# Existence results for differential equations in Banach spaces

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*Abstract.* This paper presents existence results for initial and boundary value problems for nonlinear differential equations in Banach spaces.

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## 1. Introduction and preliminaries.

In [8], [12] methods based on the topological transversality theorem of A. Granas [3] were used to establish existence results for initial and boundary value problems in Hilbert space. In two recent articles [6], [7] similar topological methods were applied to differential systems in $\Re^n$ in such a way that the existence of both classical and Carathéodory solutions could be treated simultaneously and in a classical setting. The authors of this article noticed that the general existence principles in $\Re^n$ extend very readily to the case of Hilbert space-valued solutions. Recently Granas, Guenther, Lee and O'Regan have noticed that both the classical and Carathéodory case for Banach space-valued solutions could be treated by the methods in [6], [7]; the basic ideas are presented below. A forthcoming paper [10] on differential delay equations will provide more details. In this note we use one of these existence principles to improve upon the results in [8], [12] and to provide some new existence theorems for certain specific classes of differential equations in Banach spaces.

Throughout $B$ is a real Banach space with norm $|\cdot|$. In case $B = H$ is a Hilbert space, we denote its inner product by $\langle\cdot,\cdot\rangle$ and then $|x|^2 = \langle x, x\rangle$ for $x \in H$. $C^m[a,b] = C^m([a,b],B)$ is the Banach space of functions $u : [a,b] \to B$ such that $u^{(m)}$ is continuous with norm

$$|u|_m = \max\{|u|_0, |u'|_0, \dots, |u^{(m)}|_0\}$$

where $|v|_0 = \max\{|v(t)| : t \in [a,b]\}$ for any $v \in C^0([a,b],B) = C[a,b]$.

Boundary conditions will be specified by continuous linear maps $U_i : C^k([a,b],B) \to B$ for $i = 1,2,\dots,k$ such that there is a scaler form $\widetilde{U}_i : C^k([a,b],\Re) \to \Re$ with $u_i(\theta(t)v) = \widetilde{U}_i(\theta(t))v$ for each $k-1$ times differentiable real function $\theta(t)$ and each $v$ in $B$. Given $\gamma_i$ in $B$ a function $u \in C^m[a,b]$ is said to satisfy the boundary conditions $\mathcal{B}$, denoted $u \in \mathcal{B}$, if $U_i(u) = \gamma_i$ for $i = 1,2,\dots,k$. The corresponding homogeneous boundary conditions with each $\gamma_i = 0$ are denoted by $\mathcal{B}_{\mathcal{O}}$. Given a class of functions $\mathcal{F}$ from $[a,b]$ into $B$, $\mathcal{F}_{\mathcal{B}}$ is the subclass of those functions in $\mathcal{F}$ which satisfy the boundary condition $\mathcal{B}$. Most commonly used boundary conditions satisfy the requirements above. For example a multipoint boundary condition is specified by

$$U(u) = \sum_{r=0}^{k-1} \sum_{s=0}^{q} a_{rs} u^{(r)}(c_s)$$

where $a_{rs}$ are scalers and $c_s \in [a,b]$. In this case $\widetilde{U}(u)$ is just the boundary condition of the stated form applied to scaler functions.

Let $u : [a,b] \to B$ be a measurable function. By $\int_a^b u(t)\,dt$ we understand the Bochner integral of $u$ (assuming it exists). See [4] or [17] for properties of the Bochner integral mentioned below. A measurable function $u : [a,b] \to B$ is Bochner integrable iff $|u|$ is Lebesgue integrable. Moreover, if $u : [a,b] \to B$ is measurable and $|u(t)| \le g(t)$ a.e. where $g(t)$ is integrable, then $u(t)$ is integrable. Let $u : [a,b] \to B$ be integrable and set $v(t) = \int_a^t u(s)\,ds$. The function $v : [a,b] \to B$ is absolutely continuous (according to the usual interval definition), $v$ is differentiable almost everywhere, and $v'(t) = u(t)$ almost everywhere on $[a,b]$. Finally, let $u : [a,b] \to B$ be integrable and $T : B \to B_1$ a bounded linear operator, where $B_1$ is also a Banach space. Then $Tu : [a,b] \to B_1$ is integrable and $\int_E Tu(t)\,dt = T \int_E u(t)\,dt$ for each measurable $E \subset [a,b]$. We need the following elementary consequence of these basic properties of the Bochner integral.

