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Further remarks on the Nemitskii operator in Hölder spaces

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Abstract. The paper is concerned with the Nemitskii operator in Hölder spaces. Namely conditions are given to ensure acting, continuity, Lipschitz and differentiability properties.

Keywords: Nemitskii operator, Hölder spaces

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0. Introduction.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual norm denoted by $|\cdot|$. In what follows Ω will denote an open bounded subset of $\mathbb{R}^n$ unless otherwise stated and $\overline{\Omega}$ its closure.

For $\alpha \in (0,1]$, $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ is the space of all real functions u which are α-Hölder continuous in $\overline{\Omega}$, i.e. are such that: $h_\alpha(u) := \sup\{|u(x)-u(y)|/|x-y|^\alpha,\ x,y \in \overline{\Omega},\ x \neq y\} < \infty$. $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ is a Banach space with the norm: $\|u\|_\alpha = \|u\|_\infty + h_\alpha(u)$ where $\|u\|_\infty = \sup\{|u(x)|;\ x \in \overline{\Omega}\}$.

This paper is concerned with the study in $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ of some properties of the so called Nemitskii operator, i.e. the operator $F(u)(x) = f(x,u(x))$, $x \in \overline{\Omega}$ where $f = f(x,u)$ is a real valued function defined on $\overline{\Omega} \times \mathbb{R}$.

This argument has been deeply studied mainly in eastern Europe (see [1] and [2] for a complete bibliography). Among the others we like to mention P. Drábek [4] who has found necessary and sufficient conditions for $f = f(u)$ to induce a continuous Nemitskii operator mapping $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ into itself.

Theorem 1.1 is simply a translation in words of [2, Theorem 7.3]; Theorem 3.1 extends the analogue in [2] which deals only with the case $f = f(u)$, as Theorems 2.1 and 4.1 do in relation with the ones in [5]. Finally Theorems 1.1, 2.1 and 4.1 extend our previous paper [7] since the actual assumptions are sensibly weaker.

We have now to compare our paper with the very recent one by M. Goebel [6]. First, we prove most of our results for any open bounded $\Omega \subset \mathbb{R}^n$ rather than for $\Omega = (a,b)$ as in [6]. (The extension to the case $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ is straightforward, see our final remark.)

Also, in [6] only *sufficient* conditions on f are given so that F has the various desired properties in $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$, while we prove also some *necessary* conditions (Theorems 2.2 and 3.1) which in particular — in case $\Omega = (a,b)$ — yield a characterization of the local Lipschitz property of F (Corollary 3.2).

Let us next discuss the conditions given here with those in [6]. To see this in some detail, we state here two basic assumptions — for a given function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ — to be used through the paper:

$$\text{(H)} \qquad \begin{aligned} &g = g(x,u) \text{ is continuous in } \overline{\Omega} \times \mathbb{R} \\ &\text{and } \alpha\text{-Hölder continuous in } x, \\ &\text{uniformly with respect to } u \text{ in compact intervals of } \mathbb{R}. \end{aligned}$$

$$\text{(K)} \qquad \begin{aligned} &g = g(x,u) \text{ is } \alpha\text{-Hölder continuous in } x, \\ &\text{uniformly with respect to } u \text{ in compact intervals of } \mathbb{R}, \\ &\text{and locally Lipschitz continuous in } u, \\ &\text{uniformly with respect to } x \in \overline{\Omega}. \end{aligned}$$

It is quite clear (see also the proof of Theorem 1.1) that (H) is a weaker assumption than (K).

We note that (H) is equivalent to the assumption that g be continuous and satisfy (A) of [6], while (K) is the same as (B) of [6].

