# Dense chaos

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*Abstract.* According to A. Lasota, a continuous function $f$ from a real compact interval $I$ into itself is called generically chaotic if the set of all points $(x, y)$, for which $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$, is residual in $I \times I$. Being inspired by this definition we say that $f$ is densely chaotic if this set is dense in $I \times I$. A characterization of the generically chaotic functions is known. In the paper the densely chaotic functions are characterized and it is proved that in the class of piecewise monotone maps with finite number of pieces the notion of dense chaos and that of generic chaos coincide.

*Keywords:* dense chaos, generic chaos, piecewise monotone map

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## 1. Introduction and results.

If not stated otherwise, a function in this paper will always be a function belonging to the set $C^0(I, I)$ of all continuous maps of a real compact interval $I$ into itself. The terminology and notations are taken from [2]. Speaking on neighbourhoods or one-sided neighbourhoods of $\min I$ we have only right-hand side neighbourhoods in mind. Similarly for $\max I$.

For a function $f$ define the following planar sets:

$$\begin{aligned} C_1(f) =&\{(x,y) \in I^2\colon \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0\},\\ C_2(f) =&\{(x,y) \in I^2\colon \limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0\},\\ C(f) =&C_1(f) \cap C_2(f). \end{aligned}$$

According to A. Lasota (cf. [1]) a function $f$ is called generically chaotic if the set $C(f)$ is residual in $I^2$. Similarly, we will say that $f$ is densely chaotic if $C(f)$ is dense in $I^2$.

In [2], several conditions equivalent to the generic chaos have been found. Here we recall at least the following result which follows from [2, Theorem 1.2 (a),(g) and Lemma 4.16 (i),(ii)] and will be used later.

**Theorem 1.1 ([2]).** *A function $f \in C^0(I, I)$ is generically chaotic if and only if the following two conditions are fulfilled simultaneously:*

(A-1) *there is a fixed point $x_0$ of $f$ such that for every interval $J$, $\lim_{n\to\infty} \operatorname{dist}(f^n(J), x_0) = 0$,*

(D) *there is $a > 0$ such that for every interval $J$, $\limsup_{n\to\infty} \operatorname{diam} f^n(J) > a$.*

The condition (D) is too strong to be necessary for dense chaos (see [2, Example 3.6]). By weakening this condition and adding a new one we get the following characterization of dense chaos.

**Theorem 1.2.** *A function $f \in C^0(I, I)$ is densely chaotic if and only if the following three conditions are fulfilled simultaneously:*

(A-1) *there is a fixed point $x_0$ of $f$ such that for every interval $J$, $\lim_{n\to\infty} \operatorname{dist}(f^n(J), x_0) = 0$,*
(B-1) *for every interval $J$, $\limsup_{n\to\infty} \operatorname{diam} f^n(J) > 0$,*
(C) *every one-sided punctured neighbourhood of the point $x_0$ contains points $x$, $y$ with $(x, y) \in C(f)$ and moreover, if $x_0 \in \operatorname{int} I$ then every neighbourhood of $x_0$ contains points $x < x_0 < y$ with $(x, y) \in C(f)$.*

**Remark 1.3.** Consider further the following conditions:

(A-2) for every two intervals $J_1$, $J_2$, $\liminf_{n\to\infty} \operatorname{dist}(f^n(J_1), f^n(J_2)) = 0$,
(A-3) $C_1(f)$ is dense in $I^2$,
(A-4) $C_1(f)$ is residual in $I^2$

and

(B-2) for every interval $J$, $\liminf_{n\to\infty} \operatorname{diam} f^n(J) > 0$,
(B-3) $C_2(f)$ is dense in $I^2$.

By [2, Lemma 4.3] the conditions (A-2), (A-3) and (A-4) are equivalent and by [2, Lemma 4.12 and Remark 4.14] the conditions (B-1), (B-2) and (B-3) are equivalent. The condition (A-1) is stronger than (A-2) (see Example 3.3 below) but from [2, Lemma 4.20 and Lemma 4.17 (iii)] we get that (A-2) and (B-1) together imply (A-1). All things considered we can see that for every i, j $\in \{1, 2, 3, 4\}$ and k, m $\in \{1, 2, 3\}$, the conjunctions (A-i) & (B-k) and (A-j) & (B-m) are equivalent. So Theorem 1.2 remains true if we replace (A-1) & (B-1) by (A-i) & (B-k), i $\in \{1, 2, 3, 4\}$, k $\in \{1, 2, 3\}$. Of course, the point $x_0$ in (C) is the point that is mentioned in (A-1) and whose existence follows from the fact that (A-i) & (B-k) implies (A-1).

