$\in$-representation and set-prolongations

Josef Mlček

Abstract. By an $\in$-representation of a relation we mean its isomorphic embedding to $\mathbb{E} = \{\langle x, y\rangle;\ x \in y\}$. Some theorems on such a representation are presented. Especially, we prove a version of the well-known theorem on isomorphic representation of extensional and well-founded relations in $\mathbb{E}$, which holds in Zermelo-Fraenkel set theory. This our version is in Zermelo-Fraenkel set theory false. A general theorem on a set-prolongation is proved; it enables us to solve the task of the representation in question.

Keywords: isomorphic representation, extensional relation, well-founded relation, set-prolongation

Classification: 03E70, 04A99

We prove that, in the alternative set theory, each weakly extensional and well-founded set-relation is strongly $\in$-representable. It means that there exists a set-mapping which is an isomorphism of the relation in question and a subrelation of the relation $\mathbb{E} = \{\langle x, y\rangle;\ x \in y\}$. We present a general theorem on a set-prolongation. This theorem guarantees, to a given weakly extensional and well-founded relation, its set-superrelation with the same two properties. Thus the relation in question has an $\in$-representation. Consequently, each model with absolute equality of Zermelo-Fraenkel set theory is $\in$-representable. For countable models, this result was firstly proved by Vopěnka (unpublished).

Convention. We use the usual notation of the alternative set theory. We put, having a relation R, $fld(R) = dom(R) \cup rng(R)$. We denote the class of all finite subsets of a class X as $P_f(X)$.

$\in$-representations of set-relations.

Let R be a binary relation. We shall write $R(x)$ instead of $R''\{x\}$ and $R[y]$ instead of $R^{-1''}\{y\}$.

Convention. In this paper, let R be a binary nonempty relation and let 0_R be an element from $dom(R) - rng(R)$.

We have, consequently, $R[0_R] = \emptyset$.

A mapping H is said to be an *$\in$-representation of* $\langle R, 0_R\rangle$ if we have

1) $H : fld(R) \to V$ is a one-one mapping,
2) $x, y \in fld(R) \Rightarrow (\langle x, y\rangle \in R \Leftrightarrow H(x) \in H(y)\ \ \&\ \ H(0_R) = \emptyset)$.

An $\in$-representation H is *strong* if we have, moreover,

3) $y \in rng(R) \Rightarrow H(y) = H''R[y]$,
4) $x \in dom(R) - rng(R) - \{0_R\} \Rightarrow H(x)$ is infinite.

We say that R is *weakly extensional* – formally $wex(R)$ – if we have

$$x, y \in rng(R) \ \& \ x \neq y \Rightarrow R[x] \neq R[y].$$

R is said to be *well-founded* – formally $wf(R)$ – if we have

$$(\forall u \subseteq fld(R))(u \neq \emptyset \Rightarrow (\exists y \in u)(R[y] \cap u = \emptyset)).$$

Note that having a nonempty well-founded set-relation r, we can see that $(\exists x \in dom(r) - rng(r))(r[x] = \emptyset)$. Especially, $dom(r) - rng(r) \neq \emptyset$ holds.

Theorem. *Let r be a set-relation, $0_r \in dom(r) - rng(r)$.Then r is weakly extensional and well-founded iff there exists a strong $\in$-representation of $\langle r, 0_r \rangle$ which is a set.*

Proof: The implication from the right to the left is easy. Suppose that r is weakly extensional and well-founded. Put $v = rng(r)$ and $w = dom(r) - rng(r)$. We have $0_R \in w$. We denote by $\tau(x)$ the type of a set x, i.e. $\tau(x) = min\{\alpha; x \in P_\alpha\} - 1$, where $P_0 = \emptyset$ and $P_{\alpha+1} = P(P_\alpha)$. By an $\in$-chain of the length δ we mean a set $\{z_\alpha;\ 1 \leq \alpha \leq \delta\}$ such that we have $z_\delta \in z_{\delta-1} \in \cdots \in z_1$. We denote such a chain as $z|\delta$. We say that $z|\delta$ is under x if we have $z_1 \in x$. We have for each $\delta \geq 1$: $\tau(x) = \delta$ implies that there is an $\in$-chain of the length δ which is under x. Assume $\gamma \geq 1$. Suppose, moreover, that each $\in$-chain under x has the length less than γ. Then $\tau(x) < \gamma$.

