

Valuations of lines

JOSEF MLČEK

Abstract. We enlarge the problem of valuations of triads on so called lines. A line in an e -structure $\mathbb{A} = \langle A, F, E \rangle$ (it means that $\langle A, F \rangle$ is a semigroup and E is an automorphism or an antiautomorphism on $\langle A, F \rangle$ such that $E \circ E = \mathbf{Id} \upharpoonright A$) is, generally, a sequence $\mathbb{A} \upharpoonright B, \mathbb{A} \upharpoonright U_c, c \in \mathbf{FZ}$ (where \mathbf{FZ} is the class of finite integers) of substructures of \mathbb{A} such that $B \subseteq U_c \subseteq U_d$ holds for each $c \leq d$. We denote this line as $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ and we say that a mapping H is a valuation of the line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in a line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ if it is, for each $c \in \mathbf{FZ}$, a valuation of the triad $\mathbb{A}(U_c, B)$ in $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})$. Some theorems on an existence of a valuation of a given line in another one are presented and some examples concerning equivalences and ideals are discussed. A generalization of the metrization theorem is presented, too.

Keywords: valuation, triad, metrization theorem, semigroup

Classification: 03E70, 54E35, 20M14

The problem of valuations concerns the question whether there exists a certain representation H of a given structure \mathbb{A} in another one $\hat{\mathbb{A}}$. The structures in question are semigroups with a certain automorphism or antiautomorphism i.e. structures of the form $\langle A, F, E \rangle$ where $\langle A, F \rangle$ is a semigroup and E is an automorphism or an antiautomorphism of $\langle A, F \rangle$ such that $E \circ E$ is the identity on A . The described structure is called an e -structure. The mentioned representation preserves some structural properties of the relevant e -structures; it is called a valuation of \mathbb{A} in $\hat{\mathbb{A}}$. (See [M2], [M3].) We specify, moreover, this general situation demanding to find a valuation of certain “descriptive” type in a hierarchy of classes. For example, having an e -structure $\mathbb{A} = \langle \mathcal{P}(a), \cup, \mathbf{Id} \rangle$, where a is a set, and the e -structure $\hat{\mathbb{A}} = \langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ (where \mathbf{Id} is everywhere the identity on the relevant class and \mathbf{Q}^+ is the class of all non-negative rational numbers), we look for a set-valuation H of \mathbb{A} in $\hat{\mathbb{A}}$. This means that we have $H : \mathcal{P}(a) \rightarrow \mathbf{Q}^+$ and H is a set-mapping such that $H(u \cup v) \leq H(u) + H(v)$. The relation \leq is the so called canonical relation of the e -structure $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$; it is defined in this structure by the relation $x \leq y \Leftrightarrow (\exists z)(x + z = y)$. The solution of our task is trivial: Let $r \in \mathbf{Q}^+$ be fixed. Putting $H(u) = r$ for all $u \subseteq a$ we obtain a required mapping. Assume, moreover, that $U \subseteq \mathcal{P}(a)$ is a subclass closed under operation \cup , and put $[0]^+ = \{r \in \mathbf{Q}^+; r \neq 0\}$. Then $[0]^+$ is a subclass of \mathbf{Q}^+ closed under the operation $+$. Now, the task to find a set-valuation as above with the additional property that $U = H^{-1}''[0]^+$ and $\{\emptyset\} = H^{-1}''\{0\}$ is more complicated. The required mapping is said to be a (set-) valuation of the triad $\mathbb{A}(U, \{\emptyset\})$ (i.e. of the triple $\mathbb{A}, \mathbb{A} \upharpoonright U, \mathbb{A} \upharpoonright \{\emptyset\}$ of substructures of the structure \mathbb{A}) in the triad $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle([0]^+, \{0\})$. The last triad

is the so called canonical π -one; it is denoted as \mathcal{T}_π . A canonical σ -triad is the triad $(\mathbf{N}, +, \mathbf{Id})(\mathbf{FN}, \{0\})$; we denote this triad as \mathcal{T}_σ . It is known, for example, that every triad $\mathbb{A}(U, B)$, where \mathbb{A}, B are sets and U is a π -class (σ -class resp.) has a set-valuation in \mathcal{T}_π (\mathcal{T}_σ -resp.).

In this article, we deal with so called lines. A line in a given e -structure \mathbb{A} over B is a sequence $\mathbb{A} \upharpoonright B, \mathbb{A} \upharpoonright U_c, c \in \mathbf{FZ}$ of substructures of \mathbb{A} such that $B \subseteq U_c \subseteq U_d$ holds for each $c \leq d, c, d \in \mathbf{FZ}$. We denote such a line as $\mathbb{A}(U_c, B)$. A mapping H is a valuation of a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in a line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ if it is, for each $c \in \mathbf{FZ}$, a valuation of the triad $\mathbb{A}(U_c, B)$ in $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})$.

We shall study the so called $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines. It means that \mathbb{A} and B belong to a so called standard universe \mathfrak{S} of classes, the classes U_c with c odd are $\pi(\mathfrak{S})$ -classes, i.e. classes of the form $\bigcap_{n \in \mathbf{FN}} X_n$, where $\{X_n\}_n \subseteq \mathfrak{S}$, and the classes U_c with c even are $\sigma(\mathfrak{S})$ -classes, i.e. classes of the form $\bigcup_{n \in \mathbf{FN}} X_n$, where $\{X_n\}_n \subseteq \mathfrak{S}$. We shall look for a valuation $H \in \mathfrak{S}$ of such a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in a canonical one. By a canonical line we mean a line $\mathcal{L}_{\pi-\sigma}(\zeta)$ which is defined in the section Lines and valuations; this line is a line in the e -structure $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ over $\{\emptyset\}$.

The system Sd_V of all set-theoretically definable classes is the basical example of the standard universe \mathfrak{S} of classes. Note that, generally, we cannot find a valuation H of a given $(\pi(Sd_V) - \sigma(Sd_V))$ -line in a canonical one such that H belongs to Sd_V ; see Remarks in the paragraph Valuations of $(\pi - \sigma)$ -Lines. We prove that our problem of valuations of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines is solvable in so called saturated standard universes of classes. Let us present that the revealment Sd_V of the system Sd_V is such a saturated standard universe of classes.

Let us introduce briefly two domains of applications of valuations of lines.

A sequence $\{\mathcal{E}_c\}_{c \in \mathbf{FZ}}$ of equivalences on a set z such that $\mathcal{E}_c \subseteq \mathcal{E}_{c+1}$ holds for each $c \in \mathbf{FZ}$, each \mathcal{E}_c with c odd is a π -class and each \mathcal{E}_c with c even is a σ -class, can be seen as a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line. This fact enables us to state a generalization of some known metrization theorems of equivalences ([M2],[Gui]).

Similarly, we can study “ $(\pi - \sigma)$ -lines” of ideals on a set a . Here, we must deal with so called monotonic valuations of lines, i.e. valuations which preserve canonical relations of relevant e -structures. We investigate this problems in the last paragraph.

