Semirings whose additive endomorphisms are multiplicative

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Abstract. A ring or an idempotent semiring is associative provided that additive endomorphisms are multiplicative.

Keywords: semiring, additive endomorphism
Classification: 16Y60

In [10], R.P. Sullivan posed the problem of classifying AE-rings (i.e. rings whose additive endomorphisms are ring endomorphisms) and, more or less recently, several papers were published on this thème (see [1], [2], [3], [4], [6] and [9]). The original problem and the cited papers are concerned with associative rings only. However, it is the purpose of this short note to show that, in fact, every AE-ring is associative. Besides, some classes of AE-semirings are characterized and, again, all these semirings turn out to be associative. It seems to be an open problem whether there exist non-associative AE-semirings (or AE-nearrings) at all.

1. Introduction

1.1. By a semiring we mean an algebra $S = S(+, \cdot)$ with two binary operations such that $S(+)$ is a commutative semigroup and the multiplication is both left and right distributive with respect to the addition.

1.2. Let S be a semiring. We put $\mathrm{Ida}(S) = \{x \in S; x + x = x\}$, $\mathrm{Idm}(S) = \{x \in S; xx = x\}$ and $\mathrm{Id}(S) = \mathrm{Ida}(S) \cap \mathrm{Idm}(S)$. Further, $1x = x$ and $nx = (n-1)x + x$ for all $x \in S$ and $n \geq 2$.

1.3. A semiring S is said to be

- a-idempotent if $x + x = x$ for every $x \in S$;
- m-idempotent if $xx = x$ for every $x \in S$;
- a-unipotent if $x + x = y + y$ for all $x, y \in S$;
- associative if $x \cdot yz = xy \cdot z$ for all $x, y, z \in S$;
- commutative if $xy = yx$ for all $x, y \in S$;
- a D-semiring if $x \cdot yz = xy \cdot xz$ and $zy \cdot x = zx \cdot yx$ for all $x, y, z \in S$;
- a ring if $S(+)$ is a group;
- an AE-semiring if every endomorphism of $S(+)$ is also an endomorphism of $S(\cdot)$.

1.4 Example. Let $S(+)$ be a commutative semigroup containing a unique idempotent element 0. Define a multiplication on S by $xy = 0$ for all $x, y \in S$. Then $S = S(+, \cdot)$ is an AE-semiring.

1.5 Example. Let $S(+)$ be a commutative semigroup such that $2x+y+z = x+y+z$ for all $x, y, z \in S$ (e.g. $S(+)$ idempotent or $S(+)$ nilpotent of class at most 3). Put $S(\cdot) = S(+)$. Then $S = S(+, \cdot)$ is an AE-semiring.

2. Basic properties of AE-semirings

2.1 Proposition. *Let f be an additive endomorphism of an* AE*-semiring S. Then f is an endomorphism of S, $T = f(S)$ is a subsemiring of S and T is again an* AE*-semiring.*

Proof: Let g be an additive endomorphism of T. Define a transformation h of S by $h(x) = gf(x)$ for every $x \in S$. Then h is an additive endomorphism of S and consequently $gf(x) \cdot gf(y) = h(x)h(y) = h(xy) = gf(xy) = g(f(x)f(y))$ for all $x, y \in S$. This shows that g is an endomorphism of T. $\square$

2.2 Proposition. *Let S be an* AE*-semiring. Then:*

(i) *S is a D-semiring.*
(ii) *For every $a \in S$, $S_{a,l} = aS = \{ax; x \in S\}$ is a subsemiring of S and an* AE*-semiring.*
(iii) *For every $a \in S$, $S_{a,r} = Sa = \{xa; x \in S\}$ is a subsemiring of S and an* AE*-semiring.*
(iv) *$a \cdot bc, ab \cdot c \in \mathrm{Idm}(S)$ for all $a, b, c \in S$.*

Proof: (i), (ii) and (iii): The translations $x \to ax$ and $x \to xa$ are additive endomorphisms of S and 2.1 applies.

