq L_1, \text{ or any } X \subseteq L_2.$$

As an immediate application of above facts, we show that

Proposition 2.4. *Consider frames L_1, L_2 and the coproduct maps $q_i : L_i \longrightarrow L_1 \oplus L_2$ $(i = 1, 2)$. For any $a \in L_1$ and $T \in L_1 \oplus L_2$, we have*

$$q_2^r(q_1(a) \vee T) = \bigvee\{b \in L_2 \mid e = a \vee b^T\}$$

if one of the following conditions is satisfied:

(1) *L_2 is continuous.*
(2) *L_1 is compact.*
(3) *$a \vee \bigwedge X = \bigwedge\{a \vee x \mid x \in X\}$ holds for any $X \subseteq L_1$.*

Proof: For any $b \in L_2$ with $e = a \vee b^T$, $(a, b) \in q_1(a)$ and $(b^T, b) \in T$ imply $(e, b) \in q_1(a) \vee T$, it follows that

$$q_2^r(q_1(a) \vee T) \geq \bigvee\{b \in L_2 \mid e = a \vee b^T\}.$$

On the other hand, put

$$U = q_1(a) \cup T,$$

then $\pi_1(U) = \{(x, y) \mid x \leq a \vee y^T\}$, which is fixed by $\hat{\pi}_2$.

Let $c = q_2^r(q_1(a) \vee T)$, then $(e, c) \in q_1(a) \vee T$.

(1) Assume L_2 is continuous. For any $b \ll c$, $(e, c) \in \pi(U)$ implies $(e, b) \in \pi_1(U)$ by Proposition 2.2, which means $e = a \vee b^T$. Thus

$$c \leq \bigvee\{b \in L_2 \mid e = a \vee b^T\}.$$

(2) Suppose L_1 is compact. Acting π_2 on $\pi_1(U)$, we get

$$\pi_2(\pi_1(U)) = \{(x, z) \mid z \leq \bigvee\{y \mid x \leq a \vee y^T\}\}.$$

By Proposition 2.2, $(e, c) \in \pi_2(\pi_1(U))$, which means

$$c \leq \bigvee\{y \in L_2 \mid e = a \vee y^T\}.$$

(3) If $a \vee \bigwedge X = \bigwedge\{a \vee x \mid x \in X\}$ holds for any $X \subseteq L_1$, then $\pi_1(U)$ is also fixed by π_2. It follows that

$$\pi_1(U) = q_1(a) \vee T,$$

hence c satisfies $e = a \vee c^T$.

Therefore, in each of the three cases,

$$q_2^r(q_1(a) \vee T) = \bigvee\{b \in L_2 \mid e = a \vee b^T\}.$$

$\square$

3. Separated frames.

Consider a frame L and its coproduct maps q_1, $q_2 : L \longrightarrow L \oplus L$. The codiagonal map $\nabla : L \oplus L \longrightarrow L$, given by $x \oplus y \rightsquigarrow x \wedge y$, is the coequalizer of q_1, q_2. As usual, ∇ has a dense factorization: $\nabla : L \oplus L \xrightarrow{(.)\vee s} \uparrow s \longrightarrow L$, where $s = \bigvee\{a \oplus b \mid a, b \in L, a \wedge b = 0\}$, called the *separator* of L.

We shall call a frame L *separated* if the codiagonal map ∇ is closed, that is, $\nabla \cong (.) \vee s$ for $s = \bigvee\{a \oplus b \mid a, b \in L, a \wedge b = 0\}$ (such L is also called *strongly Hausdorff* by Isbell [9]).