As remarked in [6], if f satisfies (A) and is differentiable with respect to u with f'_u continuous, then f satisfies $(B) = (K)$. On the basis of this remark, it is easy to check that the various properties of F (acting, continuity, etc.) are established in our paper under conditions on f that are weaker than those in [6]. In particular, we note that requiring existence and continuity of f'_u in order to prove the acting property of F is an unnecessarily strong assumption (compare Theorem 1.1 with [6, Theorem 1]). Theorem 2.1 and especially Theorem 2.2 below show that existence of f'_u should be required at the level of continuity of F.

We should finally mention that our proofs are sensibly different from those in [6], and in particular the proof of Theorem 4.1 (differentiability) seems to us simpler and more transparent.

1. Acting property.

Theorem 1.1. *In order that the Nemitskii operator F generated by f map $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself and be bounded, it is sufficient that f satisfies the assumption* (K). *If $\Omega = (a,b)$, this condition is also necessary.*

Proof: By Theorem 7.3 in [2] it is sufficient to prove that (K) is equivalent to:

$$\text{(1.1)} \qquad \begin{aligned} &\forall\, R > 0\ \exists\, M > 0 : \\ &|f(x,u) - f(y,v)| \le M\{|x-y|^\alpha + \frac{|u-v|}{R}\} \qquad \forall\, |u|, |v| \le R,\ \forall\, x, y \in \overline{\Omega}. \end{aligned}$$

Indeed if (1.1) holds, then f is α-Hölder in x since if $R > 0$, $|u| \le R$, and $x, y \in \overline{\Omega}$, then $|f(x,u) - f(y,u)| \le M|x-y|^\alpha$. Moreover (1.1) implies that f is locally Lipschitz in u since, given $R > 0$, $\exists\, M > 0 : |f(x,u) - f(x,v)| \le M\frac{|u-v|}{R}$, $\forall\, |u|, |v| \le R$,

$\forall\, x \in \overline{\Omega}$. Assume now that f satisfies (K); Let $R > 0$, and let L be the Lipschitz constant of f in $[-R, R]$ and k its Hölder constant in $\overline{\Omega}$. We get:

$$\begin{aligned} |f(x,u) - f(y,v)| &\le |f(x,u) - f(x,v)| + |f(x,v) - f(y,v)| \\ &\le L|u-v| + k|x-y|^{\alpha} \quad (|u|, |v| \le R,\ x, y \in \overline{\Omega}) \end{aligned}$$

and this yields (1.1) with $M = \max(LR, k)$. □

2. Continuity.

Theorem 2.1. *Let f satisfy the assumption* (K) *(so that F acts in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$). If moreover f is differentiable with respect to u and f'_u satisfies the assumption* (H)*, then F is continuous.*

Proof: Let $u, v \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$. To estimate $h_\alpha(F(u+v) - F(u))$, we write (for $x, y \in \overline{\Omega}$)

$$\begin{aligned} w(x,y) \ &\equiv f(x, u(x)+v(x)) - f(x,u(x)) - f(y, u(y)+v(y)) + f(y,u(y)) \\ &= f(x,u(x)+v(x)) - f(x,u(y)+v(y)) + f(x,u(y)+v(y)) - f(x,u(x)) \\ &\quad - f(y,u(y)+v(y)) + f(y,u(x)) - f(y,u(x)) + f(y,u(y)) \\ = (u(x) &+ v(x) - u(y) - v(y)) \int_0^1 f'_u(x, u(y)+v(y)+ \\ &\qquad \tau(u(x)+v(x)-u(y)-v(y)))\, d\tau \\ &\quad - (u(x) - u(y) - v(y)) \int_0^1 f'_u(x, u(y)+v(y)+\tau(u(x)-u(y)-v(y)))\, d\tau \\ &\quad + (u(x) - u(y) - v(y)) \int_0^1 f'_u(y, u(y)+v(y)+\tau(u(x)-u(y)-v(y)))\, d\tau \\ &\qquad - (u(x) - u(y)) \int_0^1 f'_u(y, u(y)+\tau(u(x)-u(y)))\, d\tau \\ = (u(x) &- u(y)) \int_0^1 \{ f'_u(x, u(y)+v(y)+\tau(u(x)+v(x)-u(y)-v(y))) \\ &\qquad - f'_u(x, u(y)+v(y)+\tau(u(x)-u(y)-v(y))) \\ &\qquad + f'_u(y, u(y)+v(y)+\tau(u(x)-u(y)-v(y))) \\ &\qquad - f'_u(y, u(y)+\tau(u(x)-u(y)))\}\, d\tau \\ + (v(x) &- v(y)) \int_0^1 f'_u(x, u(y)+v(y)+\tau(u(x)+v(x)-u(y)-v(y)))\, d\tau \\ &\quad + v(y) \int_0^1 \{ f'_u(x, u(y)+v(y)+\tau(u(x)-u(y)-v(y))) \\ &\qquad - f'_u(y, u(y)+v(y)+\tau(u(x)-u(y)-v(y)))\}\, d\tau. \end{aligned}$$

