

Monotonic valuations of $\pi\sigma$ -triads and evaluations of ideals

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Abstract. We develop problems of monotonic valuations of triads. A theorem on monotonic valuations of triads of the type $\pi\sigma$ is presented. We study, using the notion of the monotonic valuation, representations of ideals by monotone and subadditive mappings. We prove, for example, that there exists, for each ideal J of the type π on a set A , a monotone and subadditive set-mapping h on $P(A)$ with values in non-negative rational numbers such that $J = h^{-1''}\{r \in Q; r \geq 0 \ \& \ r \doteq 0\}$. Some analogical results are proved for ideals of the types σ , $\sigma\pi$ and $\pi\sigma$, too. A problem of an additive representation is also discussed.

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We develop problems of monotonic valuations of triads of the type $\pi\sigma$ by proving a theorem on such valuations of so called limit $\pi\sigma$ -triads. This theorem completes a list of theorems on monotonic valuations of triads of the type $\sigma\pi$ and $\pi\sigma$. (See [M3].) Moreover, we study, using the general theorems on valuations, representations of ideals on sets by monotone and subadditive mappings. We prove, for example, that there exists, for each ideal J of the type π on a set A , a monotone and subadditive set-mapping h on $P(A)$ with values in non-negative rational numbers Q^+ such that $J = h^{-1''}\{r \in Q^+; r \doteq 0\}$. Some analogical results are proved for ideals of the type σ , $\sigma\pi$ and $\pi\sigma$, too. We discuss also the existence of such additive representations; it means that the mapping in question is additive on A .

We work in the alternative set theory; we shall use the usual notations of this theory. Recall, that small Latin letters range over sets and i, j, k, l, m, n range over finite natural numbers. By a collection we mean a collection of classes which satisfies a given formula of the language FL_V . The collection of all set-theoretically definable classes is denoted by Sd_V ; it is a codable system. We say that a class X is of the type π - (σ - resp.) if there exists a set-theoretically definable relation R such that $X = \bigcap_n R''\{n\}$ ($X = \bigcup_n R''\{n\}$ resp.). Let E be a π -equivalence on a set-definable class A . We say that a class $X \subseteq A$ is E -closed in A if we have the following: $(\forall a \in A - X)(\exists U \in Sd_V)(U \cap X = \emptyset \ \& \ E''\{a\} \subseteq U)$.

MONOTONIC VALUATIONS OF $\pi\sigma$ -TRIADS

Monotonic valuations of triads.

We recall briefly some notions about triads (see [M2]). We say that a structure $\langle A, F, E \rangle$ is an e -structure if we have the following:

- (1) $\langle A, F \rangle$ is a semigroup (i.e. F is an associative operation on A),

- (2) $E \circ E$ is the identity on A ,
- (3) we have either $F(E(x), E(y)) = E(F(x, y))$ for each $x, y \in A$ or $F(E(x), E(y)) = E(F(y, x))$ for each $x, y \in A$.

Let $\langle A, F, E \rangle$ be an e -structure. We define a *canonical relation* \triangleleft_A of an e -structure $\langle A, F, E \rangle$ on A by:

$$x \triangleleft_A y \Leftrightarrow (\exists z \in A)(F(x, z) = y).$$

Let $\langle A, F, E \rangle$ be an e -structure. We can see that

- (1) E is a one-one mapping on A and, thus, E is an automorphism or an anti-automorphism of the semigroup $\langle A, F \rangle$.
- (2) \triangleleft_A is a transitive relation on A .

Assume that $\mathbb{A}, \hat{\mathbb{A}}$ are two e -structures. A mapping $H : A \rightarrow \hat{A}$ is said to be a *valuation of \mathbb{A} in $\hat{\mathbb{A}}$* if we have:

- (a) $H(F(x, y)) \triangleleft_{\hat{A}} \hat{F}(H(x), H(y))$ holds for each $x, y \in A$,
- (b) $H(F(x)) = \hat{F}(H(x))$ holds for each $x \in A$.

Let \mathbb{A} be an e -structure. Then the triple $\langle \mathbb{A}, \mathbb{A} \upharpoonright U, \mathbb{A} \upharpoonright B \rangle$, where $B \subseteq U \subseteq A$ and $\mathbb{A} \upharpoonright B, \mathbb{A} \upharpoonright U$ are substructures of \mathbb{A} , is said to be a *triad over the e -structure \mathbb{A}* . We denote it as

$$\mathbb{A}(U, B).$$

A mapping H is called a *valuation of the triad $\mathbb{A}(U, B)$ in a triad $\langle \hat{\mathbb{A}}, \hat{F}, \hat{E} \rangle(\hat{U}, \hat{B})$* , if H is a valuation of $\langle A, F, E \rangle$ in $\langle \hat{A}, \hat{F}, \hat{E} \rangle$ and we have, moreover,

- (c) $H^{-1''} \hat{U} = U, H^{-1''} \hat{B} = B$.

Let us recall the notion of monotonic valuations (see [M3]).