A continuous real function $f$ defined on a compact interval $J$ is called piecewise monotone or piecewise strictly monotone if there are $N \geq 0$ and $\min J = d_0 < d_1 < \cdots < d_N < d_{N+1} = \max J$ such that $f$ is monotone or strictly monotone, respectively, on $[d_k, d_{k+1}]$ for each $k = 0, \ldots, N$. If $f$ is piecewise strictly monotone and the points $d_k$ are chosen such that for $k = 1, \ldots, N$, $f$ is not monotone in any neighbourhood of $d_k$, the restricted functions $f|[d_k, d_{k+1}]$, $k = 0, \ldots, N$, are called the pieces of $f$.

The following theorem shows that in a class of maps which contains all piecewise monotone maps the notion of dense chaos and that of generic chaos coincide. Moreover, the characterization of such maps is simpler than those in Theorems 1.1 and 1.2.

**Theorem 1.4.** *Let $f \in C^0(I, I)$ and let every fixed point of $f$ have a neighbourhood on which $f$ is piecewise monotone. Then the following three conditions are equivalent:*

(i) *$f$ is generically chaotic,*

(ii) *$f$ is densely chaotic,*
(iii) (A-1) *and* (B-1).

**Remark 1.5.** Note that (A-1) & (B-1) is equivalent to (A-i) & (B-k) for arbitrarily chosen i $\in \{1,2,3,4\}$ and k $\in \{1,2,3\}$ (see Remark 1.3).

Recall that there is a map consisting of infinitely many strictly monotone (in fact, linear) pieces for which (i) does not hold though (ii) and consequently (iii) are fulfilled (see [2, Example 3.6]).

From the proof of Theorem 1.4 it will be seen that this theorem can be strengthened. In fact, if $f$ does not satisfy (A-1) then (i), (ii) and (iii) are not fulfilled and if $f$ satisfies (A-1) then (i), (ii) and (iii) are equivalent provided at least one of the fixed points $x_0$ of $f$ satisfying (A-1) (and not necessarily every fixed point of $f$) has a neighbourhood on which $f$ is piecewise monotone. There are maps whose dense and generic chaoticity can be proved using this result but not using Theorem 1.4 (see Example 3.4). On the other hand there is a generically chaotic function which has a unique fixed point but is not piecewise monotone in any neighbourhood of it (see Example 3.5).

The rest of the paper is organized as follows. In Section 2 the proofs of our results are given and in Section 3 some examples are presented. Concluding remarks can be found in Section 4. For open problems see Section 5.

## 2. Proofs.

Proof of Theorem 1.2: Let $f$ be densely chaotic. Then we get (A-2), (B-1) and (C) from the definition of the dense chaos trivially. Finally, (A-2) and (B-1) imply (A-1) (see Remark 1.3).

Now let (A-1), (B-1) and (C) be fulfilled. By Remark 1.3, (B-2) holds, too. Take any intervals $J_1$, $J_2$. We need to show that $(J_1 \times J_2) \cap C(f) \neq \emptyset$. By (B-2), there is $a > 0$ with $\liminf_{n\to\infty} \operatorname{diam}(f^n(J_i)) > a$, $i = 1, 2$. Consider the intervals $L = ]x_0 - \frac{a}{2}, x_0[$ and $R = ]x_0, x_0 + \frac{a}{2}[$ and suppose that $x_0$ is an interior point of $I$ (if it is an endpoint of $I$ we proceed analogously). Then by (C) there are points $(x_L, y_L) \in L^2 \cap C(f)$, $(x_R, y_R) \in R^2 \cap C(f)$ and $(\tilde{x}_L, \tilde{y}_R) \in (L \times R) \cap C(f)$. Using (A-1) we get that for sufficiently large $n$ each of the sets $f^n(J_1)$ and $f^n(J_2)$ contains at least one of the sets $\{x_L, y_L, \tilde{x}_L\}$ and $\{x_R, y_R, \tilde{y}_R\}$. Hence $(f^n(J_1) \times f^n(J_2)) \cap C(f) \neq \emptyset$ and so $(J_1 \times J_2) \cap C(f) \neq \emptyset$.

To prove Theorem 1.4 we need the following lemma. Its simple proof is omitted.

**Lemma 2.1.** *Let $f \in C^0(I, I)$ satisfy the condition* (A-1) *and let $f$ be piecewise strictly monotone on a neighbourhood of the point $x_0$ from* (A-1). *Then the point $x_0$ is isolated in the set of all fixed points of $f$.*

Proof of Theorem 1.4: (i) $\Rightarrow$ (ii) holds trivially and (ii) $\Rightarrow$ (iii) follows from Theorem 1.2. With respect to Theorem 1.1 it suffices to prove that (iii) implies (D). So let $f$ satisfy (A-1) and (B-1). To prove (D) it is sufficient to show that there exists $a > 0$ such that for every interval $J$ there is $k$ with $\operatorname{diam} f^k(J) \geq 2a$.