Let now $\varepsilon > 0$ be given; set $M = \|u\|_\alpha$, $R = M+1$. Since f'_u is uniformly continuous in $\overline{\Omega} \times [-2R, 2R]$, then:

(a) there exists a constant N such that $N = \max\{|f'_u(x,u)| : x \in \overline{\Omega},\ u \in [-2R, 2R]\}$,

(b) $\forall\, \varepsilon' > 0\ \exists\, \delta'$ such that: $|f(x,u) - f(x,v)| < \varepsilon'$ whenever $x \in \overline{\Omega}$, $u, v \in [-2R, 2R]$ and $|u - v| < \delta'$.

Moreover f'_u is α-Hölder in x, namely there exists a non negative constant L such that: $|f'_u(x,u) - f'_u(y,u)| \le L|x-y|^\alpha$ for any $x, y \in \overline{\Omega}$, and $u \in [-2R, 2R]$. Then, if $\varepsilon' = \varepsilon/2M$ and $\delta = \min\{\delta', 1, \frac{\varepsilon}{N}, \frac{\varepsilon}{L}\}$ one gets, if $\|v\|_\alpha < \delta$:

$$|w(x,y)| \le 4\varepsilon |x-y|^\alpha \qquad (x, y \in \overline{\Omega})$$

whence $h_\alpha(F(u+v) - F(u)) \le 4\varepsilon$.

To conclude, note that $f(x, u(x) + v(x)) - f(x, u(x)) = \int_0^1 f'_u(x, u(x) + \tau v(x)) v(x)\, d\tau$ and hence $\|F(u+v) - F(u)\|_\infty \le N\|v\|_\alpha < \varepsilon$. $\square$

Theorem 2.2. *Let f satisfy the assumption* (K). *If F is continuous, then f is differentiable with respect to u.*

PROOF: Since f is α-Hölder continuous in x and locally lipschitzian in u by Theorem 1.1, then f is absolutely continuous in u and hence almost everywhere differentiable with respect to u in $\mathbb{R}$ in the following sense: for every $x \in \Omega$ the set $N_x = \{u : f'_u(x,u) \text{ does not exist}\}$ has zero Lebesgue measure in $\mathbb{R}$. It follows that its complement N_x^c is dense in $\mathbb{R}$. We want to prove that $N_x^c = \mathbb{R}$ for every x.

Let us proceed by contradiction. Assume $N_{x_0} \ne \emptyset$ for some $x_0 \in \Omega$ and let $u_0 \in N_{x_0}$; thus setting

$$l_1 = \liminf_{h \to 0} \frac{f(x_0, u_0 + h) - f(x_0, u_0)}{h}$$
$$l_2 = \limsup_{h \to 0} \frac{f(x_0, u_0 + h) - f(x_0, u_0)}{h}$$

we should have $l_1 < l_2$. Let h_n and χ_n be real sequences converging to zero such that:

$$l_1 = \lim_{n \to \infty} \frac{f(x_0, u_0 + \chi_n) - f(x_0, u_0)}{\chi_n}, \qquad l_2 = \lim_{n \to \infty} \frac{f(x_0, u_0 + h_n) - f(x_0, u_0)}{h_n}$$

and let y_n and x_n be sequences in Ω such that $h_n = |y_n - x_0|^\alpha$ and $\chi_n = |x_n - x_0|^\alpha$ (take e.g. $y_n = x_0 + h_n^{\alpha^{-1}} v$, $|v| = 1$); then x_n and y_n both converge to x_0. By the density of $N_{x_0}^c$ there exists a real sequence θ_m converging to zero such that $f'_u(x_0, u_0 + \theta_m)$ exists for any m and

$$f'_u(x_0, u_0 + \theta_m) = \lim_{\xi \to 0} \frac{f(x_0, u_0 + \xi + \theta_m) - f(x_0, u_0 + \theta_m)}{\xi} \qquad (m \in \mathbb{N}).$$

Hence also:

$$f'_u(x_0, u_0 + \theta_m) = \lim_{n\to\infty} \frac{f(x_0, u_0 + h_n + \theta_m) - f(x_0, u_0 + \theta_m)}{h_n}$$
$$= \lim_{n\to\infty} \frac{f(x_0, u_0 + \chi_n + \theta_m) - f(x_0, u_0 + \theta_m)}{\chi_n}.$$

We will prove that $l_2 = \lim_{m\to\infty} f'_u(x_0, u_0 + \theta_m)$.

Let y_n be defined as above and consider, for any n, m, the following expression:

$$\begin{aligned}
&|h_n^{-1}[f(x_0, u_0 + h_n + \theta_m) \\
&\qquad - f(x_0, u_0 + h_n) - f(x_0, u_0 + \theta_m) + f(x_0, u_0)]| \\
&= |h_n^{-1}[f(y_n, u_0 + h_n + \theta_m) - f(x_0, u_0 + \theta_m) \\
&\qquad - f(y_n, u_0 + h_n) + f(x_0, u_0) \\
&\qquad - f(y_n, u_0 + h_n + \theta_m) + f(y_n, u_0 + h_n) \\
&\qquad - f(x_0, u_0 + h_n) + f(x_0, u_0 + h_n + \theta_m)]|.
\end{aligned} \tag{2.1}$$

If we define $u(x) = |x - x_0|^\alpha + u_0$, so that $u(y_n) = h_n + u_0$ and $u(x_0) = u_0$, the expression in (2.1) is less than or equal to

$$\|F(u + \theta_m) - F(u)\|_\alpha + \|F(u_0 + h_n) - F(u_0 + h_n + \theta_m)\|_\alpha .$$

Letting $n \to \infty$ and using the continuity of F in $u_0 + \theta_m$ we get for any m:

$$|l_2 - f'_u(x_0, u_0 + \theta_m)| \le \|F(u + \theta_m) - F(u)\|_\alpha + \|F(u_0) - F(u_0 + \theta_m)\|_\alpha .$$

Letting now $m \to \infty$ we get $l_2 = \lim_{m\to\infty} f'_u(x_0, u_0 + \theta_m)$. The same argument shows that $l_1 = \lim_{m\to\infty} f'_u(x_0, u_0 + \theta_m)$, so that $l_1 = l_2$: contradiction. □

Corollary 2.3. *Let $\Omega = (a, b)$ and assume that the Nemitskii operator F induced by f acts in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ is bounded and continuous. Then f is differentiable with respect to u.*

3. Lipschitz property.

Theorem 3.1. *Let f satisfy the assumption* (K). *In order that F be locally lipschitzian, it is sufficient that f be differentiable with respect to u and f'_u satisfy the assumption* (K). *If $\Omega = (a, b)$, this condition is also necessary.*

Proof: The "if" part can be proved in the same way as [7, Theorem 1.2].