(iv): This follows from (i) and [7, Theorem III 1.2 (ii)]. $\square$

2.3 Lemma. *Let S be an* AE*-semiring. Then:*

(i) *$\mathrm{Ida}(S) = \mathrm{Id}(S) \subseteq \mathrm{Idm}(S)$.*
(ii) *$nab = 2ab$ for all $a, b \in S$ and even $n \geq 2$.*
(iii) *$nab = 3ab$ for all $a, b \in S$ and odd $n \geq 3$.*

Proof: (i): If $a \in \mathrm{Ida}(S)$, then the constant transformation $S \to \{a\}$ is an additive endomorphism of S. Consequently, $\{a\}$ is a subsemiring of S and $aa = a$.

(ii): The transformation $x \to 2x$ is an additive endomorphism. Hence $4ab = 2a\,2b = 2ab$.

(iii): This follows immediately from (ii). $\square$

2.4 Lemma. *Let S be an* AE*-semiring. Then:*

(i) *$2ab \in \mathrm{Id}(S)$ for all $a, b \in S$.*
(ii) *$2a \in \mathrm{Id}(S)$ for every $a \in \mathrm{Idm}(S)$.*
(iii) *Both $\mathrm{Idm}(S)$ and $\mathrm{Id}(S)$ are ideals of the multiplicative groupoid $S(\cdot)$.*

Proof: (i): By 2.3 (ii), $2ab = 4ab$, and so $2ab \in \mathrm{Ida}(S)$. But $\mathrm{Ida}(S) = \mathrm{Id}(S)$ by 2.3 (i).

(ii): This follows from (i) for $b = a$.

(iii): If $a \in \mathrm{Idm}(S)$ and $x \in S$, then $ax \cdot ax = aa \cdot x = ax$ and $xa \cdot xa = x \cdot aa = xa$ by 2.2 (i). Further, if $a \in \mathrm{Ida}(S)$, then $ax + ax = (a+a)x = ax$ and $xa + xa = x(a+a) = xa$. □

2.5 Proposition. *Let S be an* AE*-semiring. Put $S_{(2)} = \{2x; x \in S\}$ and $S_{(3)} = \{3x; x \in S\}$. Then:*

(i) *Both $S_{(2)}$ and $S_{(3)}$ are subsemirings of S and* AE*-semirings.*
(ii) $\mathrm{Id}(S_{(2)}) = \mathrm{Id}(S_{(3)}) = \mathrm{Id}(S)$.
(iii) $uv \in \mathrm{Id}(S_{(2)})$ *for all* $u, v \in S_{(2)}$.

Proof: Use 2.1. □

3. Idempotent AE-semirings

3.1. Let $S(+)$ be a semilattice (i.e. a commutative idempotent semigroup). Define a multiplication on S by $xy = x$ ($xy = y$) for all $x, y \in S$. Clearly, $x(y+z) = x = x + x = xy + xz$ ($x(y+z) = y + z = xy + xz$) and $(y+z)x = y + z = yx + zx$ ($(y+z)x = x = x + x = yx + zx$) for all $x, y, z \in S$, and so $S = S(+, \cdot)$ is a semiring. Further, $S(\cdot)$ is a semigroup of left (right) zeros and every transformation of S is an endomorphism of $S(\cdot)$. Consequently, S is an associative a-idempotent and m-idempotent AE-semiring and we shall say that S is of type (AE1) ((AE2)).

3.2. Let $S(+)$ be a semilattice. Define a multiplication on S by $xy = x + y$ for all $x, y \in S$. Then $x(y+z) = x + y + z = x + y + x + z = xy + xz$ and $S = S(+, \cdot)$ is a semiring. Clearly, S is an associative, commutative, a-idempotent and m-idempotent AE-semiring and we shall say that S is of type (AE3).

3.3. Let $S(+)$ be a chain (i.e. $S(+)$ is a semilattice and $x + y \in \{x, y\}$ for all $x, y \in S$) and let $S(\cdot)$ be the dual chain (i.e. $xy = x$ iff $x + y = y$ and $xy = y$ iff $x + y = x$). Let $x, y, z \in S$. If $x + y = x + z = x$, then $x + y + z = x$ and $x(y+z) = y+z = xy+xz$. If $x+y = x$ and $x+z = z$, then $x+y+z = z = y+z$ and $x(y+z) = x = y+x = xy+xz$. If $x+y = y$ and $x+z = x$, then $x+y+z = y = y+z$ and $x(y+z) = x = x+z = xy+xz$. If $x+y = y$ and $x+z = z$, then $x+y+z = y+z$ and $x(y+z) = x = x+x = xy+xz$. We have checked that $S = S(+, \cdot)$ is a semiring. Clearly, S is associative, commutative, a-idempotent and m-idempotent. Now, let f be an endomorphism of $S(+)$ and let $x, y \in S$. If $x+y = x$, then $f(x) = f(x)+f(y)$ and $xy = y$, $f(xy) = f(y) = f(x)f(y)$. Similarly if $x + y = y$ and we see that f is an endomorphism of the semiring S. Thus S is an AE-semiring and we shall say that S is of type (AE4).