A valuation H of an e -structure \mathbb{A} in $\hat{\mathbb{A}}$ is called a *monotonic valuation* if we have

$$(x, y \in A \ \& \ x \triangleleft_A y) \Rightarrow (H(x) \triangleleft_{\hat{A}} H(y)).$$

By a *monotonic valuation of a triad \mathcal{T} in a triad $\hat{\mathcal{T}}$* we mean such a valuation of \mathcal{T} in $\hat{\mathcal{T}}$ which is a monotonic valuation of the relevant e -structures.

Our intention is to present a theorem on monotonic valuations of a closed (w.r.t. \triangleleft) triad $\langle A, F, E \rangle(U, B)$ in some canonical one under assumption that $\langle A, F, E \rangle$ and B belong to Sd_V and U is a $\pi\sigma$ -class. We refer to this situation as to a problem of monotonic valuations of $\pi\sigma$ -triads. We shall solve a more general problem assuming that the classes $\langle A, F, E \rangle$ and B belong to a so called saturated standard universe (see below). Note that a similar problem of monotonic valuations of π -, σ - and $\sigma\pi$ -triads is solved in [M3].

Now, let us define *canonical τ -triads \mathcal{T}_τ* , where τ is the symbol σ , π , $\sigma\pi$ or $\pi\sigma$. Put, at first,

$$[0]^+ = \{r \in Q^+; r \doteq 0\}.$$

We define

$$\mathcal{T}_\sigma = \langle N, +, Id \rangle(FN, \{0\}), \quad \mathcal{T}_\pi = \langle Q^+, +, Id \rangle([0]^+, \{0\}).$$

The triad $\mathcal{T}_{\sigma\pi}$ is defined as follows: Let ζ be fixed, $\zeta \in N - FN$. We define, on ${}^\zeta Q^+$, the mapping $+$ by the relation: $(f+g)(\alpha) = f(\alpha) + g(\alpha)$. Then $\langle {}^\zeta Q^+, +, Id \rangle$ is an e -structure and the canonical relation of this structure is the ordering \leq defined by $f \leq g \Leftrightarrow (\forall \alpha \in \zeta)(f(\alpha) \leq g(\alpha))$. Now, $\langle {}^\zeta N, +, Id \rangle$ is an e -structure, too. Put

$$0_\zeta = \zeta \times \{0\}, \quad U(\sigma\pi) = \{f \in {}^\zeta N; (\exists n)(\forall m)f(m) < n\}.$$

We can see that $U(\sigma\pi)$ is a $\sigma\pi(Sd_V)$ -class and $\langle {}^\zeta N, +, Id \rangle(U(\sigma\pi), \{0_\zeta\})$ is a triad. We define

$$\mathcal{T}_{\sigma\pi} = \langle {}^\zeta N, +, Id \rangle(U(\sigma\pi), \{0_\zeta\}).$$

Finally, let us define a *canonical $\pi\sigma$ -triad* $\mathcal{T}_{\pi\sigma}$. Let $\zeta \in N - FN$ be fixed as above. We put

$$U(\pi\sigma) = \{f \in {}^\zeta Q^+; (\forall \gamma \in \zeta - FN)f(\gamma) \doteq 0\}.$$

Then we define

$$\mathcal{T}_{\pi\sigma} = \langle {}^\zeta Q^+, +, Id \rangle(U(\pi\sigma), \{0_\zeta\}).$$

Before we formulate a theorem on monotonic valuations, let us give some useful notions. We say that an e -structure $\langle A, F, E \rangle$ is *commutative*, whenever F is a commutative operation on A .

Assume $U \subseteq A$ and let $\mathbb{A} = \langle A, F, E \rangle$ be an e -structure. The class U is said to be *closed in \mathbb{A}* if U is closed under the canonical relation, i.e. if we have

$$(\forall x \in A)(\forall y \in U)(x \triangleleft_A y \Rightarrow x \in U).$$

We say that a triad $\mathbb{A}(U, B)$ is *closed* if U and B are closed in \mathbb{A} .

A structure $\langle A, F, E, G \rangle$ is called a *u -expansion of an e -structure $\langle A, F, E \rangle$* , if G is a binary function and if we have:

- (1) $x \triangleleft_A y \Rightarrow G(y, x) = x$,
- (2) $G(x, y) \triangleleft_A x$.

We say that an e -structure \mathbb{A} has a *u -expansion*, if there exists a u -expansion $\langle \mathbb{A}, G \rangle$ of the e -structure \mathbb{A} .

Example. The structure $\langle P(a), \cup, Id, \cap \rangle$ is a u -expansion of the e -structure $\langle P(a), \cup, Id \rangle$.