§1 LINES AND VALUATIONS

We work in the alternative set theory; we shall use usual notations of this theory. Recall that small Latin letters range over sets and i, j, k, l, m, n range over finite natural numbers. By a collection we mean a collection of classes which satisfy a given formula of the language \mathbf{FL}_V . The collection of all set-theoretically definable sets is denoted by \mathbf{Sd}_V ; it is a codable system.

We say that a structure $\langle A, F, E \rangle$ is an e -structure if we have the following:

- 1) $\langle A, F \rangle$ is a semigroup (i.e. F is an associative operation on A),
- 2) $E \circ E$ is the identity on A ,
- 3) we have either, for each $x, y \in A, F(E(x), E(y)) = E(F(x, y))$ or, for each $x, y \in A, F(E(x), E(y)) = E(F(y, x))$.

Let $\langle A, F, E \rangle$ be an e -structure. We define, on A , a *canonical relation* $\triangleleft_A \langle A, E \rangle$ by:

$$x \triangleleft_A y \Leftrightarrow (\exists z \in A)(F(x, z) = y).$$

Assume that $\mathbb{A} = \langle A, F, E \rangle, \hat{\mathbb{A}} = \langle \hat{A}, \hat{F}, \hat{E} \rangle$ are two e -structures. A mapping $H : A \rightarrow \hat{A}$ is said to be a *valuation of \mathbb{A} in $\hat{\mathbb{A}}$* if we have:

- a) $H(F(x, y)) \triangleleft_{\hat{A}} \hat{F}(H(x, y))$ holds for each $x, y \in A$,
- b) $H(E(x)) = \hat{E}(H(x))$ holds for each $x \in A$.

Let \mathbb{A} be an e -structure. Then the triple $\langle \mathbb{A}, \mathbb{A} \upharpoonright U, \mathbb{A} \upharpoonright B \rangle$, where $B \subseteq U \subseteq A$ and $\mathbb{A} \upharpoonright B, \mathbb{A} \upharpoonright U$ are substructures of \mathbb{A} , is said to be a *triad over the e -structure \mathbb{A}* . We denote it as $\mathbb{A}(U, B)$. A mapping H is called a *valuation of the triad $\langle A, F, E \rangle(U, B)$ in the triad $\langle \hat{A}, \hat{F}, \hat{E} \rangle(\hat{U}, \hat{B})$* , if H is a valuation of $\langle A, F, E \rangle$ in $\langle \hat{A}, \hat{F}, \hat{E} \rangle$ and we have, moreover,

$$c) H^{-1''} \hat{U} = U, H^{-1''} \hat{B} = B.$$

We use the following notations: Let $r \in \mathbf{Q}^+$. Then $r \cdot [\mathbf{Q}^+] = \{r \cdot d; d \in \mathbf{Q}^+ \text{ \& } d \neq 0\}$, $r \cdot \mathbf{BQ}^+ = \{r \cdot b; b \in \mathbf{BQ}^+\}$. We have $r \cdot [\mathbf{Q}^+] = \{x \in \mathbf{Q}^+; (\forall n)(x \leq r \cdot n)\}$ and $r \cdot \mathbf{BQ}^+ = \{x \in \mathbf{Q}^+; (\exists n)(x \leq r \cdot n)\}$.

We can generalize the notion of the triad by the following way:

Let $\mathbb{A}, \mathbb{A} \upharpoonright B$ be two e -structures, $B \subseteq A$.

A *line in \mathbb{A} over B* is a sequence $\{\mathbb{A}, \mathbb{A} \upharpoonright U_c, \mathbb{A} \upharpoonright B\}_{c \in \mathbf{FZ}}$ such that we have

- 1) $\mathbb{A} \upharpoonright U_c$ is an e -structure and $B \subseteq U_c$ holds for each $c \in \mathbf{FZ}$,
- 2) $c < d \Rightarrow U_c \subseteq U_d$.

Let us denote such a line by the symbol

$$\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}.$$

A *valuation of a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in a line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$* is a valuation H of the structure \mathbb{A} in $\hat{\mathbb{A}}$ such that $H^{-1''} \hat{U}_c = U_c$ holds for each $c \in \mathbf{FZ}$ and $H^{-1''} \hat{B} = B$.

Example. Let $\zeta \in \mathbf{N} - \mathbf{FN}$. Put

$$\begin{aligned} U_c(\zeta) &= (2^\zeta)^c \cdot [0]^+ \text{ whenever } c \in \mathbf{FZ} \text{ is odd,} \\ U_c(\zeta) &= (2^\zeta)^{c-1} \cdot \mathbf{BQ}^+ \text{ whenever } c \in \mathbf{FZ} \text{ is even.} \end{aligned}$$

Then

$$\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle(U_c, \{0\})$$

is a line in the e -structure $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ over $\{0\}$.

Let us present the following interpretation of a line. Let $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle$ be a couple of equivalences on a class $A \in \mathbf{Sd}_V$ such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and \mathcal{E}_1 is a π -equivalence, \mathcal{E}_2 is a σ -equivalence. We can see this situation as a formalization of the phenomenon of two horizons; the horizon of “microcosmos” which is represented by the first of the equivalences in question and the horizon of “macrocosmos” represented by the second one. The couple presented is called *biequivalence*. The sequence $\{\mathcal{E}_c\}_{c \in \mathbf{FZ}}$, where each \mathcal{E}_c is an equivalence on a fixed class from \mathbf{Sd}_V , $\mathcal{E}_c \subseteq \mathcal{E}_{c+1}$ and \mathcal{E}_c is

a π -class and \mathcal{E}_{c+1} is a σ -class for each odd $c \in \mathbf{FZ}$, can be seen as a formalization of a “line of horizons”.

Note that an equivalence \mathcal{E} on a class B can be studied as an e -structure. Indeed, let $\mathbb{A} = \langle A, F, E \rangle$ be a structure defined as follows:

$$A = B^2 \cup \{\emptyset\},$$

$$F : A^2 \rightarrow A \text{ is a function defined by}$$

$$F(\langle x, y \rangle, \langle r, s \rangle) = \langle x, s \rangle \text{ (resp. } \emptyset) \Leftrightarrow y = r \text{ (resp. } y \neq r)$$

$$F(w, \emptyset) = F(\emptyset, w) = \emptyset, \text{ whenever } w \in A,$$

$$E : A \rightarrow A \text{ is a function defined by}$$

$$E(\langle x, y \rangle) = \langle y, x \rangle, E(\emptyset) = \emptyset.$$

We can see that \mathbb{A} is an e -structure and \mathcal{E} can be identified with the triad $\mathbb{A}(\mathcal{E} \cup \{\emptyset\}, \mathbf{Id} \upharpoonright B \cup \{\emptyset\})$. Thus the sequence $\{\mathcal{E}_c\}_{c \in \mathbf{FZ}}$ induces the line $\mathbb{A}(\mathcal{E}_c \cup \{\emptyset\}, \mathbf{Id} \upharpoonright B \cup \{\emptyset\})_{c \in \mathbf{FZ}}$; we denote it by the symbol $\mathcal{L}(\mathcal{E}_c)_{c \in \mathbf{FZ}}$.