Proposition 3.1. *The following are equivalent for any frame L:*

(1) *L is separated.*
(2) *In $L \oplus L$, $(e \oplus a) \vee s = (a \oplus e) \vee s$ for all $a \in L$, where s is the separator of L.*
(3) *For any $h_1, h_2 : L \longrightarrow M$, $(.) \vee t : M \longrightarrow \uparrow t$ is the coequalizer, where $t = \bigvee\{h_1(a) \wedge h_2(b) \mid a, b \in L, a \wedge b = 0\}$.*
(4) *For any $h_1, h_2 : L \longrightarrow M$, $h_1(a) \vee t = h_2(a) \vee t$ for all $a \in L$.*

Remark. In general, even if a pair of homomorphisms $h_1, h_2 : L \longrightarrow M$ has its coequalizer of the form $(.) \vee t$ for some $t \in M$, this does not guarantee that $t = \bigvee\{h_1(a) \wedge h_2(b) \mid a, b \in L, a \wedge b = 0\}$, as it is shown by the following example: Take $L = \mathbf{3}$ = the chain of 3 elements $\{0, 1, 2\}$ and M = the Boolean algebra of 4 elements $\{0, a, a', e\}$. Let $h_1 : L \longrightarrow M$ be defined by ($0 \rightsquigarrow 0$, $1 \rightsquigarrow a$, $2 \rightsquigarrow e$), $h_2 : L \longrightarrow M$ be defined by ($0 \rightsquigarrow 0$, $1 \rightsquigarrow a'$, $2 \rightsquigarrow e$). The coequalizer of h_1, h_2 is $(.) \vee e : M \longrightarrow \{e\}$ but $\bigvee\{h_1(a) \wedge h_2(b) \mid a, b \in L, a \wedge b = 0\} = 0$.

From Isbell [9], we know that regularity is strictly stronger than separatedness. In the following, we shall see that the separatedness is a well behaved property.

The next result was first proved constructively by Vermeulen [13].

Proposition 3.2.

(1) *Compact separated frames are regular.*
(2) *Continuous separated frames are regular.*

Proof: Apply Proposition 2.4 in the case $L = L_1 = L_2$ and $T = \{(x, y) \mid x, y \in L, x \wedge y = 0\}$, the separator of L.

If $a \in L$ with $q_1(a) \vee T = q_2(a) \vee T$, then $a \leq q_2^r(q_1(a) \vee T)$, and it follows that $a \leq \bigvee\{b \in L \mid e = a \vee b^*\} = \bigvee\{b \in L \mid b \prec a\}$ if L is compact, or continuous. When L is separated, every element of L has the property of a just assumed. Therefore every compact (or, continuous) separated frame is regular. □

Proposition 3.3. *A frame L is a Boolean algebra if and only if it is separated and satisfies the law: $x \vee \bigwedge S = \bigwedge\{x \vee s \mid s \in S\}$ for any $S \subseteq L$.*

Proof: Only the " if " part needs proof.

In the proof of Proposition 2.4, we have seen that

$$a \vee c^T = e \text{ if } c = q_2^r(q_1(a) \vee T).$$

Consider $T = \{(x, y) \mid x, y \in L, x \wedge y = 0\}$. If L is separated, which gives $a \leq c = q_2^r(q_1(a) \vee T)$ for all $a \in L$, therefore $a \vee a^* = e$ for all $a \in L$. □

Proposition 3.4. *Suppose we have a pushout square in* **Frm***:*

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"'] & M \arrow[d, "\bar{u}"] \\ N \arrow[r, "\bar{v}"] & P \end{tikzcd}$$

with separated L. Then:

(1) *If M is compact, then $\bar{v}$ is dense whenever v is codense.*
(2) *If N is compact, then $\bar{v}$ is codense whenever v is codense.*
(3) *If N is continuous, then $\bar{v}$ is monic whenever v is codense.*

Proof: Considering the standard construction of pushouts, we can get the pushout square as follows:

$$\begin{tikzcd} L \arrow[rr, "v"] \arrow[dd, "u"'] & & M \arrow[dl, "q_2"] \arrow[dd, "\bar{u}"] \\ & N \oplus M \arrow[dr, "p"] & \\ N \arrow[ur, "q_1"] \arrow[rr, "\bar{v}"] & & \uparrow s \end{tikzcd}$$

where q_1, q_2 are the coproduct injections, $p = (.) \vee s : N \oplus M \longrightarrow \uparrow s$ is the coequalizer of $q_1 \circ u, q_2 \circ v$, and $s = \bigvee\{u(a) \oplus v(b) \mid a \wedge b = 0\} = \bigvee\{u(a) \oplus v(a^*) \mid a \in L\}$.