Suppose that $\theta \in N$ is such a number that we have

i) $\theta > ||v||$, where $||v||$ is the set-cardinality of the set v, i.e. $||v|| \in N$ and there exists a one-one set-mapping between v and $||v||$,
ii) there exists a set $\{e_x;\ x \in w - \{0_r\}\}$ such that each e_x is infinite, $\tau(z) = \theta$ holds for each $z \in e_x$ and we have, for each $x, y \in w - \{0_r\}$, $x \neq y \Rightarrow e_x \neq e_y$.

We define sets u_α as follows: $u_0 = w$, $u_{\alpha+1} = \{x \in v;\ r[x] \subseteq u_\alpha\} \cup w$. We can see that $u_\alpha \subseteq u_{\alpha+1}$ holds for each α. We have, moreover, a number γ such that $\alpha \geq \gamma \Rightarrow u_\alpha = u_\gamma = v \cup w$.

We define, for each α, the mapping $h_\alpha : u_\alpha \to V$ by the relations: $h_0(0_r) = \emptyset$, $h_0(x) = e_x$ for each $x \in w - \{0_r\}$, $h_{\alpha+1}(y) = h_\alpha{}'' r[y]$ for each $y \in u_{\alpha+1} - w\,(= u_{\alpha+1} \cap v)$, $h_{\alpha+1}(y) = h_0(y)$ for each $y \in w$. We can easily prove that, for each α, $h_\alpha \subseteq h_{\alpha+1}$ holds.

Let us formulate two lemmas. We denote by $Univ(x)$ the universe of the set x.

Lemma. *Assume that $y \in rng(r) \cap u_\alpha$ and let $Univ(h_\alpha(y)) \cap \{e_x;\ x \in w - \{0_r\}\} = \emptyset$. Then $\tau(h_\alpha(y)) \leq ||rng(r)||$ holds.*

Proof: Let $z|\delta$ be an $\in$-chain under $h_\alpha(y)$. Let us prove that $\delta \leq ||rng(r)||$. We shall write h instead of h_α. Thus we have $z_\delta \in z_{\delta-1} \in \cdots \in z_1 \in h(y)$, where $\{z_\alpha;\ 1 \leq \alpha \leq \delta\} = z|\delta$. We deduce from the fact $h(y) = h'' r[y]$ that there exists a set y_1 such that $y_1 \in r[y]$ and $z_1 = h(y)$. Suppose that $r[y] \cap (w - \{0_r\}) \neq \emptyset$. Then $e_x \in h(y)$ holds for some $x \in w - \{0_r\}$. It follows from the formula $x \in r[y] \cap (w -$

$\{0_r\}) \Rightarrow h(x) = e_x = h(y)$. We deduce from this that $Univ(h(y)) \cap \{w - \{0_r\}\} \neq \emptyset$, which is a contradiction. Thus we have $r[y] \subseteq v \cup \{0_r\}$. Assuming $y_1 = 0_r$, we obtain that $z_1 = h(0_r) = \emptyset$. Thus $\delta = 1$. Suppose $\delta > 1$. Then $y_1 \in v$.

Assume that $1 \leq \beta \leq \delta$ and let $\{y_\alpha;\ 1 \leq \alpha \leq \beta\} \subseteq v$ be a set such that $y_\beta r y_{\beta-1} r \dots r y_1 r y$ and let $h(y_\alpha) = z_\alpha$ for each $1 \leq \alpha \leq \beta$. We have $z_{\beta+1} \in h(y_\beta) = h''r[y_\beta]$. Thus there exists a $y_{\beta+1} \in r[y_\beta]$ such that $z_{\beta+1} = h(y_{\beta+1})$. Assume that $y_{\beta+1} = 0_r$. Then $z_{\beta+1} = h(0_r) = \emptyset$ and, consequently, $\beta + 1 = \delta$ holds. Assume $\beta + 1 < \delta$. Then $y_{\beta+1} \in v$. It follows from the fact that $y_{\beta+1} \in w - \{0_r\}$ implies $z_{\beta+1} \in \{e_x;\ x \in w - \{0_r\}\} \cap Univ(h(y))$ which is a contradiction.