We can see, moreover, that every valuation $H : A \rightarrow [0, 1]_{\mathbf{Q}}$ of the triad presented in $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle([0]^+, \{0\})$ induces a rational metric $D : B \times B \rightarrow [0, 1]_{\mathbf{Q}}$ given by the relation $D(x, y) = H(\langle x, y \rangle)$. We have

$$\langle x, u \rangle \in \mathcal{E} \Leftrightarrow D(x, y) \doteq 0.$$

We conclude that a valuation H of the line $\mathcal{L}(\mathcal{E}_c)_{c \in \mathbf{FZ}}$ in the line from the previous example induces the mapping similarly as above, which can be seen as a “horizon-metric” for the line of equivalences in question. A proposition on the existence of such a valuation represents thus a generalization of the theorem on metrization (see [Gui],[M2]); such a proposition will be precisely formulated below.

We shall study, roughly speaking, “ $(\pi - \sigma)$ -lines”, i.e. lines $\mathbb{A}(U_c, B)$ where \mathbb{A}, B belong to a collection \mathfrak{S} and the classes U_c are $\pi(\mathfrak{S})$ -classes whenever c is odd and $\sigma(\mathfrak{S})$ -classes whenever c is even.

A class $\bigcap_n X_n$, where $\{X_n\} \subseteq \mathfrak{S}$, is called a $\pi(\mathfrak{S})$ -class and a class $\bigcup_n X_n$, where $\{X_n\} \subseteq \mathfrak{S}$, is called a $\sigma(\mathfrak{S})$ -class.

A line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is called a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in \mathbb{A} over B , if $\mathbb{A}, B \in \mathfrak{S}$ and, for each $c \in \mathbf{FZ}$ odd, U_c is a $\pi(\mathfrak{S})$ -class and U_{c+1} is a $\sigma(\mathfrak{S})$ -class.

The collection \mathfrak{S} plays a role of a universe of primary visual objects. The system \mathbf{Sd}_V of all set-theoretically definable classes is a good example of such a collection; we can naturally see here σ -classes, π -classes, $\sigma\pi$ -classes as secondary visual objects. We say that a collection of classes is *universe of classes* if it is closed under definition by normal formulas of the language \mathbf{FL}_V with class-parameters from this collection. Thus, having a universe \mathfrak{U} of classes and a normal formula $\varphi(x, X_1, X_2, \dots, X_k)$ of the language \mathbf{FL}_V such that the classes X_1, X_2, \dots, X_k belong to \mathfrak{U} , we see that the class $\{x; \varphi(x, X_1, X_2, \dots, X_k)\}$ belongs to \mathfrak{U} , too. Note that every universe of classes contains all sets. More generally, every set-theoretically definable class belongs to each universe of classes. By a *standard universe of classes* we call each universe of classes which contains only such non-empty subclasses of the class of natural numbers which have the first element. We can see (see [M1]) that every standard universe of classes contains only the revealed classes and does not contain a proper semiset. It satisfies all axioms of Gödel-Bernays theory of finite sets.

A standard universe \mathfrak{U} of classes is said to be *standard saturated universe of classes* if we have the following: Let $\{X_n\}_{n \in \mathbf{FN}}$ be a sequence of classes of this universe. Then there exists a relation R from \mathfrak{U} such that

$$(\forall n)R''\{n\} = X_n.$$

Convention. Throughout this paper, let \mathfrak{S} denote a standard universe of classes.

We see that the line from the first example is a $(\pi(\mathbf{Sd}_V) - \sigma(\mathbf{Sd}_V))$ -line in $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ over $\{\emptyset\}$. We denote it by

$$\mathcal{L}_{\pi-\sigma}(\zeta).$$

Recall that it is defined for each $\zeta \in \mathbf{N} - \mathbf{FN}$.

§2 VALUATIONS OF $(\pi - \sigma)$ -LINES

Theorem (on valuations of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines). *Let \mathfrak{S} be a codable saturated standard universe and let $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ be a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in \mathbb{A} over B . Then there exists a valuation of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to \mathfrak{S} .*

Let $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ be a $(\pi(\mathbf{Sd}_V) - \sigma(\mathbf{Sd}_V))$ -line in \mathbb{A} over B and let A be a set. Then there exists a set-valuation of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$.

We give a proof of the theorem in the two following sections. At first, in the section Valuations of uniform $(\pi - \sigma)$ -lines, we introduce a notion of a uniform $(\pi - \sigma)$ -line in \mathbb{A} over B w.r.t. \mathfrak{S} and prove a proposition on an existence of a valuation $H \in \mathfrak{S}$ of such a line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$. Secondly, we prove that, in a codable saturated standard universe, the notion of the $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in \mathbb{A} over B coincides with this one of the uniform $(\pi - \sigma)$ -line in \mathbb{A} over B w.r.t. \mathfrak{S} . This second step is done in the section $(\pi - \sigma)$ -uniform lines w.r.t. a saturated standard universe. Finally, our theorem is an easy consequence of the mentioned assertions. This is presented at the end of this section.

Valuations of uniform $(\pi - \sigma)$ -lines.

A relation R is called a σ -string, if we have

$$\text{dom}(R) \in \mathbf{N} \ \& \ (\forall \alpha \in \text{dom}(R) - 1)(R''\{\alpha\} \subseteq R''\{\alpha + 1\}).$$

Let $\langle A, F, E \rangle(B, B)$ be a triad. We say that a σ -string R is a σ -string in $\langle A, F, E \rangle$ over B if the following items hold:

- 1) $R(0) = B, R(\theta) = A$, where $\theta + 1 = \text{dom}(R)$,
- 2) $F''R^2(\alpha) \subseteq R(\alpha + 1), F_3''R^3(\alpha) \subseteq R(\alpha + 1)$ is true for each $\alpha \in \theta$; we denote $F_3 : A^3 \rightarrow A$ the function defined by $F_3(x, y, z) = F(F(x, y), z)$,
- 3) $E''R(\alpha) \subseteq R(\alpha)$ holds for each $\alpha \leq \theta$.

Recall the lemma on a valuation over a σ -string which is proved in [M2].