It is not difficult to prove that every triad \mathcal{T}_τ , where τ is $\sigma, \pi, \sigma\pi$ or $\pi\sigma$, is commutative and closed. The commutativity is clear. Let $U \in Sd_V$ be a relation with $dom(U) = \zeta^2$ such that

$$U(\alpha, \beta) = \{f \in {}^\zeta Q^+; (\forall \gamma \in [\beta, \zeta])f(\gamma) \leq 2^{-\alpha}\}.$$

We have, for $\alpha + 1, \beta + 1 \in \zeta$, $U(\alpha, \beta + 1) \supseteq U(\alpha, \beta) \supseteq U(\alpha + 1, \beta)$. Put $U_m = \bigcup_n U(m, n)$. Then we have $U(\pi\sigma) = \bigcap_m U_m$, $U_{m+1} \subseteq U_m$, $U_{m+1} + U_{m+1} \subseteq U_m$, $(g \in U(\alpha, \beta) \ \& \ f \leq g \ \& \ f \in {}^\zeta Q^+) \Rightarrow f \in U(\alpha, \beta)$. We deduce from this that the triad $\mathcal{T}_{\pi\sigma}$ is a closed $\pi\sigma(Sd_V)$ -triad.

We can show quite analogously that all remaining canonical triads are closed, too.

Monotonic valuation of $\pi\sigma$ -triads.

We study our problem of valuations of triads with respect to a so called standard universe of classes. It means that the relevant e -structures belong to this universe and we are looking for valuations from this universe, too. Note that the system Sd_V is such a special collection.

We say that a collection of classes is a *universe of classes* if it is closed under the definitions by normal formulas of the language FL_V with class-parameters from this collection. Thus, having a universe \mathfrak{U} of classes and a normal formula $\varphi(x, X_1, X_2, \dots, X_k)$ of the language FL_V such that the classes X_1, X_2, \dots, X_k belong to \mathfrak{U} , we see that the class $\{x; ; \varphi(x, X_1, X_2, \dots, X_k)\}$ belongs to \mathfrak{U} , too. Note that every universe of classes contains all sets. More generally, every set-theoretically definable class belongs to each universe of classes. By a *standard universe of classes* we call each universe of classes which contains only such non-empty subclasses of the class of natural numbers which have the first element. We can see that the following proposition holds (see [M1]).

Proposition. *Every standard universe of classes contains only revealed classes and does not contain any proper semiset. It satisfies all axioms of Gödel-Bernays's theory of finite sets.*

A standard universe \mathfrak{U} of classes is said to be a *saturated standard universe of classes* if we have the following: Let $\{X_n\}_{n \in FN}$ be a sequence of classes of this universe. Then there exists a relation R from \mathfrak{U} such that

$$(\forall n)R''\{n\} = X_n.$$

Example. The system Sd_V is a standard universe of classes which is not a standard saturated universe of classes. Its revelation Sd_V^* is a standard saturated universe of classes.

By a π - (σ - resp.) *string* we mean a relation R such that $dom(R) = FN$ and, for each $n \in FN$, $R''\{n+1\} \subseteq R''\{n\}$ ($R''\{n\} \subseteq R''\{n+1\}$ resp.) holds. Assume that \mathfrak{S} is a standard universe of classes. A class $\bigcap_n X_n$, where $\{X_n\}_{n \in FN} \subseteq \mathfrak{S}$, is called a $\pi(\mathfrak{S})$ -class and a class of the form $\bigcup_n X_n$, where $\{X_n\}_{n \in FN} \subseteq \mathfrak{S}$, is called a $\sigma(\mathfrak{S})$ -class.

Let $\mathbb{A} = \langle A, F, E \rangle$ be an e -structure, $\mathbb{A} \in \mathfrak{S}$. We say that a class U is a *limit $\pi\sigma(\mathfrak{S})$ -universe in \mathbb{A}* if there exists a non-increasing sequence $\{U_n\}_{n \in FN}$ of $\sigma(\mathfrak{S})$ -classes such that $U = \bigcap_m U_m$ and $F''U_{m+1}^2 \subseteq U_m$, $E''U_{m+1} \subseteq U_m$ hold. It is a *limit closed $\pi\sigma(\mathfrak{S})$ -universe in \mathbb{A}* if we have, moreover, $\triangleleft''U_{m+1} \subseteq U_m$. A triad $\mathbb{A}(U, B)$ is said to be a *limit $\pi\sigma(\mathfrak{S})$ -triad* (a *limit closed $\pi\sigma(\mathfrak{S})$ -triad* resp.) if $\mathbb{A}, B \in \mathfrak{S}$ and U is a limit $\pi\sigma(\mathfrak{S})$ -universe in \mathbb{A} (a limit closed $\pi\sigma(\mathfrak{S})$ -universe in \mathbb{A} resp.). Thus every limit closed $\pi\sigma(\mathfrak{S})$ -triad is a closed triad.

It is not difficult to prove that $\mathcal{T}_{\pi\sigma}$ is a commutative limit closed $\pi\sigma(Sd_V)$ -triad.

Theorem. *Let \mathfrak{S} be a saturated standard universe of classes. Let $\mathbb{A}(U, B)$ be a closed limit $\pi\sigma(\mathfrak{S})$ -triad such that \mathbb{A} is commutative and has a u -expansion in \mathfrak{S} .*

Then there exists a monotonic valuation of the triad $\mathbb{A}(U, B)$ in $\mathcal{T}_{\pi\sigma}$ which belongs to \mathfrak{S} .