**Theorem 1.1.** *Let $u : [a,b] \to B$ be absolutely continuous and assume $u'$ exists a.e. and is Bochner integrable. Then*

$$u(t) - u(a) = \int_a^t u'(s)\,ds.$$

PROOF: Let $b^* \in B^*$ and set $g(t) = b^*(u(t))$. Clearly $g : [a,b] \to \Re$ is absolutely continuous and consequently $g(t) - g(a) - \int_a^t g'(s)\,ds = 0$. Since $g'(s) = b^*(u'(s))$ whenever $u'(s)$ exists which is almost everywhere by assumption, we infer that $\int_a^t g'(s)\,ds = \int_a^t b^*(u'(s))\,ds = b^* \int_a^t u'(s)\,ds$ because $u'(s)$ is integrable. Thus,

$$b^*(u(t) - u(a) - \int_a^t u'(s)\,ds) = 0,$$

and the conclusion of the theorem follows because $b^*$ is arbitrary. $\square$

As usual $L^p[a,b] = L^p([a,b],B)$ for $1 \le p < \infty$ denotes the measurable functions $u : [a,b] \to B$ such that $|u|^p$ is Lebesgue integrable. $L^p[a,b]$ is a Banach space with $\|u\|_p = (\int_a^b |u|^p\,dt)^{\frac{1}{p}}$. When $p = 2$, we abbreviate $\|u\|_2$ by $\|u\|$. $L^\infty[a,b]$ is defined in the usual way and equipped with the essential supremum norm $\|\cdot\|_\infty$. In our context $[a,b]$ is a bounded interval and Hölder's inequality and an earlier remark imply that each $L^p$-function is Bochner integrable. We denote by $W^{k,p}[a,b]$ those functions $u : [a,b] \to B$ such that $u^{(k-1)}$ is absolutely continuous, $u^{(k)}$ exists almost everywhere and $u^{(k)}$ belongs to $L^p[a,b]$.

We are concerned with solutions to initial and boundary value problems of the form

$$y^{(k)}(t) = f(t, y(t), \dots, y^{(k-1)}(t)), \quad y \in \mathcal{B}, \tag{1.1}$$

where $y : [a,b] \to B$ and the differential equation is to hold either everywhere or almost everywhere, depending on the assumptions on $f$. Recall that $f : [a,b] \times B^k \to B$ is an $L^p$-Carathéodory function if

(a) $u \to f(t,u)$ is continuous in $u \in B^k$ for a.e. $t$;
(b) $t \to f(t,u)$ is measurable for all $u$;
(c) for each $r > 0$ there is a function $h_r \in L^p([a,b], \Re)$ such that $|u| \le r$ implies $|f(t,u)| \le h_r(t)$ a.e. on $[a,b]$. (Here the norm of $u$ in $B^k$ is the maximum among the norms in $B$ of its $k$ components.)

When $f$ is continuous a solution $y$ to (1.1) will mean a function $y \in C^k_{\mathcal{B}}[a,b]$ which satisfies the differential equation in (1.1) everywhere. When $f$ is an $L^p$-Carathéodory function a solution $y$ to (1.1) will mean a function $y \in W^{k,p}_{\mathcal{B}}[a,b]$ which satisfies the differential equation in (1.1) almost everywhere.