Lemma (on a valuation over a σ -string). *Let \mathfrak{S} be a standard universe. Let $\mathbb{A}(B, B) \in \mathfrak{S}$ be a triad and assume that $R \in \mathfrak{S}$ is a σ -string in \mathbb{A} over B . Suppose, moreover, that $\theta + 1 = \text{dom}(R)$.*

Then there exists a valuation $H \in \mathfrak{S}$ of a triad $\mathbb{A}(B, B)$ in $\langle \mathbf{N}, +, \mathbf{Id} \rangle(\{0\}, \{0\})$ such that, for each $\alpha \in \theta$, we have

$$R(\alpha) \subseteq \{x \in A; H(x) \leq 2^\alpha\} \subseteq R(\alpha + 1).$$

Natural number $\zeta \notin \mathbf{FN}$ is called a *distance* in a σ -string R , whenever there exists a number $\eta \in \mathbf{N} - \mathbf{FN}$ so that we have $\text{dom}(R) - 1 = 2\zeta \cdot \eta$. Suppose that $\zeta \notin \mathbf{FN}$ is a distance in a σ -string R and let $\eta \in \mathbf{N} - \mathbf{FN}$ be a number such that we have $\text{dom}(R) - 1 = 2\zeta \cdot \eta$. We define the function $z_{R,\zeta} = z$ on $[-\eta, \eta]$ with values in $\text{dom}(R)$ as follows: Let us denote $\theta = \text{dom}(R) - 1$. Then $z(0) = \frac{\theta}{2}$, $z(c) = z(0) + c \cdot \zeta$ whenever $c \in [-\eta, \eta] - \{0\}$ holds. Assuming R is a σ -string in \mathbb{A} over B , we can see that $R(z(-\eta)) = B$, $R(z(\eta)) = A$ hold.

Let ζ be a distance in a σ -string R in \mathbb{A} over B .

We say that a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a $\langle R, \zeta \rangle$ -line in \mathbb{A} over B if we have:

$$U_c = \bigcap_n R''\{z(c) - n\}, \text{ whenever } c \in \mathbf{FZ} \text{ is odd,}$$

$$U_c = \bigcup_n R''\{z(c - 1) + n\}, \text{ whenever } c \in \mathbf{FZ} \text{ is even.}$$

Let ζ be a distance in a σ -string R in \mathbb{A} over B and let $R \in \mathfrak{S}$, $\text{dom}(R) = \theta + 1$, $\mathbb{A}, B \in \mathfrak{S}$. Then there exists a valuation H over a σ -string R , $H \in \mathfrak{S}$. We have $(\forall x \in B)H(x) \leq 2^{-\theta}$. We have, moreover, for each $c \in \mathbf{FZ}$ odd,

$$x \in U_c \Leftrightarrow (\forall n)H(x) \leq 2^{z(c)-n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^\zeta)^c \cdot [0]^+,$$

$$x \in U_{c+1} \Leftrightarrow (\exists n)H(x) \leq 2^{z(c)+n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^\zeta)^c \cdot \mathbf{BQ}^+.$$

We say that a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a *uniform $(\pi - \sigma)$ -line* in \mathbb{A} over B w.r.t. \mathfrak{S} , if there are $\mathbb{A}, B \in \mathfrak{S}$ and a σ -string $R \in \mathfrak{S}$ in \mathbb{A} over B and a distance ζ in this string such that $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is an $\langle R, \zeta \rangle$ -line in \mathbb{A} over B . Saying a uniform $(\pi - \sigma)$ -line w.r.t. \mathfrak{S} we mean a uniform $(\pi - \delta)$ -line in an \mathbb{A} over a B w.r.t. \mathfrak{S} .

We can see that the line $\mathcal{L}_{\pi-\sigma}(\zeta)$ is a $(\pi - \sigma)$ -line in $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ over $\{0\}$ w.r.t. \mathbf{Sd}_V . Indeed, let $\eta \in \mathbf{N} - \mathbf{FN}$. We define the relation R on $\theta + 1 = 2\zeta\eta + 1$ by $R(\alpha) = \{x \in \mathbf{Q}^+; x \leq 2^{\alpha-\zeta\eta}\}$ whenever $0 < \alpha\theta$ and we put in addition $R(0) = \{0\}$, $R(\theta) = \mathbf{Q}^+$. Then $R \in \mathfrak{S}$ and R is a σ -string in $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ over $\{0\}$. Let $z : [-\eta, \eta] \rightarrow \theta$ be a function defined by the relation $z(0) = \theta/2$, $z(\gamma) = z(0) + \gamma\zeta = \zeta\eta + \gamma\zeta$ for $\gamma \in [-\eta, \eta] - \{0\}$. We have, for each $c \in \mathbf{FZ}$ odd and $x \in \mathbf{Q}^+$, the following:

$$x \in \bigcap_n R(z(c) - n) \Leftrightarrow (\forall n)(x \leq 2^{z(c)-n-\zeta\eta}) \Leftrightarrow (\forall n)(x \leq 2^{c\zeta-n}) \Leftrightarrow x \in 2^{c\zeta} \cdot [0]^+,$$

$$x \in \bigcup_n R(z(c) + n) \Leftrightarrow (\exists n)(x \leq 2^{z(c)+n-\zeta\eta}) \Leftrightarrow (\exists n)(x \leq 2^{c\zeta+n}) \Leftrightarrow x \in 2^{c\zeta} \cdot \mathbf{BQ}^+.$$

Thus $\mathcal{L}_{\pi-\sigma}(\zeta)$ is an $\langle R, \zeta \rangle$ -line in $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle$ over $\{0\}$. We can deduce from the fact above that the following theorem holds.

Theorem (on a valuation of a uniform $(\pi - \sigma)$ -line). *Let \mathfrak{S} be a standard universe. Then every uniform $(\pi - \sigma)$ -line w.r.t. \mathfrak{S} has a valuation in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to \mathfrak{S} .*

Let us clarify some questions about uniform lines.

Lemma. *Let \mathfrak{S} be a standard universe and let \mathcal{L} be a line in a structure from \mathfrak{S} . Let $H \in \mathfrak{S}$ be a valuation of \mathcal{L} in a uniform $(\pi - \sigma)$ -line w.r.t. \mathfrak{S} . Then \mathcal{L} is a uniform $(\pi - \sigma)$ -line w.r.t. \mathfrak{S} .*

PROOF: Suppose that $\mathcal{L} = \mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ has the valuation $H \in \mathfrak{S}$ in $\hat{\mathcal{L}} = \hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ and let $\hat{\mathcal{L}}$ be an $\langle R, \zeta \rangle$ -uniform line in $\hat{\mathbb{A}}$ over \hat{B} , $R \in \mathfrak{S}$. Put $z = z_{R, \zeta}$ as above. Let us define the string S on $\theta + 1 = 2\eta\zeta + 1 = \text{dom}(R)$ by the relation $S(\alpha) = H^{-1''}R(\alpha)$. We can see that $S \in \mathfrak{S}$ and that it is a σ -string in \mathbb{A} over B . Indeed, assuming $\mathbb{A} = \langle A, F, E \rangle$, we have $x, y \in S(\alpha) \Rightarrow H(x), H(y) \in R(\alpha)$ whenever $\alpha + 1 \in \text{dom}(S)$. Further, the relation $E''S(\alpha) \subseteq S(\alpha)$ is easy.