To prove the "only if" part, note that by assumption

$$\begin{aligned}
&\forall R > 0\ \exists k(R) \ge 0 : \\
&\|F(u) - F(v)\|_\alpha \le k(R)\|u - v\|_\alpha \qquad \forall\, \|u\|_\alpha, \|v\|_\alpha \le R.
\end{aligned} \tag{3.1}$$

Let $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ with $\|u\|_\alpha = M$, $R = M+1$ and $\lambda \in (0,1)$, so that $\|u+\lambda\|_\alpha < R$. Let us consider, for any $x \in [a, b]$, the function: $g(x, \lambda) = \lambda^{-1}[f(x, u(x) + \lambda) - f(x, u(x))]$. As a consequence of (3.1) the function g has the following properties:

(i) $|g(x, \lambda) - g(y, \lambda)| \le k(R)|x - y|^\alpha$ $(x, y \in [a, b],\ \lambda \in (0, 1))$

(ii) $|g(x, \lambda)| \le k(R)$ $(x, y \in [a, b],\ \lambda \in (0, 1))$.

Then the set $\{g_\lambda\} := \{g(\cdot, \lambda),\ \lambda \in (0,1)\}$ is a subset of real continuous functions defined on $[a,b]$ which satisfies the assumptions of Ascoli-Arzelà's theorem; hence there exists a sequence λ_n such that:

$\lambda_n \to 0$
$g_{\lambda_n} \to g$ for some g continuous. Observe that, since F is continuous, from Theorem 2.2 we get the differentiability of f with respect to u. Hence for any $x \in [a,b]$ we have $g(x) = f'_u(x, u(x))$.

The rest of the proof consists in showing that the Nemitskii operator G induced by f'_u maps $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself and is bounded, so that we can apply Theorem 1.1 to prove the claim. For $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ with $\|u\|_\alpha \le R$ we have $|g_{\lambda_n}(x)| \le k(R)$, and thus passing to the limit as $n \to \infty$, we get $|g(x)| \le k(R)$, which implies $\|G(u)\|_\infty \le k(R)$. Likewise, letting $n \to \infty$ in the inequality $|x-y|^{-\alpha}|g_{\lambda_n}(x) - g_{\lambda_n}(y)| \le k(R)$, we get $|x-y|^{-\alpha}|g(x) - g(y)| \le k(R)$, whence $h_\alpha(G(u)) \le k(R)$. We conclude that $\|G(u)\|_\alpha \le 2k(R)$ and finish the proof. □

Corollary 3.2. *Let $\Omega = (a,b)$. Then F maps $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself and is locally lipschitzian if and only if both f and f'_u satisfy the assumption* (K).

4. Differentiability.

Theorem 4.1. *Let f be twice differentiable with respect to u and assume that both f and f'_u satisfy the assumption* (K), *while f''_u satisfies the assumption* (H). *Then F is continuously differentiable.*

Proof: From the assumptions and Theorem 2.1 the Nemitskii operator G induced by f'_u is continuous. Let us compute:

$$\begin{aligned} w(x,u,v) &= f(x, u(x) + v(x)) - f(x, u(x)) - f'_u(x, u(x))v(x) \\ &= \int_0^1 [f'_u(x, u(x) + \xi v(x)) - f'_u(x, u(x))v(x)]\, d\xi \\ &= \int_0^1 [G(u + \xi v) - G(u)](x) v(x)\, d\xi \end{aligned}$$

whence

$$\|F(u+v) - F(u) - G(u)v\|_\alpha \le \int_0^1 \|G(u+\xi v) - G(u)v\|_\alpha\, d\xi.$$