It is well known that, in contrast to systems in $\Re^n$, even the initial value problem may have no solution in the Banach space case when $f$ in (1.1) is merely continuous. Various additional compactness conditions are needed to assume existence in the infinite dimensional setting. For our purposes the following added compactness property will suffice. We say a function $g : [a,b] \times B^k \to B$ satisfies $(*)$ if

$$(*) \quad \begin{cases} \text{for each bounded set } S \subset C^{k-1}([a,b], B) \text{ and each } t \in [a,b] \text{ the set} \\ \{\int_a^t g(s, u(s)), \dots, u^{(k-1)}(s)\, ds : u \in S\} \text{ is relatively compact.} \end{cases}$$

Note that $(*)$ holds if $g : [a,b] \times B^k \to B$ is completely continuous. (See the proof of Theorem 2.1)

Let $\Lambda : C^k_{\mathcal{B}_\mathcal{O}}([a,b], B) \to C([a,b], B)$ be the linear operator defined by $\Lambda y = y^{(k)}$. Assume that $f$ is continuous or $L^p$-Carathéodory. Now observe that (Theorem 1.1) any solution to (1.1) of the sort we seek is also a solution in $C^{k-1}[a,b]$ to the integro-differential equation with boundary conditions

$$y^{(k-1)}(t) - y^{(k-1)}(a) = \int_a^t f(t, y(s), \dots y^{(k-1)}(s))\, ds, \quad y \in \mathcal{B}. \tag{1.2}$$

Conversely, any solution $y \in C^{k-1}[a,b]$ to (1.2) defines a solution to (1.1) of the required sort. In this sense (1.1) and (1.2) are equivalent; however, (1.1) may have "solutions" which are not in $C^k[a,b]$ or $W^{k,p}[a,b]$. Considerations based on the equivalence of (1.1) and (1.2) just described lead to the following existence result from [7]; see [10] for more details.

**Theorem 1.2.** *Let $f : [a,b] \times B^k \to B$ be $L^p$-Carathéodory (respectively, continuous). Assume $\varepsilon$ is not an eigenvalue of $\Lambda : C^k_{\mathcal{B}_\mathcal{O}} \to C$ and that $f(t, u_1, \dots, u_{k-1}) - \varepsilon u_1$ satisfies $(*)$. Consider the family of problems*

$$y^{(k)} - \varepsilon y = \lambda[f(t, y, \dots, y^{(k-1)}) - \varepsilon y], \quad y \in \mathcal{B} \tag*{$(1.3)_\lambda$}$$

*for $\lambda \in (0,1)$. Then* (1.1) *has a solution in $W^{k,p}[a,b]$ (respectively, $C^k[a,b]$) provided there is a constant $M$ independent of $\lambda$ in $(0,1)$ such that any solution $y$ in $W^{k,p}[a,b]$ (respectively, $C^k[a,b]$) to $(1.3)_\lambda$ satisfies $|y|_{k-1} \le M$.*

For the analysis in the remaining sections we will need a variant [5] of the standard change-of-variables theorem for the Lebesgue integral which is helpful in establishing *a priori* bounds.

**Theorem 1.3.** *Let $g : [a,b] \to [A,B]$ and $h : [A,B] \to \Re$, where $g$ is absolutely continuous, $h$ is measurable, and $(h \circ g)g'$ is Lebesgue integrable on $[a,b]$. Then $h$ is integrable on the interval with endpoints $g(a)$ and $g(b)$ and $\int_{g(a)}^{g(b)} h(u)\,du = \int_a^b h(g(t))g'(t)\,dt$.*

We conclude this section with a proof of Wirtinger's inequality in a real Hilbert space $H$.