We have, for each $c \in \mathbf{FZ}$ odd: $U_c = H^{-1''}\hat{U}_c = \bigcap_n H^{-1''}R(z(c) - n) = \bigcap_n S(z(c) - n)$, $U_{c-1} = H^{-1''}\hat{U}_{c-1} = H^{-1''}\bigcup_n R(z(c) + n) = \bigcup_n H^{-1''}R(z(c) + n) = \bigcup_n S(z(c) + n)$. Thus, \mathcal{L} is an $\langle S, \zeta \rangle$ -line in \mathbb{A} over B w.r.t. \mathfrak{S} . \square

Lemma. *Let \mathcal{L} be an $\langle R, \zeta \rangle$ -line in \mathbb{A} over B , $\tilde{\zeta} \in \zeta - \mathbf{FN}$. Then there is an \tilde{R} such that \mathcal{L} is an $\langle \tilde{R}, \tilde{\zeta} \rangle$ -line in \mathbb{A} over B . If $R \in \mathfrak{S}$ then \tilde{R} can be chosen from \mathfrak{S} .*

PROOF: Let $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ be an $\langle R, \zeta \rangle$ -uniform line and assume that $R \in \mathfrak{S}$. Let $z = z_{R, \zeta}$ be as above and let $\tilde{\eta} \in \eta - \mathbf{FN}$. We define a function $\tilde{z} : [-\tilde{\eta}, \tilde{\eta}] \rightarrow [0, \tilde{\theta}]$, where $\tilde{\theta} = 2 \cdot \tilde{\eta} \cdot \tilde{\zeta}$, by $\tilde{z}(\gamma) = \tilde{\eta} \cdot \tilde{\zeta} + \gamma \cdot \tilde{\zeta}$. We define the relation \tilde{R} with $\text{dom}(\tilde{R}) = [0, \tilde{\theta}]$ as follows: Put $\tilde{R}(0) = B$, $\tilde{R}(\tilde{\theta}) = A$. Put, for each $\gamma \in [-\tilde{\theta} + 1, \tilde{\theta} - 1]$ and $\alpha \in [-\tilde{\zeta}, \tilde{\zeta} - 1]$ such that $\tilde{\zeta}(\gamma) + \alpha > 0$, $\tilde{R}(\tilde{z}(\gamma) + \alpha) = R(z(\gamma) + \alpha)$. We wish to prove that \tilde{R} and $\tilde{\zeta}$ have the required properties.

Let us prove, at first, that \tilde{R} is a σ -string in \mathbb{A} over B . Assume that $\mathbb{A} = \langle A, F, E \rangle$. It is easy that $E''\tilde{R}(\delta) \subseteq \tilde{R}(\delta)$ holds for each $\delta \leq \tilde{\theta}$. Assume that $\delta = \tilde{z}(\gamma) + \alpha$ holds for some $\gamma \in [-\tilde{\theta} + 1, \tilde{\theta} - 1]$, $\alpha \in [-\tilde{\zeta}, \tilde{\zeta} - 1]$. We have, for each $x, y \in \tilde{R}(\delta)$, $F(x, y) \in R(z(\gamma) + \alpha + 1)$. Thus $F(x, y) \in \tilde{R}(\delta + 1)$ holds. We can similarly prove that $x, y, z \in \tilde{R}(\delta) \Rightarrow F_3(x, y, z) \in \tilde{R}(\delta + 1)$.

It remains to prove that, for each $c \in \mathbf{FZ}$ odd, $U_c = \bigcap_n \tilde{R}(\tilde{z}(c) - n)$ and $U_{c-1} = \bigcup_n \tilde{R}(\tilde{z}(c) + n)$ hold. But it follows immediately from the definition of \tilde{R} . Finally, assuming $R \in \mathfrak{S}$, we can see that $\tilde{R} \in \mathfrak{S}$. \square

The following proposition is an easy consequence of the above results.

Proposition (on valuation in $\mathcal{L}_{\pi-\sigma}(\zeta)$). *Let \mathfrak{S} be a standard universe and let \mathcal{L} be a line in an \mathbb{A} which belongs to \mathfrak{S} . Then \mathcal{L} has a valuation from \mathfrak{S} in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ iff \mathcal{L} is a uniform $(\pi - \sigma)$ -line w.r.t. \mathfrak{S} .*

$(\pi - \sigma)$ -uniform lines w.r.t. a saturated standard universe.

We give a criterion of the uniformity of lines.

We shall use the following notation: Let $S \subseteq \mathbf{V}^3$. We denote, for $\langle x, y \rangle \in \text{dom}(S)$, the class $S''\{\langle x, y \rangle\}$ by $S(x, y)$.

Lemma (on uniform lines). *Let \mathfrak{S} be a saturated standard universe and suppose that $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a line, $\mathbb{A}, B \in \mathfrak{S}$. Let S be a relation and $\xi \in \mathbf{N} - \mathbf{FN}$ be such that we have*

- 1) $\text{dom}(S) = \mathbf{FZ}$,

- 2) $S''\{c\}$ is, for each $c \in \mathbf{FZ}$, a σ -string in \mathbb{A} over B , $\text{dom}(S''\{c\}) = \xi + 1$ and $S''\{c\} \in \mathfrak{S}$,
- 3) the relations $U_c = \bigcap_n S(c, \xi - n)$, $U_{c+1} = \bigcup_n S(c, n)$ hold for each $c \in \mathbf{FZ}$ odd.

Then $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a uniform $(\pi - \sigma)$ -line in \mathbb{A} over B w.r.t. \mathfrak{S} .

PROOF: We are looking for a σ -string $R \in \mathfrak{S}$ and a number β such that the line in question is an $\langle R, \beta \rangle$ -line in \mathbb{A} over B . We shall say that a σ -string R is a *weak σ -string in \mathbb{A} over B* if there exists a relation R on natural numbers with values in A such that the extension of each number from its domain contains B as a subclass and

- a) $F''R^2(\alpha) \subseteq R(\alpha + 1)$, $F_3''R^3(\alpha) \subseteq R(\alpha + 1)$ holds for each $\alpha \in \theta$, where $F_3 : A^3 \rightarrow A$ is a function defined by the relation $F_3(x, y, z) = F(F(x, y), z)$.
- b) $E''R(\alpha) \subseteq R(\alpha)$ holds for each $\alpha \in \theta + 1$.

We say sometimes that the domain of a σ -string is the *length of the string* in question.

The classes $U_c, c \in \mathbf{FZ}$, are linearly ordered by the inclusion. We deduce from this and by using the prolongation axiom that there exists a number $\delta \in \mathbf{N} - \mathbf{FN}$ and numbers

$$\begin{aligned} l_c &\in \mathbf{FN}, c \in \mathbf{FZ} \text{ odd,} \\ n_c &\in \mathbf{FN}, c \in \mathbf{FZ} \text{ even,} \end{aligned}$$

so that we have for each $c \in \mathbf{FZ}$ odd

$$S(c, \xi - (n_c + \delta)) \subseteq S(c, \xi - n_c) \subseteq S(c + 1, l_c) \subseteq S(c + 1, l_c + \delta) \subseteq S(c + 2, \xi - (n_{c+2} + \delta)).$$

We have, for each $c \in \mathbf{FZ}$ odd:

$$F_3''(S(c, \xi - n_c - 1))^3 \subseteq S(c, \xi - n_c) \subseteq S(c + 1, l_c + 1)$$

and

$$\begin{aligned} F_3''(S(c + 1, l_c + \delta - 1))^3 &\subseteq S(c + 1, l_c + \delta) \subseteq S(c + 2, \xi - (n_{c+2} + \delta)) \\ &\subseteq S(c + 2, \xi - (n_{c+2} + \delta - 1)). \end{aligned}$$