PROOF: Writing $[F, E, \triangleleft](X, Y)$ we mean that $F''X^2 \subseteq Y$, $E''X \subseteq Y$, $\triangleleft''X \subseteq Y$ hold. We define a mapping $F_3 : A^3 \rightarrow A$ by $F_3(x, y, z) = F(F(x, y), z)$. By a matrix we mean a relation M such that $\text{dom}(M) = \xi^2$ for some $\xi \in N - FN$. Let us use the following notation: $\mathcal{H}_{\pi\sigma}(M) = \bigcap_m \bigcup_n M(m, n)$. We deduce from the proposition in 2.1.2, [M3], that there exists a matrix $T \in \mathfrak{S}$ such that (1) $B \subseteq T(\alpha, \beta) \subseteq A$ holds for each $\alpha, \beta \in \text{dom}(T)$, (2) $\mathcal{H}_{\pi\sigma}(T) = U$ and (3) for each $m \in FN$, $[F, E, \triangleleft](\bigcup_n T(m+1, n), \bigcup_n T(m, n))$. We can construct as in [M3, 2.2.3] a matrix $R \in \mathfrak{S}$ such that (1), (2) hold for R instead of T and $[F, E, \triangleleft](R(\alpha+1, \beta), R(\alpha, \beta))$ is satisfied for each $\alpha+1, \beta \in \xi$, where $\xi = \text{dom}(R)$. Let $2\theta \leq \xi$, $\theta \notin FN$. Let $M \in \mathfrak{S}$ be such a matrix that we have $\text{dom}(M) = \theta^2$, $M(0, \beta) = B$, $M(\theta - 1, \beta) = A$ for each $\beta \in \theta$ and $M(\alpha, \beta) = P(2\alpha, \beta)$ for each $\alpha, \beta \leq \theta$ where $P(\gamma, \beta) = \triangleleft''(R(\gamma, \beta) \cap E''R(\gamma, \beta))$. We can see similarly as in [M3, 1.1.0] that $F''M^2(\alpha+1, \beta) \subseteq M(\alpha, \beta)$ holds for each $\alpha+1, \beta \in \theta$, $E''M(\alpha, \beta) \subseteq M(\alpha, \beta)$ holds for each $\alpha, \beta \subseteq \theta$ and $\triangleleft''M^2(\alpha+1, \beta) \subseteq M(\alpha, \beta)$. We have, moreover, $R(2\alpha, \beta+1) \subseteq M(\alpha, \beta+1) \subseteq R(2\alpha, \beta)$. We deduce from this that $\mathcal{H}_{\pi\sigma}(M) = \mathcal{H}_{\pi\sigma}(R) = U$. We have, for $\gamma+1, \beta \in \theta$, $F''P^2(\gamma+1, \beta) \subseteq P(\gamma, \beta)$ and, consequently, for $\gamma+2, \beta \in \theta$ holds the following: $F_3''P^3(\gamma+2, \beta) \subseteq F''(F''P^2(\gamma+2, \beta))^2 \subseteq F''P^2(\gamma+1, \beta) \subseteq P(\gamma, \beta)$. Thus $F_3''M^3(\alpha+1, \beta) = F_3''P^3(2(\alpha+1), \beta) \subseteq P(2\alpha, \beta) = M(\alpha, \beta)$ holds for each $\alpha+1, \beta \in \theta$.

Put, for each $\beta \in \theta$,

$$S(\beta) = \{\langle \alpha, x \rangle; x \in M(\alpha, \beta)\}.$$

Then each $S(\beta)$ has the following properties:

- (a) $S(\beta) \in \mathfrak{S}$,
- (b) $\alpha+1 \in \theta \Rightarrow [F, F_3](S(\beta)(\alpha+1), S(\beta)(\alpha))$ ($[F, F_3](\dots)$ has a similar meaning as in the previous proof),
- (c) $\alpha \in \theta \Rightarrow E''S(\beta)(\alpha) \subseteq S(\beta)(\alpha)$,
- (d) $\alpha \in \theta \Rightarrow \triangleleft''(E''S(\beta)(\alpha)) \subseteq S(\beta)(\alpha)$.

It is not difficult to see that we can assume that $\text{dom}(S) = \zeta^2$ with some $\zeta \in N - FN$ and that, for each $\beta \in \zeta$, $S(\beta)(0) = A$, $S(\beta)(\zeta - 1) = B$ hold. Such $S(\beta)$ is called a monotonic $\pi^{\mathfrak{S}}$ -string in \mathbb{A} over B . We can see, similarly as in the proof of the theorem on monotonic valuations of σ^{m} - and π^{m} -triads in [M3, p. 383-384], that there exists a normal formula $\Psi(x, y, X, Y)$ of the language FL_V such that the following holds:

Let $\mathbb{A}(B, B) \in \mathfrak{S}$ be a triad and let $D \in \mathfrak{S}$ be a monotonic $\pi^{\mathfrak{S}}$ -string in \mathbb{A} over B . Then $H = \{\langle x, y \rangle; \Psi(x, y, \mathbb{A}, D)\}$ is a monotonic valuation of $\mathbb{A}(B, B)$ in $\langle Q^+, +, Id \rangle(\{0\}, \{0\})$ and $D(\alpha+1) \subseteq \{x \in A; H(x) \leq 2^{-\alpha}\} \subseteq D(\alpha)$ holds for each $\alpha \in \text{dom}(D)$.