Moreover,

$$\begin{aligned} &|x-y|^{-\alpha}|w(x,u,v) - w(y,u,v)| \le \\ &\quad \le \int_0^1 |x-y|^{-\alpha}|(G(u+\xi v) - G(u))(x)v(x) - (G(u+\xi v) - G(u))(y)v(y)|\, d\xi \end{aligned}$$

whence

$$h_\alpha[F(u+v) - F(u) - G(u)v] \le \int_0^1 h_\alpha[G(u+\xi v) - G(u)v]\, d\xi.$$

We conclude that

$$\begin{aligned}\|F(u+v)-F(u)-G(u)v\|_\alpha &\le \int_0^1 \|(G(u+\xi v)-G(u))v\|_\alpha\, d\xi \\ &\le m\|v\|_\alpha \int_0^1 \|G(u+\xi v)-G(u)\|_\alpha\, d\xi.\end{aligned}$$

Now let $\varepsilon > 0$. By the continuity of G there exists $\delta > 0$ such that $\|G(u+\xi v) - G(u)\|_\alpha < \varepsilon$ whenever $\|v\|_\alpha < \delta$. Therefore,

$$\|F(u+v)-F(u)-G(u)v\|_\alpha \le \varepsilon\|v\|_\alpha$$

whenever $\|v\|_\alpha < \delta$, showing that F is differentiable at u with derivative $F'(u)[v] = G(u)v$. Finally, to show that the derivative is continuous, let $\mathcal{L}$ denote the Banach space of all linear bounded mappings of $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself, equipped with its usual norm $\|T\|_{\mathcal{L}} = \sup\{\|T[v]\|_\alpha : \|v\|_\alpha = 1\}$. Since

$$\|F'(u+w)[v]-F'(u)[v]\|_\alpha = \|G(u+w)v-G(u)v\|_\alpha \le m\|G(u+w)-G(u)\|_\alpha\|v\|_\alpha$$

we have

$$\|F'(u+w)-F'(u)\|_{\mathcal{L}} \le m\|G(u+w)-G(u)\|_\alpha$$

and the conclusion follows again from the continuity of G. □

Remark. If Ω denotes, as before, an open bounded subset of $\mathbb{R}^n$, the conditions stated in Sections 1, 2, 3, 4 are sufficient also in the case $f = f(x,u) = f(x,u_1,\dots,u_m)$ is a real valued function defined in $\overline{\Omega} \times \mathbb{R}^m$, $(m \ge 1)$. In this case f'_u denotes the gradient of f with respect to the variable $u \in \mathbb{R}^m$, while f''_u will denote the $m \times m$ Hessian matrix $(f''_{u_i u_j})$ $(i,j = 1,\dots,m)$ of f with respect to the same variable. As a norm in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ we take $\|u\|_{\alpha,m} = \sum_{i=1}^m \|u\|_\alpha$, $(u = (u_1,\dots,u_m))$.

References

[1] Appell J., *The superposition operator in function spaces – A survey*, Expo. Math. **6** (1975), 209–270.

[2] Appell J., Zabreiko P., *Nonlinear superposition operators*, Cambridge Tracts in Mathematics 95, Cambridge University Press, 1990.

[3] Bondarenko V.A., Zabreiko P., *The superposition operator in Hölder function spaces*, Soviet Math. Dokl. **16** (1975), 739–743.

[4] Drábek P., *Continuity of Němickij's operator in Hölder spaces*, Comment. Math. Univ. Carolinae **16** (1975), 37–57.

[5] Goebel M., *On Fréchet differentiability of Nemitskij operators acting in Hölder spaces*, Glasgow Math. J. **33** (1991).

[6] ———, *Continuity and Fréchet differentiability of Nemitskij operators in Hölder spaces*, Monatshefte für Mathematik **113** (1992), 107–119.

[7] Nugari R., *Continuity and differentiability properties of the Nemitskij operator in Hölder spaces*, Glasgow Math. J. **30** (1988), 59–65.

[8] Valent T., *Boundary value problems in finite elasticity*, Springer Tracts in Natural Philosophy 31, Springer-Verlag, New York, 1988.

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