**Theorem 1.4.** (i) *Let $u : [0,1] \to H$ have a continuous derivative and satisfy $u(0) = u(1) = 0$. Then*

$$\pi^2 \int_0^1 |u(t)|^2\,dt \le \int_0^1 |u'(t)|^2\,dt$$

*with equality only if $u(t) = (\sin \pi t)e$ for some $e \in H$.*

(ii) *Let $u : [0,1] \to H$ have a continuous derivative and satisfy $u(0) = 0$ or $u(1) = 0$. Then*

$$\pi^2 \int_0^1 |u(t)|^2\,dt \le 4 \int_0^1 |u'(t)|^2\,dt.$$

PROOF: (i) The compact set $S = u([0,1]) \cup u'([0,1])$ is separable in $H$ as is the smallest closed subspace $H'$ of $H$ which contains $S$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $H'$. By Parseval's relation, Wirtinger's inequality when $H = \Re$, and two applications of the Monotone Convergence Theorem,

$$\begin{aligned}\pi^2 \int_0^1 |u(t)|^2\,dt &= \pi^2 \int_0^1 \sum_n |\langle u(t), e_n\rangle|^2\,dt = \sum_n \pi^2 \int_0^1 |\langle u(t), e_n\rangle|^2\,dt \\ &\le \sum_n \int_0^1 |\langle u'(t), e_n\rangle|^2\,dt = \int_0^1 \sum_n |\langle u'(t), e_n\rangle|^2\,dt = \int_0^1 |u'(t)|^2\,dt\end{aligned}$$

with equality iff there exist constants $c_n$ with $\langle u(t), e_n\rangle = c_n \sin \pi t$ for $n = 1, 2, \dots,$ in which case $u(t) = \sum_n \langle u(t), e_n\rangle e_n = \sum_n (c_n \sin \pi t)e_n$, i.e. $u(t) = (\sin \pi t)e$ where $e = \sum_n c_n e_n$. □

## 2. Some first order problems.

Consider the initial value problem

$$y' = \eta(t) f(t,y), \quad y(0) = r, \tag{2.1}$$

where $\eta : [0,T] \to [0,\infty)$, $f : [0,T] \times B \to B$, $B$ is a Banach space with norm $|\cdot|$, and $r \in B$. We seek global solutions to (2.1); that is, solutions defined for all $t$ in $[0,T]$. Of course, such global results require appropriate growth restrictions on $\eta$ and $f$. The following theorem establishes existence under conditions of Wintner-type [16].

**Theorem 2.1.** *Let* $\frac{1}{p}+\frac{1}{q}=1$, $f:[0,T]\times B\to B$ *be an* $L^p$*-Carathéodory (respectively, continuous) function which is completely continuous,* $\eta\in L^q([0,T],\Re)$ *(respectively, continuous) be nonnegative, and* $r\in B$. *Assume that* $\psi:[0,\infty)\to(0,\infty)$ *is a nondecreasing Borel function,* $\alpha\in L^p([0,T],\Re)$, *and that*

$$|f(t,y)|\le\alpha(t)\psi(|y|)$$

*for almost all* $t$ *in* $[0,T]$ *and* $y\in B$. *Then* (2.1) *has a solution* $y$ *in* $W^{1,p}[0,T]$ *(respectively,* $C^1[0,T]$*) provided*

$$\int_0^T\alpha(t)\eta(t)\,dt<\int_{|r|}^{\infty}\frac{du}{\psi(u)}\,. \tag{2.2}$$

Proof: The differential operator $\Lambda:C^1_{\mathcal{B}_O}[0,T]\to C[0,T]$ defined by $\Lambda y=y'$ and where $u\in\mathcal{B}$ means $U(u)\equiv u(0)=r$ is clearly invertible. Also, by Hölder's inequality $\eta f$ is $L^1$-Carathéodory. Furthermore, $\eta f$ satisfies $(*)$. Indeed, let $S\subset C[0,T]$ be bounded. By complete continuity of $f$ there is a compact subset $K$ of $B$ such that $f(t,y(t))\in K$ for all $t\in[0,T]$ and $y\in S$. Fix $t$ and consider the set