Let

$$W = \{\langle \beta, \langle x, y \rangle \rangle; \Psi(x, y, \mathbb{A}, S(\beta)) \ \& \ \beta \in \zeta\}.$$

Put $W_\beta = W''\{\beta\}$. Then W_β is a monotonic valuation of $\mathbb{A}(B, B)$ in $\langle Q^+, +, Id \rangle(\{0\}, \{0\})$ and $W_\beta \in \mathfrak{S}$. We have

$$x \in U \Leftrightarrow (\forall \beta \in \zeta - FN)(x \in \bigcap_m M(m, \beta)) \Leftrightarrow (\forall \beta \in \zeta - FN)W_\beta \doteq 0.$$

Let $H : A \rightarrow {}^\zeta Q^+$ be defined by

$$H(x) = \{\langle \alpha, W_\alpha(x) \rangle; \alpha \in \zeta\}.$$

We have, for each $x, y \in A$, $H(F(x, y))(\alpha) = W_\alpha(F(x, y)) \leq W_\alpha(x) + W_\alpha(y)$. Thus $H(F(x, y)) \leq H(x) + H(y)$ holds. We can see similarly that $H(E(x)) = H(x)$ and $x \triangleleft_A y \Rightarrow H(x) \leq H(y)$ hold, too. Thus H is a monotonic valuation of \mathbb{A} in $\langle {}^\zeta Q^+, +, Id \rangle$. It is easy to see that $H(x) = \zeta \times \{0\}$ iff $x \in B$. Finally, we have $x \in U \Leftrightarrow (\forall \beta \in \zeta - FN)(W_\beta \doteq 0) \Leftrightarrow (\forall \beta \in \zeta - FN)H(x)(\beta) \doteq 0 \Leftrightarrow H(x) \in U_{\pi\sigma}$, which completes the proof. \square

Remark. Let $\mathbb{A} = \langle A, F, E \rangle$ be an e -structure. We say that a class $U \subseteq A$ is a *limit $\pi\sigma^\mathfrak{S}$ -universe* in \mathbb{A} if there exists a matrix $M \in \mathfrak{S}$ such that

- (1) $\mathcal{H}_{\pi\sigma}(M) = U$,
- (2) $(\forall m \in FN)([F, E](\bigcup_n M(m+1, n), \bigcup_n M(m, n)))$.

U is a *limit closed $\pi\sigma^\mathfrak{S}$ -universe* in \mathbb{A} if we have, moreover,

$$\triangleleft'' \bigcup_n M(m+1) \subseteq \bigcup_n M(m, n).$$

A triad $\mathbb{A}(U, B)$ is said to be a *limit $\pi\sigma^\mathfrak{S}$ -triad* (a *limit closed $\pi\sigma^\mathfrak{S}$ -triad* resp.) if $\mathbb{A}, B \in \mathfrak{S}$ and U is a limit $\pi\sigma^\mathfrak{S}$ -universe in \mathbb{A} (a limit closed $\pi\sigma^\mathfrak{S}$ -universe in \mathbb{A} resp.). Thus every limit closed $\pi\sigma^\mathfrak{S}$ -triad is a closed triad. The triad $\mathcal{T}_{\pi\sigma}$ is a limit closed $\pi\sigma^\mathfrak{S}$ -triad.

We can see that the last proof guarantees that if we assume, in the last theorem, that \mathfrak{S} is only a standard universe and that the triad in question is a limit closed $\pi\sigma^\mathfrak{S}$ -triad, we obtain a true proposition.

Proposition. *There exists a $\pi\sigma(Sd_V)$ -triad $\mathbb{A}(U, B)$ (i.e. $\mathbb{A} \in Sd_V$, $B \in Sd_V$ and U is a $\pi\sigma$ -class) which is not a limit $\pi\sigma(Sd_V)$ -triad.*

PROOF: Let (\mathcal{E}) be an equivalence on N defined by

$$\langle \alpha, \beta \rangle \in (\mathcal{E}) \Leftrightarrow (\exists n)(\alpha, \beta < n) \vee (\forall n)(\alpha, \beta > n).$$

Let $\langle N^2 \cup \{\emptyset\}, F, E \rangle$ be the e -structure defined by the following relations:

$$\begin{aligned} F(\langle x, y \rangle, \langle \tilde{y}, z \rangle) &= \langle x, z \rangle \Leftrightarrow y = \tilde{y} \\ &= \emptyset \Leftrightarrow y \neq \tilde{y}, \\ F(u, \emptyset) &= F(\emptyset, u) = \emptyset \Leftrightarrow u \in A. \end{aligned}$$

The function $E : N^2 \cup \{\emptyset\} \rightarrow N^2 \cup \{\emptyset\}$ is defined by $E(\langle x, y \rangle) = \langle y, x \rangle$ for each $\langle x, y \rangle$ and $E(\emptyset) = \emptyset$.