Thus, there exists a set $\{y_\alpha;\ 1 \leq \alpha < \delta\} \subseteq v$ such that $y_{\delta-1} r y_{\delta-2} r \dots y_1 r y$ holds. The relation r is well-founded. We deduce from this that $\delta \leq ||v||$. Thus each $\in$-chain under $h(y)$ has the length less or equal to $||v||$. Consequently, $\tau(h(y)) \leq ||v||$ holds. □

Lemma. *Each mapping h_α is a one-one mapping.*

Proof: We shall prove it by induction on α. If $\alpha = 0$ then the assertion holds. Assume that h_α is a one-one mapping; we shall prove that $h_{\alpha+1}$ has the same properties. Suppose that $x, y \in u_{\alpha+1}$ are such that $h_{\alpha+1}(x) = h_{\alpha+1}(y)$.

a) $x, y \in w$. Then $x = y$ follows directly from the definition of $h_{\alpha+1}$.

b) $x, y \in v$. Then $h_\alpha{}''r[x] = h_{\alpha+1}(x) = h_{\alpha+1}(y) = h_\alpha{}''r[y]$. We deduce from the induction hypothesis that $r[x] = r[y]$. The equality $x = y$ follows from this by using the weak extensionality of r.

c) $x \in v$, $y \in w$. Assume, at first, that $y = 0_r$. We have $h_{\alpha+1}(y) = \emptyset$, $h_{\alpha+1}(x) = \emptyset$. But $h_{\alpha+1}(x) = h_\alpha{}''r[x] \neq \emptyset$, which is a contradiction. Assume, secondly, that $y \neq 0_r$. We have $h_{\alpha+1}(x) = h_{\alpha+1}(y) = e_y$. Suppose that $Univ(h_{\alpha+1}(x)) \cap \{e_z;\ z \in w - \{0_r\}\} \neq \emptyset$. Then $\tau(h_{\alpha+1}(x)) > \tau(e_y)$, which is a contradiction. Suppose that $Univ(h_{\alpha+1}(x)) \cap \{e_z;\ z \in w - \{0_r\}\} \neq \emptyset$. We deduce from this assumption and by using the previous lemma that $\tau(h_{\alpha+1}(x)) \leq ||v|| < \tau(e_y)$, which is impossible. □

Let us finish the proof of our theorem. Choose δ such that $u_\delta = v \cup w$ $(= dom(r) \cup rng(r))$ and put $u = u_\delta$ and $h = h_\delta$. Now, we have the following: h is a one-one mapping such that $x \in rng(r) \Rightarrow h(x) = h''r[x]$, $x \in dom(r) - rng(r) - \{0_r\} \Rightarrow h(x)$ is infinite, $h(0_r) = \emptyset$ and $\langle x, y\rangle \in r \Rightarrow h(x) \in h(y)$. Thus, only the following must be proved:

$$x, y \in dom(r) \cup rng(r) \Rightarrow (h(x) \in h(y) \Rightarrow \langle x, y\rangle \in r).$$

Suppose that $x, y \in dom(r) \cup rng(r)$ and let $h(x) \in h(y)$. We have $y \neq 0_r$.

α) $x, y \in w$. Then $h(y) = e_y$ and, consequently, $h(x) \in h(y)$ is false. (Indeed, we have $h(x) = e_x$ or $h(x) = 0_r$. But neither $e_x \in e_y$ for some $x, y \in w - \{0_r\}$ nor $\emptyset \in e_y$ holds.)

β) $y \in v$. We have $h(x) \in h''r[y]\,(= h(y))$. Thus $h(x) = h(z)$ holds for some $z \in r[y]$. The mapping h is a one-one. Consequently $z = x$ is satisfied and we have $\langle x, y\rangle \in r$.

γ) $x \in v$, $y \in w$. Suppose that

$$(*) \qquad Univ(h(x)) \cap \{e_z;\ z \in w - \{0_r\}\} \neq \emptyset$$

We deduce from this that $\tau(h(x)) > \tau(h(y))$. But it is a contradiction with our assumption that $h(x) \in h(y) = e_y$. Suppose that $(*)$ is not true. We have $\tau(h(x)) \leq ||v||$. But the relation $\tau(h(x)) = \theta$ follows from the assumption that $h(x) \in e_y$. We have $\theta > ||v||$, which is a contradiction.