3.4 Theorem. *Let S be a non-trivial a-idempotent* AE*-semiring. Then S if of just one of the types* (AE1), (AE2), (AE3), (AE4).

Proof: Firstly, let $u, v \in S$ be such that $u \neq v$ and $u + v = u$. For every ordered pair $p = (a, b) \in S^{(2)}$, $a \neq b$, choose a congruence r_p of $S(+)$ maximal with respect to $p \notin r_p$. Then $S(+)/r_p$ is a two-element semilattice, and hence there is an additive endomorphism f_p of S such that $f_p(S) = T = \{u, v\}$ and $f_p(a) \neq f_p(b)$. The endomorphism f_p is then multiplicative and consequently T is a subsemiring of S. Since $\bigcap r_p = \mathrm{id}_S$, S is a subdirect product of copies of T. On the other hand, S and T are m-idempotent by 2.3 (i) and we have only the following four possibilities for the multiplication on T:

$$\begin{array}{c|cc} & u & v \\ \hline u & u & u \\ v & v & v \end{array} \qquad \begin{array}{c|cc} & u & v \\ \hline u & u & v \\ v & u & v \end{array} \qquad \begin{array}{c|cc} & u & v \\ \hline u & u & u \\ v & u & v \end{array} \qquad \begin{array}{c|cc} & u & v \\ \hline u & u & v \\ v & v & v \end{array}$$

In the first case, $T(\cdot)$ is a semigroup of left zeros, and hence the same is true for $S(\cdot)$, since it is a subdirect product of copies of $T(\cdot)$. The second case is dual and, in the third case, T satisfies the identity $x + y = xy$. Then S satisfies also this identity, which means that $S(+) = S(\cdot)$. Now, finally, let the fourth case take place. Then $S(\cdot)$ is neither a semigroup of left zeros nor that of right zeros and $S(+) \neq S(\cdot)$. If $w, z \in S$ are such that $w + z = w$, then $wz = z$ (see the first part of the proof). Further, if $r, s, t \in S$ and $r + s \neq r$, then $r + s + t \neq r$ (otherwise $r + s = r + s + t + s = r + s + t = r$).

Suppose, for a moment, that $S(+)$ is not a chain. Then there are $a, b \in S$ such that $a \neq c = a + b \neq b$ and we have $a \neq b$, $a + c = c = b + c$, $a = ac = a + d$, $b = bc = b + d$, where $d = ab$. Define a transformation f of S as follows: If $x \in S$ and $a + x \neq a$, then $f(x) = c$; if $a + x = a$, $b + x \neq b$, then $f(x) = a$; if $a + x = a$, $b + x = b$, then $f(x) = d$. We are going to check that f is an additive endomorphism of $S(+)$, i.e. that $f(x + y) = f(x) + f(y)$ for all $x, y \in S$. With respect to the commutativity of $+$, it is sufficient to consider only the following cases:

(1) $a + x \neq a$. Then $a + x + y \neq a$, $f(x) = f(x + y) = c$ and $f(y) \in \{a, c, d\}$. However, $c + c = c + a = c$ and $c + d = c + a + d = c + a + c$. Thus $f(x + y) = f(x) + f(y)$.

(2) $a + x = a = a + y$, $b + x \neq b$. Then $a + x + y = a + y = a$, $b + x + y \neq b$, $f(x) = f(x + y) = a$ and $f(y) \in \{a, d\}$. However, $a + a = a = a + d$, and hence $f(x + y) = f(x) + f(y)$.

(3) $a + x = a = a + y$, $b + x = b$, $b + y \neq b$. Then $a + x + y = a$, $b + x + y \neq b$, $f(x) = f(x + y) = a$, $f(y) = a$, $f(x + y) = a = a + a = f(x) + f(y)$.