Assume that the fixed point $x_0$ from (A-1) is an interior point of $I$ (if it is an endpoint of $I$, the proof is similar). Take sufficiently small $\delta > 0$ such that both $f$

and $g = f^2$ are piecewise monotone on $M = [x_0 - \delta, x_0 + \delta]$. From (B-1) we have that $f$ and hence also $g$ are piecewise strictly monotone on $M$. Since $f$ satisfies (A-1), $g$ satisfies an analogous condition in which the point $x_0$ is the same and $f$ is replaced by $g$. So by Lemma 2.1, the point $x_0$ is isolated in the set of all fixed points of $g$. Take $a > 0$ such that the interval $[x_0 - 3a, x_0 + 3a]$ contains no fixed point of $g$ (and hence no fixed point of $f$) different from $x_0$ and both $f$ and $g$ are strictly monotone on each of the intervals $L = [x_0 - 3a, x_0[$ and $R = ]x_0, x_0 + 3a]$.

Take $h \in \{f, g\}$. If $x < h(x) < x_0$ for all $x \in L$ then $\lim_{n\to\infty} \operatorname{diam} h^n(L) = 0$, whence $\lim_{n\to\infty} \operatorname{diam} f^n(L) = 0$. But this is impossible since $f$ satisfies (B-1). So either

(1) $h(x) > x_0$ for all $x \in L$, or
(2) $h(x) < x$ for all $x \in L$.

For the analogous reasons either

(3) $h(x) < x_0$ for all $x \in R$, or
(4) $h(x) > x$ for all $x \in R$.

Now take any interval $J$. To finish the proof we are going to show that for some $k$, $\operatorname{diam} f^k(J) \geq 2a$.

Denote $L_1 = [x_0 - a, x_0]$, $L_2 = [x_0 - 3a, x_0 - a]$, $R_1 = [x_0, x_0 + a]$ and $R_2 = [x_0 + a, x_0 + 3a]$. Since $f$ satisfies (A-1), there is $N$ such that $f^n(J) \cap (L_1 \cup R_1) \neq \emptyset$ for all $n \geq N$. To complete the proof it suffices to show that for some $k \geq N$, $f^k(J)$ meets $I \setminus (L \cup R \cup \{x_0\})$.

Take a point $z \in f^N(J) \cap L$ (if no such point exists then $f^N(J)$ meets $R$ and one can proceed similarly). Consider three possible cases:

Case 1. $h = f$ satisfies (2). Then it is easy to see that for some $r \geq 1$, $f^r(z) < \min L$ and it suffices to take $k = N + r$.

Case 2. $h = f$ satisfies (1) and (4). Then $f^{N+1}(J)$ meets $R$ and one can proceed in the same way as in Case 1.

Case 3. $h = f$ satisfies (1) and (3). Then $h = g$ satisfies (2) and in the same way as in Case 1 we get some $r \geq 1$ with $g^r(z) = f^{2r}(z) < \min L$. Take $k = N + 2r$.

## 3. Examples.

We want to present some examples to illustrate our results. Let $\min I = a_0 < a_1 < \cdots < a_n = \max I$ and $b_i \in I$, $i = 0, 1, \ldots, n$. Then by $\langle (a_0, b_0), \ldots, (a_n, b_n) \rangle$ will be denoted the piecewise linear map which sends $a_i$ to $b_i$, $i = 0, 1, \ldots, n$ and is linear on each interval $[a_i, a_{i+1}]$, $i = 0, 1, \ldots, n-1$.

Recall that the map $f$ from [2, Example 3.6] is densely chaotic but is not generically chaotic. Now also Theorem 1.2 shows that $f$ is densely chaotic with $x_0 = 1$ in (A-1).

**Example 3.1.** Let $I = [0, 2]$. Take the map $g \in C^0(I, I)$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0,1] \\ 2 - f(2 - x) & \text{if } x \in [1,2] \end{cases}$$

where $f$ is the map from [2, Example 3.6]. Though $f$ is densely chaotic and $g$ satisfies the condition (A-1) with $x_0 = 1$ and the condition (B-1), $g$ is not densely chaotic since it does not satisfy the condition (C).