Put $A = \{(u(a), v(a^*)) \mid a \in L\}$, thus $s = \pi(\downarrow A)$ and $\downarrow A$ is fixed by μ.

(1) Consider any $x \in N$ such that $\bar{v}(x) = 0$. Then $x \oplus e \leq s = \pi(\downarrow A)$, hence $(x, e) \in \pi_1(\downarrow A)$ by the compactness of M and Proposition 2.2. Now there exists some $K \subseteq A$ such that

$$x \leq \bigvee p_1[K] \text{ and } e = \bigwedge p_2[K].$$

For each $(u(a), v(a^*)) \in K$, $v(a^*) = e$ implies $a^* = e$ since v is codense, hence $a = 0$. Therefore $x = 0$. This proves $\bar{v}$ is dense.

(2) Suppose $\bar{v}(x) = (x \oplus e) \vee s = e_{N \oplus M}$, that is,

$$(x \oplus e) \vee (\bigvee\{u(a) \oplus v(a^*) \mid a \in L\}) = e_{N \oplus M}.$$

By Proposition 2.3, we get $\bigvee\{v(a^*) \mid x \vee u(a) = e\} = e$ in M. Hence $\bigvee\{a^* \mid x \vee u(a) = e\} = e$ in L since v is codense, which implies $\bigvee\{u(a^*) \mid x \vee u(a) = e\} = e$ in N. But $x = x \vee u(a \wedge a^*) = (x \vee u(a)) \wedge (x \vee u(a^*)) = x \vee u(a^*)$, which implies $x \geq u(a^*)$, and therefore $x = e$. Hence $\bar{v}$ is codense, as claimed.

(3) Suppose $\bar{v}(x) = \bar{v}(y)$. For any $z \ll x$, from $x \oplus e \leq (y \oplus e) \vee \pi(\downarrow A)$ and Proposition 2.2, we get $(z, e) \in \pi_2 \circ \hat{\pi}_1(\downarrow (y, e) \cup \downarrow A)$. Then, by Lemma 2.4,

$$\begin{aligned} e &= \bigvee \{ \bigwedge p_2[K] \mid \bigvee p_1[K] \geq z \text{ for some finite } K \subseteq \{(y, e)\} \cup A \} \\ &\leq \bigvee \{ v(a^*) \mid y \vee u(a) \geq z \}, \end{aligned}$$

implying $e = \bigvee \{ a^* \mid y \vee u(a) \geq z \}$ since v is codense, hence $e = \bigvee \{ u(a^*) \mid y \vee u(a) \geq z \}$, which implies $z \leq y$. This shows $x \leq y$, hence $x = y$ by symmetry. □

Corollary. *If L is separated and N is spatial, then the pushout along every $u : L \longrightarrow N$ preserves monomorphisms.*

Proof: From Proposition 3.4, we know that pushouts along every $u : L \longrightarrow 2$ preserve monomorphisms, which leads to the claimed fact. □

Moreover, Proposition 3.4 provides a constructive proof of the following:

Proposition 3.5. *Pushouts preserve monomorphisms in the category* **KRegFrm** *of compact regular frames and also in the category* **RegConFrm** *of regular continuous frames and frame homomorphisms.*

4. Closed frame homomorphisms.

Definition 4.1. *A frame homomorphism $h : L \longrightarrow M$ is called closed if*

$$h^r(h(x) \vee y) = x \vee h^r(y) \quad \text{for any} \quad x \in L, y \in M.$$

Among various properties of closed homomorphisms, we only present those involving binary coproducts.