$$\Big\{\int_0^t\eta(s)f(s,y(s))\,ds:y\in S\Big\}. \tag{2.3}$$

If $\eta(s)=0$ a.e. on $[0,t]$, then the set is compact; otherwise $\int_0^t\eta(s)\,ds>0$ and

$$\frac{1}{\int_0^t\eta(s)\,ds}\int_0^t f(s,y(s))\eta(s)\,ds\in\overline{co}\,(range\ f(s,y(s)))\subset\overline{co}\,(K)$$

which is compact by Mazur's theorem. Thus the set in (2.3) is relatively compact and $\eta f$ satisfies $(*)$. Consequently, Theorem 1.2 is applicable with $\varepsilon=0$ and existence of a solution to (2.1) in $W^{1,p}[0,T]$ (respectively, $C^1[0,T]$) will follow if there is an *a priori* bound $M$ independent of $\lambda$ in $(0,1)$ on $|y|_0$ for all $y\in W^{1,p}[0,T]$ (respectively, $C^1[0,T]$) which satisfy

$$y'=\lambda\eta(t)f(t,y),\quad y(0)=r. \tag{2.4$_\lambda$}$$

Let $y=y(t)\in W^{1,p}[0,T]$ solve $(2.4)_\lambda$ for some $\lambda\in(0,1)$. Since $\eta f$ is an $L^1$-Carathéodory function, $(2.4)_\lambda$ shows that $y'$ is integrable. Since $y\in W^{1,p}[0,T]$, it is absolutely continuous and so Theorem 1.1 implies $y(t)-y(0)=\int_0^t y'(s)\,ds$, which yields

$$|y(t)|\le|r|+\int_0^t|y'(s)|\,ds\equiv\varrho(t).$$

Clearly $\varrho(t)$ is absolutely continuous with $\varrho'(t)=|y'(t)|$ almost everywhere. Now from $(2.4)_\lambda$ and the nondecreasing nature of $\psi$ we obtain

$$\varrho'(t)=|y'(t)|\le\eta(t)\alpha(t)\psi(|y(t)|)\le\eta(t)\alpha(t)\psi(\varrho(t))$$

almost everywhere on $[0,T]$. Next, Theorem 1.3 yields

$$\int_{|r|}^{\varrho(t)} \frac{du}{\psi(u)} = \int_0^t \frac{\varrho'(s)}{\psi(\varrho(s))}\,ds \le \int_0^t \alpha(s)\eta(s)\,ds \le \int_0^T \alpha(s)\eta(s)\,ds$$

and then

$$\int_{|r|}^{\varrho(t)} \frac{du}{\psi(u)} \le \int_0^T \alpha(s)\eta(s)\,ds < \int_{|r|}^{\infty} \frac{du}{\psi(u)}$$

by (2.2). This chain of inequalities entails the existence of a constant $M$ (independent of $\lambda$) such that $|y(t)| \le \varrho(t) \le M$ for $t \in [0,T]$. Thus, $|y|_0 \le M$ and existence of a solution to (2.1) in the required class follows. □

**Remarks.** (i) The classical result of Wintner is Theorem 2.1 when $f$ and $\psi$ are continuous, $\eta = \alpha \equiv 1$, $B = \Re^n$, and $\int^\infty \frac{du}{\psi(u)} = +\infty$. In this context, $\psi$ need not be increasing; see below.

(ii) Various extensions of Wintner's theorem in $\Re$ are given in [2], [6], [9], [11].

(iii) Notice that if $|f(t,y)| \le a(t)|y| + b(t)$ with $a, b \in L^p([0,T],\Re)$, then $|f(t,y)| \le \alpha(t)\psi(|y|)$ where $\alpha(t) = 1 + a(t) + b(t)$ and $\psi(u) = u+1$. Since $\int_{|r|}^\infty \frac{du}{u+1} = +\infty$, (2.2) holds and a solution exists on $[0,T]$ for any $T > 0$. The same conclusion holds if $|f(t,y)| \le a(t)h(|y|) + b(t)$ where $h : [0,\infty) \to (0,\infty)$ is increasing and $\int_{|r|}^\infty \frac{du}{h(u)} = +\infty$.