(4) $a + x = a = a + y$, $b + x = b = b + y$. Then $a + x + y = a$, $b + x + y = b$ and $f(x) = f(y) = f(x + y) = d$. Thus $f(x + y) = f(x) + f(y)$.

We have checked that f is an additive endomorphism of S. On the other hand, $a + a = a + d = a \neq a + b$, $b + d = b$, $f(a) = a$, $f(b) = c$, $f(a)f(b) = ac = a$, $f(ab) = f(d) = d$ and $b + d = b \neq c = b + a$. Consequently, $d \neq a$, $f(ab) \neq f(a)f(b)$ and f is not multiplicative, a contradiction.

We have proved that $S(+)$ is a chain. If $x, y \in S$, then either $x + y = x$ and $xy = y$ or $x + y = y$ and $xy = y$. This means that $S(\cdot)$ is the dual chain. □

3.5 Corollary. *Every a-idempotent* AE*-semiring is m-idempotent and associative.*

3.6 Remark. In the proof of 3.4 we did not use the fact that the multiplication is distributive with respect to the addition. Hence, the same result remains true for algebras $S = S(+, \cdot)$ with two binary operations such that $S(+)$ is a semilattice and every endomorphism of $S(+)$ is also an endomorphism of $S(\cdot)$.

3.7 Corollary. *Let S be an m-idempotent* AE*-semiring. Then $S_{(2)} = \mathrm{Id}(S)$ and at least one of the following cases takes place:*

(i) *$S_{(2)}(\cdot)$ is a semigroup of left zeros and $2ab = a \cdot 2b = 2a \cdot b = 2a$ for all $a, b \in S$.*
(ii) *$S_{(2)}(\cdot)$ is a semigroup of right zeros and $2ab = a \cdot 2b = 2a \cdot b = 2b$ for all $a, b \in S$.*
(iii) *$S_{(2)}(+) = S_{(2)}(\cdot)$ and $2ab = 2ba = 2a + 2b$ for all $a, b \in S$.*
(iv) *$S_{(2)}(+)$ is a chain, $S_{(2)}(\cdot)$ is the dual chain and $2ab = 2ba \in \{2a, 2b\}$ for all $a, b \in S$.*

4. Some consequences

4.1 Proposition. *Let S be an* AE*-semiring such that $uv \in \mathrm{Id}(S)$ for all $u, v \in S$. Then S is associative.*

Proof: Let $a, b, c \in S$, $d = a \cdot bc$ and $e = ab \cdot c$. Then both $S_{d,l}$, $S_{d,r}$ are contained in $\mathrm{Id}(S)$ (see 2.2) and they are a-idempotent AE-semirings. By 3.2, they are also m-idempotent and associative. Now, $d = dd = d(a \cdot bc) = (da)(db \cdot dc) = (da \cdot db)(dc) = d(ab \cdot c) = de = (a \cdot bc)e = (ae)(be \cdot ce) = (ae \cdot be)(ce) = (ab \cdot c)e = ee = e$. □

4.2 Corollary. *Let S be an* AE*-semiring. Then:*

(i) *$S_{(2)}$ is an associative* AE*-semiring.*
(ii) *$2a \cdot bc = 2ab \cdot c$ for all $a, b, c \in S$.*
(iii) *S is associative, provided that the transformation $x \to 2x$ is either injective or projective.*

4.3 Theorem. *Let S be an* AE*-ring. Then S is associative.*

Proof: By 2.2 (i), S is a D-ring, and hence S is the ring direct sum of two subrings S_1 and S_2 such that S_1 is m-idempotent and S_2 is nilpotent of class at most 3 (see [8, Theorem 2.16]). Clearly, S_2 is associative and S_1 is an AE-ring by 2.1. Now, without loss of generality, we can assume that S is m-idempotent. Then, by 2.4 (ii), $2x = 0$ for every $x \in S$.