Proposition 4.1. *For frames L_1 and L_2, the coproduct injection $q_2 : L_2 \longrightarrow L_1 \bigoplus L_2$ is closed when one of the following conditions is satisfied:*

(1) *L_1 is compact.*

(2) *L_2 satisfies the law $x \vee \bigwedge S = \bigwedge \{ x \vee s \mid s \in S \}$ for any $S \subseteq L_2$.*

Proof: That $q_2 : L_2 \longrightarrow L_1 \bigoplus L_2$ is closed means, for any $T \in L_1 \bigoplus L_2$, $a \in L_2$,

$$q_2^r(q_2(a) \vee T) = a \vee q_2^r(T). \tag{1}$$

Because $q_2^r(T) = e^T$ for the unit $e \in L_1$, the equality (1) holds if and only if $(e, y) \in q_2(a) \vee T$ implies $y \leq a \vee e^T$.

To analyze (1), it is natural to start with

$$U = q_2(a) \cup T.$$

We have $\pi_2(U) = \{ (x, y) \mid y \leq a \vee x^T \}$, which is a downset fixed by μ.

Obviously, $(e, y) \in \pi_2(U)$ if and only if $y \leq a \vee e^T$. Therefore, (1) holds if and only if

$$(e, y) \in \pi(U) \text{ implies } (e, y) \in \pi_2(U). \tag{2}$$

(1) By Proposition 2.2, $(e, y) \in \pi(U) = \pi(\pi_2(U))$ implies $(e, y) \in \pi_2(U)$ since L_1 is compact.

(2) Now $\pi(U) = \pi_2(U)$. □

Furthermore, as a counterpart of Kuratowski-Mrówka theorem of general topology, the following fact has been obtained by Pultr and Tozzi [12]. By applying the results of binary coproducts developed in this paper, we can present a constructive proof.

Proposition 4.2 (Pultr and Tozzi)**.** *The frame M is compact if and only if $q_2 : L \longrightarrow M \oplus L$ is closed for any frame L.*

Proof: One direction is actually Proposition 4.1 (1). Another direction can be shown by the following modification of the corresponding proof in [12].

Suppose U is an updirected cover of M. Take M as an underlying set and define

$$\mathfrak{O}(M) = \{S \subseteq M \mid e \in S \ \text{ implies } \ \uparrow u \subseteq S \ \text{ for some } \ u \in U\},$$

then $\mathfrak{O}(M)$ is a topology on M.

Let $L = \mathfrak{O}(M)$, $q_2 : L \longrightarrow M \oplus L$ is closed means that

$$q_2^r((e \oplus a) \vee A) = a \vee q_2^r(A) \ \text{ for any } \ a \in L, \ A \in M \oplus L.$$

Now consider $a = M - \{e\}$ and $A = \bigvee\{u \oplus \uparrow u \mid u \in U\}$.

$$\begin{aligned}(e \oplus a) \vee A &= \bigvee\{u \oplus a \mid u \in U\} \vee \bigvee\{u \oplus \uparrow u \mid u \in U\}\\ &= \bigvee\{u \oplus (a \cup \uparrow u) \mid u \in U\}\\ &= \bigvee\{u \oplus e_L \mid u \in U\}\\ &= e_{M \oplus L}.\end{aligned}$$

Hence $a \vee q_2^r(A) = e_L$ since q_2 is closed, which implies $e \in q_2^r(A)$. Therefore there exists an element $v \in U$ such that $\uparrow v \subseteq q_2^r(A)$, that is

$$e \oplus \uparrow v \leq \bigvee\{u \oplus \uparrow u \mid u \in U\}.$$

By taking the meet with $e \oplus \downarrow v$ on both sides, we obtain

$$e \oplus \{v\} \leq \bigvee\{u \oplus [u, v] \mid u \in U\}.$$

where $[u, v] = \{x \in M \mid u \leq x \leq v\}$. Notice that $[x, v] \in \mathfrak{O}(M)$ for any $x \in M$.