**Example 3.2.** Let $I = [0,1]$. Consider the tent maps $f_\lambda \in C^0(I,I)$ defined by $f_\lambda(x) = \frac{\lambda}{2}(1-|2x-1|)$ for $\lambda \in [0,2]$. The point $f^3_{\sqrt{2}}(\frac{1}{2})$ is the positive fixed point of $f_{\sqrt{2}}$. If $\lambda \geq \sqrt{2}$ then for every interval $J \subset I$ there is $n$ with $f^n_\lambda(J) \supset [f^2_\lambda(\frac{1}{2}), f_\lambda(\frac{1}{2})]$. If $\lambda < \sqrt{2}$ then either $f_\lambda$ has a periodic interval of period 2 or $f_\lambda$ is identical on an interval or $\lim_{k\to\infty} f^k_\lambda(x) = 0$ for every $x$. So $f_\lambda$ is densely or, equivalently, generically chaotic if and only if $\lambda \geq \sqrt{2}$.

**Example 3.3.** Let $I = [0,1]$. We are going to define a map $f \in C^0(I,I)$ which satisfies the condition (A-2) but does not satisfy (A-1). Take $f = \langle(0,0),(1/4,1/2),(1/2,0),(3/4,a),(1,a)\rangle$ where $a$ is any point from $[0,1/2]$ such that for $g = f|[0,1/2]$, $g^n(a)$ does not converge to a fixed point of $g$ when $n\to\infty$.

**Example 3.4.** Let $I = [-1,1]$. Take a map $f \in C^0(I,I)$ such that $f(x) = 1 - |2x-1|$ for $x \in [0,1]$, $f([-1,0]) \subset [0,1]$ and $f$ is constant in no interval. Then for every interval $J$ there is $n$ with $f^n(J) = [0,1]$ and so, by Theorem 1.1, $f$ is generically chaotic. If, additionally, $f$ is piecewise monotone in no left-hand side neighbourhood of the point 0, Theorem 1.4 cannot be used. The stronger version of this theorem mentioned in the last but one paragraph of Section 1 works (take the larger from two fixed points of $f$ as the required point $x_0$).

**Example 3.5.** Recall the folk result that there is a continuous self-map $g$ of the interval $[0,1]$ such that for every $x$, $x^2 \leq g(x) \leq \sqrt{x}$ and $g$ is topologically transitive. (Then for every $\varepsilon > 0$ and for every interval $J$ there is $N$ with $f^n(J) \supset [\varepsilon, 1-\varepsilon]$ whenever $n \geq N$, but it is not true that for every $J$ there is $N$ with $f^N(J) = [0,1]$. Such maps can consist of infinitely many linear pieces and cannot be piecewise monotone in any neighbourhood of the point 0 or 1.) Now take $I = [-1,1]$ and $f \in C^0(I,I)$ such that $f(x) = -x$ for $x \in [0,1]$, $x^2 \leq f(x) \leq \sqrt{-x}$ for $x \in [-1,0]$ and $g = f^2|[0,1]$ is a map with the above mentioned properties. Then $f$ is generically chaotic (even topologically transitive) but is not piecewise monotone in any neighbourhood of its unique fixed point 0.

## 4. Concluding remarks.

Some results concerning the generic chaos can be carried over to the case of continuous self-maps of the compact metric spaces.

For example, if for every two balls $B_1$ and $B_2$, $\liminf_{n\to\infty} \operatorname{dist}(f^n(B_1), f^n(B_2)) = 0$ and if there is an $a > 0$ such that for every ball $B$, $\limsup_{n\to\infty} \operatorname{diam} f^n(B) > a$, then $f$ is generically chaotic (the definition of the generic chaos is analogous to that on the interval). On the other hand, an irrational rotation of the circle is topologically transitive but not densely chaotic, which is impossible on the interval.

Let $I$ be an interval and $0 < \alpha \leq \beta < \operatorname{diam} I$. We say that $f \in C^0(I,I)$ is generically or densely $(\alpha,\beta)$-chaotic if the set of all $(x,y) \in I^2$ with $\liminf_{n\to\infty} |f^n(x) - f^n(y)| \leq \alpha$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \beta$ is residual or dense in $I^2$, respectively. A characterization of such maps is analogous to that of generically chaotic maps and can be found in [3].

## 5. Open problems.

We finish our paper with the following problems:

(1) Find inf {topological entropy of $f$ : $f$ is densely chaotic} (cf. [2, Theorem 1.3]).
(2) What is the position of densely chaotic functions in the Šarkovskii ordering? (Cf. [2, Theorem 1.4].)
(3) Take $I = [0, 1]$ and $f_\lambda \in C^0(I, I)$ defined by $f_\lambda(x) = \lambda x(1-x)$ for $\lambda \in [0, 4]$. What can be said about the parameters $\lambda$ for which $f_\lambda$ is densely chaotic? Is there a countable or an uncountable number of such parameters? Are such parameters dense in some interval $[\lambda_0, 4]$?

It is easy to see (cf. [2]) that every generically chaotic map has sensitive dependence on initial conditions. It would be interesting to know whether a kind of converse holds for the maps $f_\lambda(x) = \lambda x(1-x)$.

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