Set-prolongation.

Our aim is to present a method of a prolongation of a given class, say X, to a set, say d, such that $X \subseteq d$ and the set d has some properties as X. We see that this purpose is essentially limited by the fact that d is a formally finite set. Thus, only some properties of X can be transferred on d.

We formulate a theorem on set-prolongation below. Before we give it, let us introduce one definition.

Let X be a class and let Γ be a class of set formulas of the language FL_V with exactly one free-variable x. We say that Γ is an *f-type over* X if we have for each finite set $\{\varphi_1 \dots, \varphi_k\} \subseteq \Gamma$ the following

$$(\forall u \in P_f(X))(\exists v \in P_f(X))(u \subseteq v \ \& \ \varphi_1(v) \ \& \ \dots \varphi_k(v)),$$

where $\varphi_i(v)$ denotes the formula which is obtained from φ by replacing all of the occurrence of the variable x by v.

Theorem (on set-prolongation). *Let Γ be an f-type over a class X. Then there exists an endomorphism $\mathcal{F}$ and a set d such that we have:*

1) $\mathcal{F}''X = \mathcal{F}''V \cap d$.
2) *If $\varphi(x, p_1, p_2, \dots, p_l) \in \Gamma$ and $\varphi(x, x_1, x_2, \dots, x_n)$ is a formula of the language FL, then $\varphi(d, \mathcal{F}(p_1), \mathcal{F}(p_2), \dots, \mathcal{F}(p_l))$,*
3) *Let $\varphi(x, p_1, p_2, \dots, p_l)$ be a set-formula of the language FL_V with exactly one set-variable x and suppose that $(\exists u \in P_f(X))(\forall v \in P_f(X))(u \subseteq v \Rightarrow \varphi(v, p_1, p_2, \dots, p_l)$. Then $\varphi(d, \mathcal{F}(p_1), \mathcal{F}(p_2), \dots, \mathcal{F}(p_l))$ holds.*

Proof: We sketch a proof by using the notion of the coherency [V] which states the following. Let $\mathfrak{M}$ be an ultrafilter on the ring Sd_V of all set-theoretically definable classes. Then $\mathcal{F}, \mathfrak{M}, d$ are coherent if $\{x;\, \varphi(x, p_1, p_2, \dots, p_l)\} \in \mathfrak{M} \Leftrightarrow \varphi(d, \mathcal{F}(p_1), \mathcal{F}(p_2), \dots, \mathcal{F}(p_l))$ holds for each set-formula $\varphi(x_0, p_1, p_2, \dots, p_l)$ of FL_V with exactly one free-variable x_0 and such that $p_1, p_2, \dots, p_l \in dom(\mathcal{F})$.

Let

$$\mathfrak{M}_0 = \{\{x;\, \varphi(x, p_1, p_2, \dots, p_l)\};\ \varphi(x_0, p_1, p_2, \dots, p_l) \in \Gamma \text{ or } \varphi(x_0, p_1, p_2, \dots, p_l) \text{ is a set-formula of } FL_V \text{ with exactly one free-variable } x_0 \text{ such that } (\exists u \in P_f(X))(\forall v \in P_f(X))(u \subseteq v \Rightarrow \varphi(v, p_1, p_2, \dots, p_l))\}.$$

Then $\mathfrak{M}_0$ is a centered system of set-theoretically definable classes. Let $\mathfrak{M}$ be an ultrafilter on Sd_V such that $\mathfrak{M}_0 \subseteq \mathfrak{M}$. There exists an endomorphism $\mathcal{F}$ and a set d such that $\mathcal{F}, \mathfrak{M}$, d are coherent. It follows from the first theorem of Section 2, Chapter V in [V]. We can see that 2), 3) hold. Let us prove 1). We have

$\{x;\, y \in x\} \in \mathfrak{M} \Leftrightarrow y \in X$ and $\{x;\, y \in x\} \in \mathfrak{M} \Leftrightarrow \mathcal{F}(y) \in d$. Thus $\mathcal{F}(y) \in d \Leftrightarrow y \in X$ holds. $\square$

$\in$-representations.