Put $W = \{\bigvee K \mid K \subseteq U\}$ and $S = \downarrow\{(w, [w, v]) \mid w \in W\}$. Then S is fixed by π_1 and $e \oplus \{v\} \leq \pi(S)$. Since $\{v\} \in L$ is an atom, by Proposition 2.2, $(e, \{v\}) \in S$, which means $e = w$, and $\{v\} \subseteq [w, v]$, for some $w \in W$, therefore v must be e. □

The following result is more general than Proposition 4.1 (1).

Proposition 4.3. *If $h : L \longrightarrow M$ is closed and L has a basis B such that $h(B)$ consists of some compact elements of M, then $h \oplus id_N : L \oplus N \longrightarrow M \oplus N$ is closed for every frame N.*

Proof: We need to show, for any $T \in L \oplus N$, $S \in M \oplus N$,

$$(h \oplus id_N)^r((h \oplus id_N)(T) \vee S) \leq T \vee (h \oplus id_N)^r(S).$$

We assume $T = \bigvee\{a \oplus b \mid (a, b) \in A\}$ for some $A \subseteq L \times N$ and $S = \bigvee\{u \oplus v \mid (u, v) \in B\}$ for some $B \subseteq M \times N$ such that $p_1[A]$ and $p_1[B]$ are updirected. Then $(h \oplus id_N)(T) = \bigvee\{h(a) \oplus b \mid (a, b) \in A\}$.

Consider any

$$x \oplus y \leq (h \oplus id_N)^r((h \oplus id_N)(T) \vee S) \quad \text{with} \quad x \in B.$$

Then $h(x) \oplus y \leq (h \oplus id_N)(T) \vee S$. By the compactness of $h(x)$, Proposition 2.2 and Lemma 2.4, we have

$$\begin{aligned} y &\leq \bigvee\{b \wedge v \mid h(a) \vee u \geq h(x) \text{ with } (a, b) \in A \text{ and } (u, v) \in B\} \\ &= \bigvee\{b \wedge v \mid a \vee h^r(u) \geq x \text{ with } (a, b) \in A \text{ and } (u, v) \in B\}, \end{aligned}$$

which implies

$$x \oplus y \leq \bigvee\{a \oplus b \mid (a, b) \in A\} \vee \bigvee\{h^r(u) \oplus v \mid (u, v) \in B\} \leq T \vee (h \oplus id_N)^r(S).$$

□

Remark. Recall that a continuous mapping $f : X \longrightarrow Y$ is called perfect if the map $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is closed for every space Z. We know that *$f : X \longrightarrow Y$ is perfect if and only if f is closed and the fibre $f^{-1}(y)$ is compact for each $y \in Y$.* To some extent, the above proposition can be regarded as a frame counterpart of this topological fact.

For any homomorphism $h : L \longrightarrow M$, there exists uniquely an onto homomorphism $G(h) : L \oplus M \longrightarrow M$ such that $G(h) \circ q_1 = h$ and $G(h) \circ q_2 = id_M$. $G(h)$ is given by $x \oplus y \rightsquigarrow h(x) \wedge y$ and is the coequalizer of

$$q_1 : L \longrightarrow L \oplus M \quad \text{and} \quad q_2 \circ h : L \longrightarrow M \longrightarrow L \oplus M.$$

Consider the factorization $h : L \xrightarrow{q_1} L \oplus M \xrightarrow{G(h)} M$. If L is separated, then $G(h)$ is closed by Proposition 3.1. If M is compact, then q_1 is closed by Proposition 4.1. Therefore, we have proved

Proposition 4.4. *For separated L and compact M, any frame homomorphism $h : L \longrightarrow M$ is closed.*

Finally, let us allow the Axiom of Choice, so we can talk about cardinalities.