The ideas in [12] permit a version of Theorem 2.1 in a real Hilbert space setting where the assumption that $\psi$ is increasing can be relaxed.

**Theorem 2.2.** *In Theorem* 2.1 *assume $B = H$ is a real Hilbert space and delete the requirement that $\psi$ be nondecreasing. Then the conclusion to Theorem* 2.1 *holds.*

PROOF: The proof is essentially the same as that in Theorem 3.2 of [12] except we now use Theorems 1.2 and 1.3. The establishment of *a priori* bounds relies on

$$|y(t)|' \le |y'(t)| \tag{2.5}$$

whenever $y'(t)$ exists and $y(t) \ne 0$. □

**Remark.** Theorem 2.2 also holds for Banach spaces $B$, such as the $L^p$-spaces for $1 < p < \infty$, for which $B^*$ is uniformly convex. Indeed, if this is so, then the norm $|\cdot|$ in $B$ is Fréchet differentiable for any $u \ne 0$ with derivative $F_u \in B^*$ the unique functional with norm 1 such that $F_u(u) = |u|$. Thus, if $y'(t)$ exists and $y(t) \ne 0$ it follows that $|y(t)|' = F_{y(t)}(y'(t))$. Since $F_{y(t)}$ has norm 1, we find that $|y(t)|' \le |y'(t)|$ which is (2.5). Given this, the proof mentioned above applies and Theorem 2.2 holds with $B$ a Banach space whose dual $B^*$ is uniformly convex.

**Remark.** Of course, there is an extensive literature on various existence results for first order differential equations in a Banach space; see for example [1], [15] and the references therein. The treatment here typically involves fewer technical assumptions and the proofs themselves also lead quickly and naturally to reasonably general existence results.

**Remark.** Theorem 2.1 is sharp relative to the full class of problems covered. That is, given $\alpha(t)$, $\eta(t)$, $\psi(u)$ and $r$ as in Theorem 2.1 a solution to (2.1) will exist on $[0,T]$ provided $T$ satisfies (2.2). Conversely, given such data there is a differential equation in the class covered by Theorem 2.1 for which (2.2) must hold if the solution exists on $[0,T]$. To see this, take $f(t,y) = \alpha(t)\psi(|y|)e$ where $e = \frac{r}{|r|}$ when $r \neq 0$ and $e$ is any convenient unit vector when $r = 0$. Suppose

$$y' = \eta(t)f(t,y) = \eta(t)\alpha(t)\psi(|y|)e, \quad y(0) = r$$

has a solution on $[0,T]$. Integration from 0 to $t$ shows that $y$ has the form $y(t) = z(t)e$ for some scaler function $z(t)$. Moreover,

$$z' = \eta(t)\alpha(t)\psi(|z|), \quad z(0) = |r|.$$

It follows that $z(t) \geq |r| \geq 0$ and that

$$\int_0^T \eta(t)\alpha(t)\,dt = \int_0^T \frac{z'(t)}{\psi(z(t))}\,dt = \int_{|r|}^{z(T)} \frac{du}{\psi(u)} < \int_{|r|}^{\infty} \frac{du}{\psi(u)},$$

which is just (2.2).