Then $\langle N^2 \cup \{\emptyset\}, F, E \rangle ((\mathcal{E}) \cup \{\emptyset\}, Id \upharpoonright N^2 \cup \{\emptyset\})$ is a $\pi\sigma(Sd_V)$ -triad which is not a limit $\pi\sigma(Sd_V)$ -one.

Indeed, assume, contrariwise, that it is. Then there exists its valuation $H \in Sd_V^*$ in $\mathcal{T}_{\pi\sigma}$ (where Sd_V^* is a revelation of Sd_V). This follows from 2.1.2 and 3.0.4 in [M3]. Put $D = H \upharpoonright N^2$. We have $(\mathcal{E}) = \bigcap \{D^{-1''}U(m, \beta); m \in FN \ \& \ \beta \in \zeta - FN\}$, where $U(\alpha, \beta)$ are as above. Put, for $\alpha, \beta < \zeta$, $W(\alpha, \beta) = D^{-1''}U(\alpha, \beta)$. We can see that $W(m+1, \beta) \circ W(m+1, \beta) \subseteq W(m, \beta)$. Let $\langle a, b \rangle \in N^2 - (\mathcal{E})$, $a \in FN$, $b \in N - FN$. Then there exist $m \in FN$ and $\beta \in \zeta - FN$ such that $\langle a, b \rangle \notin W(m, \beta)$. Thus $W(m+1, \beta)''\{a\} \cap W(m+1, \beta)''\{b\} = \emptyset$. Put $A = W(m+1, \beta)''\{a\}$ and $B = W(m+1, \beta)''\{b\}$. We have $FN = (\mathcal{E})''\{a\} \subseteq A$, $N - FN = (\mathcal{E})''\{b\} \subseteq B$ and, moreover, $A \cap B = \emptyset$, $A \in Sd_V^*$, $B \in Sd_V^*$. The class A is a fully revealed class and $A \cap (N - FN) = \emptyset$, which is impossible. \square

EVALUATIONS OF IDEALS

Evaluations of ideals of the type $\sigma\pi$ and $\pi\sigma$.

Throughout this section, let A be a non-empty set and let $\zeta \in N - FN$ be fixed.

We say that J is an *ideal on A* if we have: $J \subseteq P(A)$, $A \notin J$, $u \in J \ \& \ v \in J \Rightarrow u \cup v \in J$ and $v \subseteq u \in J \Rightarrow v \in J$.

Let $H : P(A) \rightarrow {}^\zeta Q^+$ be a mapping. We say that H is *monotone on $P(A)$* if $u \subseteq v \Rightarrow H(u) \leq H(v)$ holds for each $u, v \subseteq A$. The mapping H is said to be *subadditive on $P(A)$* if we have, for each $u, v \subseteq A$, $H(u \cup v) \leq H(u) + H(v)$. We say that H is an *evaluation on $P(A)$ in ${}^\zeta Q^+$* if it is a monotone and subadditive mapping on $P(A)$ and $H^{-1''}\{0_\zeta\} = \{\emptyset\}$. (Recall that $0_\zeta = \zeta \times \{0\}$.)

The presented definitions can be naturally applied to a mapping $H : P(A) \rightarrow K$, where K is N or Q^+ . (We identify K with the subclass $\{\zeta \times \{x\}; x \in K\}$ of ${}^\zeta Q^+$.)

Theorem. *Let J be an ideal on a non-empty set A .*

- (1) *Let J be a σ -class. Then there exists a set-evaluation h on $P(A)$ in N such that $h^{-1''}FN = J$.*
- (2) *Let J be a π -class. Then there exists a set-evaluation h on $P(A)$ in Q^+ such that $h^{-1''}[0]^+ = J$.*
- (3) *Let ζ be fixed. Let J be a $\sigma\pi$ -class. Then there exists a set-evaluation h on $P(A)$ in ${}^\zeta Q^+$ such that $h^{-1''}U(\sigma\pi) = J$.*

PROOF: Let J be an ideal on A . We see that $\langle P(A), \cup, Id \rangle (J, \{\emptyset\})$ is a closed triad and $\langle P(A), \cup, Id \rangle$ is commutative. Moreover, $\langle P(A), \cup, Id, \cap \rangle$ is a u -expansion of the e -structure in question. We can find, for τ equal to π, σ or $\sigma\pi$, a monotonic set-valuation h of the triad $\langle P(A), \cup, Id \rangle (J, \{\emptyset\})$ in the canonical τ -triad \mathcal{T}_τ .

The existence of the mapping h is guaranteed by the following proposition on monotonic set-valuations of π -, σ - and $\sigma\pi$ -triads, which follows easily from the theorems on monotonic valuations in [M3].