We say that a binary relation R is *without cycles* if there is no sequence $\{x_1, x_2, \dots, x_n\} \subseteq fld(R)$ such that $x_1 \, R \, x_n \, R \, x_{n-1} \, \dots \, R \, x_1$ holds.

Theorem. *Let R be a weakly extensional relation without cycles and let $0_R \in dom(R) - rng(R)$. Then we have:*

1) *There exist a relation S and 0_S such that $\langle R, 0_R\rangle$ is isomorphic to $\langle S, 0_S\rangle$ and there exists a weakly extensional and well-founded set-relation r such that $S \subseteq r$ and $0_S \in dom(r) - rng(r)$.*
2) *There exists a class K such that $\emptyset \in K$ and $\langle fld(R), R, 0_R\rangle$ is isomorphic to $\langle K, \mathbb{E} \cap K^2, \emptyset\rangle$.*

Proof: Let us prove, at first, that $\{wex(x),\, wf(x)\}$ is an f-type over R. Assume that $s \subseteq R$ is finite. It is easy to see that s is well-founded. We must find a finite weakly-extensional relation r such that $s \subseteq r \subseteq R$. Put $v = rng(s)$ and, for each $\{x, y\} \in [v]^2$, let $d_{xy} \in \triangle(R[x], R[y])$, where $\triangle$ is the symmetric difference. Put $r = s \cup \{\langle d_{xy}, x\rangle \in R;\, \{x, y\} \in [v]^2\}$. We have $rng(r) = v$ and $\{x, y\} \in [v]^2$ implies $d_{xy} \in \triangle(r[x], r[y])$. Thus r is weakly extensional.

We can easily see that $\{x;\, (\exists y, z)(x = \{1\} \times y \cup \{2\} \times z \;\&\; wex(y) \;\&\; wf(y) \;\&\; z \in dom(y) - rng(y))\}$ is an f-type over $\{1\} \times R \cup \{2\} \times \{0_R\}$.

Now, we deduce from the previous theorem that there exist an endomorphism F, a set-relation r and a set e such that $F''(\{1\} \times R \cup \{2\} \times \{0_R\}) = F''V \cap (\{1\} \times r \cup \{2\} \times \{e\})$. Put $S = F''R$. We have $\langle x, y\rangle \in R \Leftrightarrow \langle F(x), F(y)\rangle \in S$, i.e. F is an isomorphism of R and S. Put $0_S = F(0_R)$. We have $0_S \in dom(F''R) - rng(F''R)$. Thus $\langle S, 0_S\rangle$ has the required properties.

2) We know that there exists a strong $\in$-representation h of $\langle r, 0_S\rangle$. Let us define a mapping $H : fld(R) \to V$ by $H(x) = h(F(x))$ and put $K = H''fld(R)$. Then H is an isomorphism of R and $\mathbb{E} \cap K^2$. We have, moreover, $H(0_R) = h(0_S) = \emptyset$. $\square$

Corollary. *Let $\langle A, R\rangle$ be a model of ZF with absolute equality and let $0_R \in A$ be such that $\langle A, R\rangle \models$ "0_R is the empty set". Then there exists a class M such that the structures $\langle A, R, 0_R\rangle$ and $\langle M, \mathbb{E} \cap M^2, \emptyset\rangle$ are isomorphic.*

Proof: It is clear that R is an extensional relation and, consequently, weakly extensional one. R is without cycles, too. We have $dom(R) - rng(R) = \{0_R\}$. We deduce from the previous theorem that there exists a class M with the required properties. $\square$

Note: The just presented assertion can be strengthened. We can find the class M in question such that, in addition, some gödelian operations are absolute for the model $\langle M, \mathbb{E} \cap M^2, \emptyset\rangle$. Naturally, the transitivity of M cannot be guaranteed.

A publication of these results is in preparation.

References

[V] Vopěnka P., *Mathematics in the Alternative Set Theory*, TEUBNER TEXTE, Leipzig, 1979.

Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 110 00 Praha 1, Czechoslovakia

(*Received* May 13, 1992)