Let κ be a regular cardinal. A frame L is called a $D(\kappa)$-frame if it satisfies the following law:

$$D(\kappa): \qquad a \vee \bigwedge S = \bigwedge\{a \vee s \mid s \in S\} \text{ for } |S| < \kappa.$$

In the definitions for compact elements and updirected sets, replacing "a finite subset" by "a subset with cardinality strictly smaller than κ", we get the definitions for κ-compact elements and κ-updirected sets.

Now we continue the observation launched by Proposition 4.1.

Proposition 4.5. *The coproduct map $q_2 : L_2 \longrightarrow L_1 \oplus L_2$ is closed if one of the following holds:*

(1) *L_1 is κ-compact and regular, L_2 is a $D(\kappa)$-frame.*

(2) *L_1 has a basis B with $| B |< \kappa$, L_2 is a $D(\kappa)$-frame.*

PROOF: Continue to consider the implication (2) in the proof of Proposition 4.1.

(1) Suppose $X \times \{y\} \subseteq \pi_2(U)$ for some $X \subseteq L_1$ with $|X| < \kappa$. Then $y \leq a \vee x^T$ for each $x \in X$, which implies $y \leq \bigwedge\{a \vee x^T \mid x \in X\} = a \vee \bigwedge\{x^T \mid x \in X\} = a \vee (\bigvee X)^T$ since L_2 satisfies the law of $D(\kappa)$. It follows that $(\bigvee X, y) \in \pi_2(U)$. It turns out

$$\pi_1(\pi_2(U)) = \{(\bigvee D, y) \mid \kappa - \text{updirected } D \text{ and } D \times \{y\} \subseteq \pi_2(U)\}.$$

We claim that $\pi_1(\pi_2(U))$ is also fixed by π_2: Consider $\{x\} \times Y \subseteq \pi_1(\pi_2(U))$. For each $y \in Y$, suppose $x = \bigvee D_y$ with $D_y \times \{y\} \subseteq \pi_2(U)$. Then

$$(\bigwedge_{y \in Y} d_y, \bigvee Y) \in \pi_2(U) \text{ for } d = (d_y)_{y \in Y} \in \prod_{y \in Y} D_y,$$

so

$$(\bigvee_d \bigwedge_{y \in Y} d_y, \bigvee Y) \in \pi_1(\pi_2(U)).$$

Now,

$$x = \bigwedge_{y \in Y} \bigvee D_y = \bigvee_d \bigwedge_{y \in Y} d_y$$

since the κ-compact regular frame L_1 must satisfy this distributive law. Therefore $(x, \bigvee Y) \in \pi_1(\pi_2(U))$, as expected. This shows $\pi(U) = \pi_1(\pi_2(U))$.

Thus, when $(e, y) \in \pi(U)$, there is a κ-updirected D such that $e \leq \bigvee D$ and $(d, y) \in \pi_2(U)$ for $d \in D$. That e is κ-compact implies $e \in D$, hence $(e, y) \in \pi_2(U)$.

(2) Consider any $X \subseteq L_1$ and $y \in L_2$ satisfying $X \times \{y\} \subseteq \pi_2(U)$. Take $B_1 = \{b \in B \mid b \leq x \text{ for some } x \in X\}$, then $\bigvee B_1 = \bigvee X$. We have

$$y \leq \bigwedge\{a \vee b^T \mid b \in B_1\} = a \vee (\bigvee B_1)^T = a \vee (\bigvee X)^T,$$

which means $(\bigvee X, y) \in \pi_2(U)$. Therefore $\pi(U) = \pi_2(U)$. $\square$

5. Homomorphisms from separated to continuous frames.

Lemma 5.1. *Let L be separated and M continuous. If $h : L \longrightarrow M$ is dense onto, then the set $\{x \in L \mid h(x) = e\}$ has a least element.*

Proof: Put $m = \bigvee\{h^r(c) \mid c \ll e \text{ in } M\}$.