Let $\mathbb{A}(U, B)$ be a triad such that \mathbb{A}, B are sets and let \mathbb{A} be commutative and have a u -expansion which is a set. If U is a τ -class, where τ is π, σ or $\sigma\pi$, then $\mathbb{A}(U, B)$ has a monotonic set-valuation in \mathcal{T}_τ .

Now, let h be a monotonic valuation of the triad $\langle P(A), \cup, Id \rangle (J, \{\emptyset\})$ in \mathcal{T}_τ , where τ is σ , π or $\sigma\pi$. We see that h is an evaluation on $P(A)$ in K_τ , where $K_\sigma = N$, $K_\pi = Q^+$ and $K_{\sigma\pi} = {}^\zeta Q^+$. The required equalities from the items (1), (2) and (3) clearly hold. \square

Now, we shall formulate a theorem on evaluations of $\pi\sigma$ -ideals.

Theorem. *Let J be an ideal on a nonempty set A and let J be a $\pi\sigma$ -class. Then there exists a set-evaluation h on $P(A)$ in ${}^\zeta Q^+$ such that $h^{-1''}U(\pi\sigma) = J$ iff there exists a non-increasing sequence $\{J_n\}_{n \in FN}$ of σ -classes such that we have $\bigcap_m J_m = J$, $\{u \cup v; u, v \in J_{m+1}\} \subseteq J_m$ and $v \subseteq u \in J_{m+1} \Rightarrow v \in J_m$ hold for each m .*

PROOF: We deduce quite analogously as in the previous proof, by using the theorem on monotonic valuations of $\pi\sigma$ -triads, that there exists a monotone valuation $H \in Sd_V^*$ of $\langle P(A), \cup, Id \rangle (J, \{\emptyset\})$ in $\mathcal{T}_{\pi\sigma}$. The mapping H is necessarily a set and $h = H$ has the required properties.

Let us prove the implication from the left to the right. Let U_m be, for each $m \in FN$, as above. Assume that $h : P(A) \rightarrow {}^\zeta Q^+$ is such that $h^{-1''}\{0_\zeta\} = \{\emptyset\}$ and $h^{-1''}U(\pi\sigma) = J$. Put $J_m = h^{-1''}U_m$. Then the classes J_m have the required properties. \square

Additive evaluations of π -ideals.

In this section, let A be a set which has at least two elements.

A mapping $h : P(A) \rightarrow Q^+$ is said to be *additive on $P(A)$* if we have for each $u, v \subseteq A$:

$$u \cap v = \emptyset \Rightarrow h(u \cup v) = h(u) + h(v).$$

Then $h(\emptyset) = 0$ and, for each $u \subseteq A$, the equality $h(u) = \sum_{x \in u} h(\{x\})$ holds. Thus, h is monotone.

We shall describe a class of π -ideals on A of such a kind that, having such an ideal J , there is no additive set-mapping $h : P(A) \rightarrow Q^+$ such that $J = h^{-1''}[0]^+$. At first, we denote by $|u|$ the set-cardinality of the set u . It means that there exists a one-one set-mapping between u and a natural number α .

A partition p of A is said to be *relatively bounded*, whenever $(\forall t \in p)(|t|/|p| \in BQ)$ holds.

By a *set-selector* on a partition p on A we mean a set $u \subseteq A$ such that $(\forall t \in p)(|t \cap u| = 1)$.

Proposition. *Let A be a non-empty set. Put $J = w^{-1''}[0]^+$ and let w be an additive mapping on $P(A)$. Let p be a relatively bounded partition on A such that $J \cap p = \emptyset$. Then there exists a set-selector on p which does not belong to J .*

PROOF: Put, for each $x \in A$, $\tilde{w}(x) = w(\{x\})$. There exists a set $u = \{a_t; t \in p\}$ such that $a_t \in t$ holds for each $t \in p$ and $\tilde{w}(a_t) = \max(\tilde{w}''t)$. We have, for each $t \in p$: $0 \neq \tilde{w}(t) \leq |t| \cdot \tilde{w}(a_t)$. Thus, there exists a number $k \in FN$ such that $1/k \leq |t| \cdot \tilde{w}(a_t)$ holds for each $t \in p$.

Put $\theta = \max\{|t|; t \in p\}$. We deduce from the assumption that the partition p is relatively bounded that there exists a number $m \in FN$ such that $\theta/|p| \leq m$. Thus we have the following: $w(u) = \sum_{t \in p} \tilde{w}(a_t) \geq 1/k \cdot \sum_{t \in p} 1/|t| \geq 1/k \cdot |p|/\theta \geq 1/(k \cdot m)$. We can conclude that $u \notin J$. \square

We say that a partition p of a set A is *relatively non-zero* if we have $(\forall t \in p)(|t|/|p| \neq 0)$.