We denote, for each $c \in \mathbf{FZ}$ odd, $I_c = \{\alpha; \xi - (n_c + \delta) < \alpha < \xi - n_c\}$ and $I_{c+1} = \{\alpha; l_c < \alpha < l_c + \delta\}$. Every set I_c has $\delta - 1$ elements. Let \tilde{S}_c be the relation which is obtained from $S(c) \upharpoonright I_c$ by the natural renumbering of its domain I_c (starting from the number 0). We see that \tilde{S}_c is a weak σ -string in \mathbb{A} over B of the length $\delta - 1$, which belongs, moreover, to \mathfrak{S} . We can see that, for each $c < d$ from \mathbf{FZ} and $\alpha, \beta \leq \delta - 1$, the relation $F_3''(\tilde{S}_c(\alpha))^3 \subseteq \tilde{S}_d(\beta)$ holds. Put $\beta = \delta - 1$. We have, for $c \in \mathbf{FZ}$ odd,

$$U_c = \bigcap_n \tilde{S}_c(\beta - n), \quad U_{c+1} = \bigcup_n \tilde{S}_c(n).$$

There exists a relation \tilde{S} on \mathbf{FZ} such that we have for each $c \in \mathbf{FZ}$: $\tilde{S}(c) = \tilde{S}_c$. There exists, moreover, a number $\gamma \in \mathbf{N} - \mathbf{FN}$ and a relation $T \in \mathfrak{S}$ such that

- 1) $\text{dom}(T) = [-\gamma, \gamma]$,

- 2) $\tilde{S} \subseteq T$,
- 3) $c \in \text{dom}(T) \Rightarrow T''\{c\}$ is a weak σ -string in \mathbb{A} over B of the length β ,
- 4) $c, d \in \text{dom}(T) \ \& \ c < d \Rightarrow (\alpha, \alpha' \in \beta \Rightarrow T(c, \alpha) \subseteq T(d, \alpha') \ \& \ F_3''(T(c, \alpha))^3 \subseteq T(d, \alpha') \ \& \ F''(T(c, \alpha))^2 \subseteq T(d, \alpha'))$.

Especially, we have for each $c \in \mathbf{FZ}$ odd:

$$U_c = \bigcap_n T(c, \beta - n), \quad U_{c+1} = \bigcup_n T(c, n).$$

Let us define the function $z : [-\gamma, \gamma] \rightarrow [0, 2\beta\gamma]$ by $z(c) = \beta\gamma + c\gamma$. Let R be the relation with domain $[0, 2\beta\gamma]$ such that $R \in \mathfrak{S}$, $R(0) = B$, $R(2\beta\gamma) = A$ and, for each $c \in [-\gamma, \gamma - 1]$ and $0 \leq \alpha \leq \beta - 1$, holds

$$R(z(c) + \alpha) = T(c, \alpha).$$

Thus R is a σ -string in \mathbb{A} over B and β is a distance in R . We have for each $c \in \mathbf{FZ}$: $1 \leq \alpha \leq \beta \Rightarrow R(z(c) - \alpha) = R(z(c - 1) + \beta - 1)$. Thus, for $c \in \mathbf{FZ}$ odd, it holds

$$\begin{aligned} \bigcap_n R(z(c) - n) &= \bigcap_n R(z(c - 1) + \beta - n) = \bigcap_n T(c, \beta - n) = U_c, \\ \bigcup_n R(z(c) + n) &= \bigcup_n T(c, n) = U_{c+1}. \end{aligned}$$

The proof is finished. □

Lemma. *Let \mathfrak{S} be a codable saturated standard universe and let $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ be a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in \mathbb{A} over B . Then there exist a relation S and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that the items 1) - 3) from the lemma on uniform lines hold.*

PROOF: Let U be a relation such that $\text{dom}(U) = \mathbf{FZ}$ and, for each $c \in \mathbf{FZ}$, $U_c = U''\{c\}$ holds. Let $\langle T, K \rangle$ be a codable pair for \mathfrak{S} , i.e. we assume that $\mathfrak{S} = \{T''\{z\}; z \in K\}$ holds. Take $\xi \in \mathbf{N} - \mathbf{FN}$. Let L be the relation on \mathbf{FZ} defined by the formula

$$\begin{aligned} L''\{c\} &= \{z; z \text{ is a code of a } \sigma\text{-string } R \in \mathfrak{S} \text{ in } \mathbb{A} \text{ over } B \text{ of the length } \xi \notin \mathbf{FN} \text{ and} \\ &\quad \bigcap_n R''\{\xi - n\} = U''\{c\} \text{ holds whenever } c \text{ is odd and} \\ &\quad \bigcup_n R''\{n\} = U''\{c\} \text{ holds whenever } c \text{ is even}\}. \end{aligned}$$

Note that the existence of the relation R , mentioned in the definition of L , follows from the proposition on σ -string in e -structure (see [M2]) and from our assumption that \mathfrak{S} is a saturated standard universe.

Let G be such a function on \mathbf{FZ} which satisfies: $c \in \mathbf{FZ} \Rightarrow G(c) \in L''\{c\}$. Let S be the relation with domain \mathbf{FZ} , such that we have for each $c \in \mathbf{FZ}$:

$$S''\{c\} = T''G(c).$$

Then S has the required properties. □

We deduce from this lemma and by using the lemma on uniform lines that the following proposition holds.

Proposition. *Let \mathfrak{S} be a codable saturated standard universe. Then the collection of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines in \mathbb{A} over B is equal to this one of uniform $(\pi - \sigma)$ -lines in \mathbb{A} over B w.r.t. \mathfrak{S} .*

Now, we can finish the proof of the theorem on valuations of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines. The first part is an immediate consequence of the previous results and of the lemma on a valuation of uniform $(\pi - \sigma)$ -line. The second part follows from the first one. It is because the line in question is a $(\pi(\mathbf{Sd}_{\mathbf{V}^*}) - \sigma(\mathbf{Sd}_{\mathbf{V}^*}))$ -line in \mathbb{A} over B and that $\mathbf{Sd}_{\mathbf{V}^*}$ is a codable saturated standard universe. Thus there exists a valuation from $\mathbf{Sd}_{\mathbf{V}^*}$ of the line in question in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which has the domain equal to the set A . Consequently, this valuation is a set. \square

Remarks.

1. We cannot omit, in the theorem on valuations of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines, the assumption, that \mathfrak{S} is saturated. Indeed, the equivalence $\overset{\circ}{=}$ defined on \mathbf{V} by the relation $x \overset{\circ}{=} y \Leftrightarrow (\varphi(x) \Leftrightarrow \varphi(y))$ holds for each set-formula φ of the language \mathbf{FL}_{\emptyset} is a $\pi(\mathbf{Sd}_{\mathbf{V}})$ -equivalence which is no $\pi^{\mathbf{V}}$ -class. (See [M1] .) Thus there is no valuation $H \in \mathbf{Sd}_{\mathbf{V}}$ of $\mathcal{L}(\overset{\circ}{=})$ in an $\mathcal{L}_{\pi-\sigma}(\zeta, 1)$.