First, $h(m) = \bigvee\{hh^r(c) \mid c \ll e\} = \bigvee\{c \mid c \ll e\} = e$.

Second, since M is regular continuous, $c \ll e$ means that $\uparrow c^*$ is compact. The composite $L \longrightarrow M \longrightarrow \uparrow c^*$ is closed by Proposition 4.4, hence $h^r(h(a) \vee c^*) = a \vee h^r(c^*)$ for any $a \in L$. Consider any $x \in L$ with $h(x) = e$. We get $e = h^r(h(x) \vee c^*) = x \vee h^r(c^*)$, thus $h^r(c) = (x \wedge h^r(c)) \vee (h^r(c^*) \wedge h^r(c)) = x \wedge h^r(c)$, that is $h^r(c) \leq x$. This shows $m \leq x$. Therefore m is the required least element.

□

Lemma 5.2. *Let L be separated and M continuous. If $h : L \longrightarrow M$ is dense, codense and onto, then h is an isomorphism.*

Proof: Let $k = id_L \oplus h : L \oplus L \longrightarrow L \oplus M$, and $s = \bigvee\{x \oplus y \mid x \wedge y = 0\}$ in $L \oplus L$. Then, since L is separated, for $a, b \in L$,

$$a \oplus e \leq (e \oplus a) \vee s, \text{ and } e \oplus b \leq (b \oplus e) \vee s \text{ in } L \oplus L.$$

Now, suppose $h(a) = h(b)$.

Acting k on the above two inequalities, we obtain

$$a \oplus e \leq (e \oplus h(a)) \vee k(s) = (e \oplus h(b)) \vee k(s) \leq (b \oplus e) \vee k(s).$$

Since h is dense onto, $h(x^*) = h(x)^*$ for any $x \in L$. Thus $h(y) = z$ implies $y^* \leq h^r(z^*)$. Therefore

$$\begin{aligned} a \oplus e &\leq \bigvee\{x \oplus h(y) \mid x \leq b \text{ or } x \wedge y = 0\} \\ &\leq \bigvee\{x \oplus h(y) \mid x \leq b \vee y^*\} \\ &\leq \bigvee\{x \oplus z \mid x \leq b \vee h^r(z^*)\}. \end{aligned}$$

Let

$$T = \{(x, z) \mid x \leq b \vee h^r(z^*)\},$$

which is closed under π_1 and $\hat{\pi}_2$. By Proposition 2.2, $(a, e) \in \pi(T)$ and $c \ll e$ imply $(a, c) \in T$, that is $a \leq b \vee h^r(c^*)$, then $h^r(c) \wedge a \leq b$. Thus $(\bigvee\{h^r(c) \mid c \ll e\}) \wedge a \leq b$. On the other hand, $h(\bigvee\{h^r(c) \mid c \ll e\}) = \bigvee\{c \mid c \ll e\} = e$, which implies $\bigvee\{h^r(c) \mid c \ll e\} = e$ by h codense, hence $a \leq b$. By symmetry, we also have $b \leq a$. Thus h is one-one. □

Proposition 5.1. *For separated L and continuous M, if M is an image of L, then there exist two elements $s, m \in L$ such that $[s \wedge m, m] \cong M$.*

Proof: Given an onto homomorphism $h : L \longrightarrow M$, let $m \in L$ be the least element such that $h(m) = e$ by Lemma 5.1, and $s \in L$ the largest element such that $h(s) = 0$. Then h can be factored as

$$h : \xrightarrow{(((.) \vee s) \wedge m)} [s \wedge m, m] \xrightarrow{\bar{h}} M.$$

Now $\bar{h}$ is dense, codense and onto, therefore $\bar{h}$ is an isomorphism. □

Remark. This is the frame version of the topological fact that, *in a T_2 space X, every locally compact subspace A is locally closed, that is, A is the intersection of an open subset and a a closed subset of X* (see [5]).

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