Take $0 \neq w \in S$ and, for every ordered pair $p = (a, b) \in S^{(2)}$, $a \neq b$, choose a congruence r_p of $S(+)$ maximal with respect to $p \notin r_p$. Then $S(+)/r_p$ is a subdirectly irreducible 2-elementary group, and so there is an additive endomorphism f_p of S such that $f_p(S) = T = \{0, w\}$ and $f_p(a) \neq f_p(b)$. The endomorphism is multiplicative,and hence T is a subring of S. Clearly, T is associative, $\bigcap r_p = \mathrm{id}_S$, S is a subdirect product of copies of T, and therefore S is associative. □

5. Unipotent AE-semirings

5.1 Lemma. *Let S be an a-unipotent semiring and $0 = x + x$, $x \in S$. Then $\mathrm{Ida}(S) = \{0\}$ and $x0 = 0 = 0x$ for every $x \in S$.*

Proof: $x0 = x(0 + 0) = x0 + x0 = 0$ and $0x = (0 + 0)x = 0x + 0x = 0$. □

5.2. Let $S(+)$ be a semigroup with zero addition (i.e. there is an element $0 \in S$ such that $x + y = 0$ for all $x, y \in S$). Put $S(\cdot) = S(+)$. Then $S = S(+, \cdot)$ becomes an associative and commutative a-unipotent AE-semiring and we shall say that S is of type (AE5).

5.3. Consider the following two-element algebra $S = S(+, \cdot)$:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \qquad\qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Clearly, S is an associative, commutative, m-idempotent and a-unipotent AE-semiring and we shall say that S is of type (AE6).

5.4 Theorem. *Let S be a non-trivial* AE-*semiring such that $S(+)$ is a semigroup with zero addition. Then S is of just one of the types* (AE5), (AE6).

PROOF: Take $0 \neq w \in S$ and, for every ordered pair $p = (a, b) \in S^{(2)}$, $a \neq b$, choose a congruence r_p of $S(+)$ maximal with respect to $p \notin r_p$. Then $S(+)/r_p$ is a two-element semigroup with zero addition, and hence there is an additive endomorphism f_p of S such that $f_p(S) = T = \{0, w\}$ and $f_p(a) \neq f_p(b)$. The endomorphism is multiplicative and T is a subsemiring of S. Since $\bigcap r_p = \mathrm{id}_S$, S is a subdirect product of copies of T. With regard to 5.1, we have only the following two possibilities for the multiplication in T:

$$\begin{array}{c|cc} & 0 & w \\ \hline 0 & 0 & 0 \\ w & 0 & 0 \end{array} \qquad\qquad \begin{array}{c|cc} & 0 & w \\ \hline 0 & 0 & 0 \\ w & 0 & w \end{array}$$

In the first case, $T(+) = T(\cdot)$ and consequently $S(+) = S(\cdot)$, i.e. S is of type $(AE5)$. Now, let $ww = w$. Since T is commutative and m-idempotent, S is so and, moreover, S is of type (AE6), provided that S contains just two elements. Assume that this is not true and take $a, b \in S$, $a \neq 0 \neq b \neq a$. Then either $ab \neq a$ or $ab \neq b$ and we can assume that $ab \neq a$, the other case being similar. Define a transformation f of S by $f(a) = 0$ and $f(x) = x$ for every $x \in S$, $x \neq a$. Since $f(0) = 0$, f is an endomorphism of $S(+)$, and hence f is an endomorphism of $S(\cdot)$. Now, $0 = 0b = f(a)f(b) = f(ab) = ab$. Finally, define an endomorphism g of $S(+)$ by $g(a) = g(b) = a$ and $g(x) = 0$ for every $x \in S$, $a \neq x \neq b$. Again, g is multiplicative and $a = aa = g(a)g(b) = g(ab) = g(0) = 0$, a contradiction. □

5.5 Corollary. *Let S be an* AE-*semiring such that $S(+)$ is a semigroup with zero addition. Then S is associative and commutative.*

5.6 Proposition. *Let S be an a-unipotent* AE-*semiring. Then:*

(i) *$S_{(3)}$ is an* AE-*ring, and hence $S_{(3)}$ is associative.*
(ii) *$3a \cdot bc = 3ab \cdot c$ for all $a, b, c \in S$.*

PROOF: We have $3x + 0 = x + 2x + 0 = x + 0 + 0 = x + 0 = x + 2x = 3x$ for every $x \in S$ and this shows that $S_{(3)}$ is a ring. □

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(*Received* November 30, 1992)