2. By a finite $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line we mean a finite sequence $\{\mathbb{A}, \mathbb{A} \upharpoonright U_n, \mathbb{A} \upharpoonright B\}_{n \in [1, l]}$ of triads (where $1 \leq l$) such that $1 \leq m < n \leq l \Rightarrow B \subseteq U_m \subseteq U_n$ and, for each $1 \leq m \leq l$ odd, U_m is a $\pi(\mathfrak{S})$ -class, U_{m+1} is a $\sigma(\mathfrak{S})$ -class (whenever $m + 1 \leq 1$) and $\mathbb{A}, B \in \mathfrak{S}$. We denote this line as $\mathbb{A}(U_c, B)_{c \in [1, l]}$; l is called the length of the presented line. The notion of a valuation of a finite line of the length l in another one of the length l is defined naturally. Put, for each $c \in \mathbf{FZ}$, $c \leq 0$, $U_c = B$ and, for each $c > l$, $U_c = A$. Then $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line. Having such a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ and assuming that \mathfrak{S} is a standard saturated universe we can see that there exists a relation $S \in \mathfrak{S}$ and $\zeta \in \mathbf{N} - \mathbf{FN}$ such that the items 1) - 3) from the lemma on uniform lines hold. We deduce from this that there exists a valuation $H \in \mathfrak{S}$ of $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in $\mathcal{L}_{\pi-\sigma}(\zeta)$.

Denoting $\mathcal{L}_{\pi-\sigma}(\zeta, l)$, for $l \geq 1$, the finite line $\langle \mathbf{Q}^+, +, \mathbf{Id} \rangle(U_n(\zeta), \{0\})_{n \in [1, l]}$, we can formulate the following proposition.

Proposition. *Let \mathfrak{S} be a standard saturated universe. Assume that \mathcal{L} is a finite $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line of the length l . Then there exists a valuation $H \in \mathfrak{S}$ of \mathcal{L} in an $\mathcal{L}_{\pi-\sigma}(\zeta, l)$.*

Metritzation.

Let $\{\mathcal{E}_c\}_{c \in I}$, where $I = \mathbf{FZ}$ or $I = [1, l]$ for some $l \geq 1$, be a sequence of equivalences on a class $Z \in \mathfrak{S}$. Suppose that, for each $c < d$, $c, d \in I$, the relation $\mathcal{E}_c \subseteq \mathcal{E}_d$ holds and we have, for each $c \in I$ odd (even, resp.) that U_c is a $\pi(\mathfrak{S})$ -class ($\sigma(\mathfrak{S})$ -class resp.). We say that $\{\mathcal{E}_c\}_{c \in I}$ is a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line of equivalences.

Theorem (on a metrization of a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line of equivalences). *Let $\{\mathcal{E}_c\}_{c \in I}$ be a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line of equivalences on a class $Z \in \mathfrak{S}$, where \mathfrak{S} is a standard saturated universe.*

1) Assume that $I = \mathbf{FZ}$ and let \mathfrak{S} be a codable system. Then there exist

a rational metric $D : Z^2 \rightarrow \mathbf{Q}^+$ and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that we have, for each $c \in \mathbf{FZ}$ odd:

$$(*) \langle x, y \rangle \in \mathcal{E}_c \Leftrightarrow D(x, y) \in (2^{c-\zeta}) \cdot [0]^+, \quad \langle x, y \rangle \in \mathcal{E}_{c+1} \Leftrightarrow D(x, y) \in (2^{c-\zeta}) \cdot \mathbf{BQ}^+.$$

- 2) Assume that $I = [1, l]$. Then there exist a rational metric $D : Z^2 \rightarrow \mathbf{Q}^+$ and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that (*) holds for each c odd, provided $c \in [1, l]$.

PROOF: 1) Let $\mathcal{L}(\mathcal{E}_c)_{c \in \mathbf{FZ}}$ be the $(\pi - \sigma)$ -line described above. We deduce from the last theorem that there exists a valuation H in \mathfrak{S} of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$. The metric in question can be defined on Z^2 by $D(x, y) = H(\langle x, y \rangle)$. 2) can be proved quite analogously by using the last proposition. □

§3 MONOTONIC VALUATIONS OF $(\pi - \sigma)$ -LINES

A valuation H of an e -structure \mathbb{A} in $\hat{\mathbb{A}}$ is called a *monotonic valuation* if we have $(x, y \in A \ \& \ x \triangleleft_A y) \Rightarrow (H(x) \triangleleft_{\hat{A}} H(y))$. By a *monotonic valuation* of a line in another one we mean a valuation of the first line in the second one such that it is a monotonic valuation of the relevant e -structures. More explicitly, a mapping H is said to be a monotonic valuation of the line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in the line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ if H is a valuation of $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ such that H is a monotonic valuation of \mathbb{A} in $\hat{\mathbb{A}}$.

By a *closed line* in \mathbb{A} over B we mean a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ such that all U_c and B are closed under \triangleleft . We can naturally modify the definitions above to obtain the relevant ones which are connected with the notion of a closed line. A σ -string S is said to be a *closed σ -string* in \mathbb{A} over B if it is a σ -string in \mathbb{A} over B and we have for each $\alpha \in \text{dom}(S)$: $\triangleleft'' S(\alpha) \subseteq S(\alpha)$. By an $\langle R, \zeta \rangle$ -closed line in \mathbb{A} over B we mean an $\langle R, \zeta \rangle$ -line in \mathbb{A} over B such that R is a closed σ -string in \mathbb{A} over B . $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is said to be a *uniform $(\pi - \sigma)$ -closed line* in \mathbb{A} over B w.r.t. \mathfrak{S} if there exist an $R \in \mathfrak{S}$ and $\zeta \in \mathbf{N} - \mathbf{FN}$ such that the line in question is an $\langle R, \zeta \rangle$ -closed line in \mathbb{A} over B and $\mathbb{A}, B \in \mathfrak{S}$.

Recall (see [M3]) that an e -structure $\langle A, F, E \rangle \in \mathfrak{S}$ has a *u-expansion* in \mathfrak{S} if there exists a mapping $G : A^2 \rightarrow A, G \in \mathfrak{S}$ such that 1) $x \triangleleft_A y \Rightarrow G(y, x) = x$ and 2) $G(x, y) \triangleleft_A x$ holds for each $x, y \in A$ and that it is *commutative* if F is a commutative mapping on A .