Proposition. *Let A be an infinite set and let p be an infinite relatively bounded and relatively non-zero set-partition on A . Put*

$$J = \{u \subseteq A; (\forall t \in p)(|t \cap u|/|p| \doteq 0)\}.$$

Then we have:

- (1) *The class J is an ideal on A of the type π and $[A]^1 \subseteq J$.*
- (2) *Every set-selector on p belongs to J .*
- (3) *There is no additive set-mapping h on $P(A)$ such that $J = h^{-1}''[0]^+$.*

PROOF: The items (1) and (2) are easy. The item (3) follows from (2) and from the previous proposition. \square

It is clear that, on each infinite set A , there exists an infinite relatively bounded and relatively non-zero set-partition p .

Now, let us briefly pay attention to some classes of evaluations. Let A be a non-empty set. We put

$$Ev(A) = \{h; h \text{ is a set-evaluation on } P(A) \text{ in } Q^+\}$$

and let \sim be a relation on $Ev(A)$ defined as follows:

$$f \sim g \Leftrightarrow f^{-1}''[0]^+ = g^{-1}''[0]^+.$$

Proposition. *Let A be an infinite set. Then \sim is a non-compact equivalence on $Ev(A)$ of the type $\pi\sigma$. The system $\mathcal{W} = \{W_{\kappa, m}; \kappa \in N - FN \ \& \ m \in FN\}$, where for each α, β ,*

$$\begin{aligned} W_{\alpha, \beta} &= \{\langle f, g \rangle \in (Ev(A))^2; \\ f(u) < 2^{-\alpha} &\Rightarrow g(u) < 2^{-\beta} \ \& \ g(u) < 2^{-\alpha} \Rightarrow f(u) < 2^{-\beta}\}, \end{aligned}$$

is a uniformity basis on $Ev(A)$ over \sim (that is, $\sim = \bigcap \mathcal{W}$ and \mathcal{W} has the usual properties).

PROOF: It is clear that \sim is an equivalence on $Ev(A)$. Let us prove that \sim is not compact. Let $\delta \notin FN$ be such that there exists a set-partition $\{A_\alpha; \alpha < \delta\}$ of A and $\delta \leq |A_\alpha| \leq 2\delta$ holds. Let $w_\alpha : A \rightarrow Q^+$ be a function such that we have, for each $x \in A$, $w_\alpha(x) = \delta^{-1}$ and w_α is equal to zero on $A - A_\alpha$. Let $h_\alpha : P(A) \rightarrow Q^+$ be a function defined by $f_\alpha(u) = \sum_{x \in u} w_\alpha(x)$ whenever

$\emptyset \neq u \subseteq A$ and let $f_\alpha(\emptyset) = 0$. Then $\{f_\alpha; \alpha < \delta\}$ is an infinite \sim -net which guarantees that \sim is not compact. We prove that \sim is a $\pi\sigma$ -class. The explicit definition of \sim has the form $(\forall u \subseteq A)(\forall k)(\exists m)\varphi$ where φ is a set-formula of the language FL_V . We deduce from this that the formula in question is equivalent to a formula $(\forall k)(\exists n)\psi$ where ψ is a set-formula of the language FL_V .

Let us prove that \mathcal{W} has the required properties. We can see that

$$f \sim g \Leftrightarrow (\forall \kappa \in N - FN)(\forall m)(\forall u \subseteq A)(f(u) < 2^{-\kappa} \Rightarrow \\ g(u) < 2^{-m} \ \& \ g(u) < 2^{-\kappa} \Rightarrow f(u) < 2^{-m})$$

and, consequently, $\sim = \bigcap \mathcal{W}$ holds. We see also that the system $\mathcal{W} \subseteq Sd_V$ is a system of reflexive and symmetric relations on $Ev(A)$ such that $(\forall W_1, W_2 \in \mathcal{W})(\exists W \in \mathcal{W})(W \subseteq W_1 \cap W_2)$, $(\forall W \in \mathcal{W})(\exists W_0 \in \mathcal{W})(W_0 \circ W_0 \subseteq W)$. \square

Let us define, finally,

$$Ev_0(A) = \{h \in Ev(A); h''[A]^1 \subseteq [0]^+\} \\ Ad_0(A) = \{h \in Ev_0(A); h \text{ is an additive set-mapping on } P(A)\}.$$

Proposition. *Let A be an infinite set. Then*

$$Ad_0(A) \subsetneq \sim'' Ad_0(A) \subsetneq Ev_0(A)$$

and the classes $Ad_0(A)$ and $Ev_0(A)$ are π -classes.

PROOF: Let us prove the first inclusion. Assume that $h \in Ad_0(A)$ and let $r \doteq 0, r > 0$. Put, for each $u \subseteq A, u \neq \emptyset, h_r(u) = h(u) + r$ and $h_r(\emptyset) = 0$. Then $h_r \sim h$ and $h_r \notin Ad_0(A)$.

Let us prove the second inclusion. Let J be an ideal from the last but one proposition and let h be an evaluation on $P(A)$ in Q^+ such that $h^{-1''}[0]^+ = J$. Then $h \in Ev_0(A)$. Let $w \in Ad_0(A)$ and suppose that $w \sim h$. We have $J = w^{-1''}[0]^+$, which is a contradiction. \square

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