Theorem (on monotonic valuation of uniform $(\pi - \sigma)$ -closed lines). *Let \mathfrak{S} be a standard universe. Assume that $\mathcal{L} = \mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$, \mathbb{A} is commutative and \mathbb{A} has a u-expansion in \mathfrak{S} . Then every uniform $(\pi - \sigma)$ -closed line w.r.t. \mathfrak{S} has a monotonic valuation in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to \mathfrak{S} .*

PROOF: We can state, observing the proof of the theorem on monotonic valuation of $\sigma^{\mathfrak{E}}$ - and $\pi^{\mathfrak{E}}$ -triads in [M3], p.383-384, that there exists a normal formula $\Phi(x, y, X, Y)$ of the language $\mathbf{FL}_{\mathbf{V}}$ such that we have: Let $\mathbb{A}(B, B) \in \mathfrak{S}$ be a closed commutative triad and suppose that \mathbb{A} has a u-expansion in \mathfrak{S} . Let S be a closed σ -string in \mathbb{A} over B . Then the mapping $H = \{\langle x, y \rangle; \Phi(x, y, \mathbb{A}, S)\}$ is a monotonic

valuation of $\mathbb{A}(B, B)$ in $\langle \mathbf{N}, +, \mathbf{Id} \rangle(\{0\}, \{0\})$ such that, for each $\alpha + 1 \in \text{dom}(S)$, the relation $S(\alpha) \subseteq \{x; H(x) \leq 2^\alpha\} \subseteq S(\alpha + 1)$ holds.

Let $R \in \mathfrak{S}$ be a closed π -string in \mathbb{A} over B and let $\zeta \in \mathbf{N} - \mathbf{FN}$ be such that the line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in question is an $\langle R, \zeta \rangle$ -closed line in \mathbb{A} over B w.r.t. \mathfrak{S} . Let H be defined as above. Assuming that $\text{dom}(S) = \theta + 1$ we have, for each $c \in \mathbf{FZ}$ odd, similarly as it is mentioned before the theorem on valuation of uniform $(\pi - \sigma)$ -lines: a) $x \in U_c \Leftrightarrow (\forall n) H(x) \leq 2^{z(c)-n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^\zeta)^c \cdot [0]^+$, b) $x \in U_{c+1} \Leftrightarrow (\exists n) H(x) \leq 2^{z(c)+n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^\zeta)^c \cdot \mathbf{BQ}^+$. Thus the mapping H is a monotonic valuation of the line in question in $\mathcal{L}_{\pi-\sigma}(\zeta)$. \square

We shall clarify a question what kinds of lines are closed uniform ones. Replacing, in the lemma on uniform lines, the assumption “ $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a line, $\mathbb{A}, B \in \mathfrak{S}$ ” by “ $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a closed line, $\mathbb{A}, B \in \mathfrak{S}$ ” and the assumption “ $S(c)$ is a σ -string in \mathbb{A} over B ” by “ $S(c)$ is a closed σ -string in \mathbb{A} over B ” we obtain, replacing the conclusion by “ $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a closed uniform $(\pi - \sigma)$ -line in \mathbb{A} over B w.r.t. \mathfrak{S} ”, a true proposition. We define similarly as above that a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a closed $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in \mathbb{A} over B , if $\mathbb{A}, B \in \mathfrak{S}$ and such that, for each $c \in \mathbf{FZ}$ odd, U_c is a $\pi(\mathfrak{S})$ -class and U_{c+1} is a $\sigma(\mathfrak{S})$ -class and, moreover, each U_c is closed under the canonical relation of \mathbb{A} . We can see, analyzing the proof of the lemma on monotonic valuations of $\sigma^\mathfrak{S}$ - and $\pi^\mathfrak{S}$ -triads in [M3] that the following holds:

Let \mathfrak{S} be a standard universe. Let $\mathbb{A}(B, B) \in \mathfrak{S}$ be a closed triad and suppose that $S \in \mathfrak{S}$ is a σ -string in \mathbb{A} over B . Then there exists a closed σ -string R in \mathbb{A} over B such that $\bigcup_n R(n) = \bigcup_n S(n)$.

We can prove, by using this assertion, quite analogously as above that the next proposition holds:

Let \mathfrak{S} be a codable saturated standard universe. Then the collection of closed $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -lines in \mathbb{A} over B is equal to this one of closed uniform $(\pi - \sigma)$ -lines in \mathbb{A} over B w.r.t. \mathfrak{S} .

We obtain immediately as a consequence:

Theorem (on monotonic valuations of $(\pi - \sigma)$ -lines). *Let \mathfrak{S} be a codable saturated standard universe and assume that $\mathbb{A} \in \mathfrak{S}$ is a commutative e -structure which has a u -expansion in \mathfrak{S} . Let $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ be a closed $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$ -line in \mathbb{A} over B . Then there exists a monotonic valuation of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to \mathfrak{S} .*

Let $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ be a closed $(\pi(\mathbf{Sd}_\mathbf{V}) - \sigma(\mathbf{Sd}_\mathbf{V}))$ -line in \mathbb{A} over B and let A be a set. Then there exists a monotonic set-valuation of this triad in an $\mathcal{L}_{\pi-\sigma}(\zeta)$.

Let us present an application of the last theorem.

Theorem. *Let A be a set and let $\{J_c\}_{c \in \mathbf{FZ}}$ be a class of ideals on A such that $J_{c+1} \subseteq J_c$ holds for each $c \in \mathbf{FZ}$ and, for each $c \in \mathbf{FZ}$ odd, J_c is a π -class and J_{c+1} is a σ -class. Then there exist a monotone and subadditive set-mapping $h : \mathcal{P}(A) \rightarrow \mathbf{Q}^+$ and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that we have $h^{-1''}\{0\} = \{\emptyset\}$ and, for each $c \in \mathbf{FZ}$ odd, $U_c = h^{-1''}(2^{\zeta \cdot c}) \cdot [0]^+$, $U_{c-1} = h^{-1''}(2^\zeta \cdot c) \cdot \mathbf{BQ}^+$.*

A proof of this theorem follows from the fact that $\langle \mathcal{P}(A), \cup, \mathbf{Id} \rangle_{c \in \mathbf{FZ}}$ is a closed $(\pi^{\mathbf{V}} - \sigma^{\mathbf{V}})$ -line and the structure $\langle \mathcal{P}(A), \cup, \mathbf{Id} \rangle$ is a commutative e -structure which has a u -expansion in $\mathbf{Sd}_{\mathbf{V}}$.

REFERENCES

- [Gui] Guide to Alternative set theory, (in Proc. of the 1st Symp. Mathematics in the alternative set theory), 1989.
- [M1] Mlček J., *Approximations of σ -classes and π -classes*, Comment. Math. Univ. Carolinae **20** (1979), 669–679.
- [M2] ———, *Valuations of structures*, Comment. Math. Univ. Carolinae **20** (1979), 681–695.
- [M3] ———, *Monotonic valuations and valuations of triads of higher types*, Comment. Math. Univ. Carolinae **22** (1981), 377–398.
- [V] Vopěnka P., *Mathematics in the Alternative Set Theory*, TEUBNER TEXTE, Leipzig, 1979.

FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, MALOSTRANSKÉ NÁM. 25,
110 00 PRAHA 1, CZECHOSLOVAKIA

(Received May 13, 1992)