**3. Some second order problems.**

The existence principle in Theorem 1.2 can be used in place of Theorems 2.1 and 2.2 in [8] to sharpen the results obtained there where problems of the form

$$y'' = f(t,y,y'), \quad y \in \mathcal{B} \tag{3.1}$$

were considered with $f : [0,1] \times H \times H \to H$ continuous, $H$ a real Hilbert space, and $\mathcal{B}$ boundary conditions of Sturm-Liouville type:

$$-\alpha y(0) + \beta y'(0) = r, \quad ay(1) + by'(1) = s \tag{3.2}$$

where $r, s \in H$, $\alpha, \beta, a, b \geq 0$, $\alpha+\beta > 0$, $a+b > 0$, and in addition $(\alpha+a)(\beta+b) > 0$, $r = 0$ if $\alpha = 0$, and $s = 0$ if $a = 0$. The additional conditions exclude pure Dirichlet data at both ends, exclude pure Neumann data at both ends, and require that any pure Neumann condition be homogeneous. The special nature of problems with either pure Dirichlet or Neumann data is discussed further in [8]. Two principle assumptions were made in [8] in order to invoke the general existence principles given there:

$$f(t,u,p) \ \text{ is completely continuous on } \ [0,1] \times H \times H, \tag{3.3}$$

and

$$\begin{cases} \text{given a bounded subset } U \text{ of } C^2([0,1],H) \text{ there exist constants } \gamma > 0 \\ \text{and } A \text{ such that } |f(t,u(t),u'(t)) - f(s,u(s),u'(s))| \leq A|t-s|^\gamma \text{ for all } u \\ \text{in } U \text{ and } t,s \in [0,1]. \end{cases} \tag{3.4}$$

When Theorem 1.2 is used for existence purposes the assumption (3.4) is not needed. Furthermore, the reasoning used in [8] to establish *a priori* bounds never uses (3.4). Therefore, all the results in [8] hold with assumption (3.4) deleted from all hypotheses. For example, we have the following result of Bernstein-Nagumo type.

**Theorem 3.1.** *Let $f : [0,1] \times H \times H \to H$ be continuous and completely continuous. Assume*

$$\begin{cases} \text{there is a constant } M > 0 \text{ such that } |u| > M \text{ and } \langle u, p\rangle = 0 \\ \text{implies } \langle u, f(t,u,p)\rangle > 0 \end{cases}$$

*and*

$$\begin{cases} \text{there is a Borel function } \psi : [0,\infty) \to (0,\infty) \text{ such that } |f(t,u,p)| \le \\ \psi(|p|) \text{ for } (t,|u|) \in [0,1] \times [0,M_0], \text{ and } \int_c^\infty \frac{dx}{\psi(x)} > 1, \text{ where } M_0 = \\ \max\{M, \frac{|r|}{\alpha}, \frac{|s|}{a}\} \text{ and } c = \min\{\beta^{-1}(|r| + \alpha M_0), b^{-1}(|s| + aM_0)\}. \end{cases}$$

*Then* (3.1) *has a solution $y \in C^2[0,1]$.*

**Remark.** If $\alpha, \beta, a$ or $b$ equals 0 in the expressions for $M_0$ and $c$ above then the corresponding term is omitted from that expression.

We now broaden the class of differential equations covered in [8] by considering the following analogue of Theorem 3.1 for singular second order boundary value problems of the form

$$y'' = \eta(t) f(t,y,y'), \quad y \in \mathcal{B} \tag{3.5}$$

with $\eta : [0,1] \to [0,\infty)$. Here the boundary conditions $\mathcal{B}$ denote (3.2) with $\alpha \neq 0$ and $a \neq 0$.

**Theorem 3.2.** *Let $f : [0,1] \times H \times H \to H$ be continuous and completely continuous. Let $\mathcal{B}$ denote* (3.2) *with $\alpha \neq 0$ and $a \neq 0$. In addition suppose $\eta \in L^1[0,1]$ and*

$$\begin{cases} \text{there is a constant } M > 0 \text{ such that } t \in (0,1),\ |u| > M \text{ and} \\ \langle u, p\rangle = 0 \text{ implies } \langle u, \eta(t) f(t,u,p)\rangle > 0 \end{cases}$$

*and*

$$\begin{cases} \text{there is a Borel function } \psi : [0,\infty) \to (0,\infty) \text{ and a continuous func-} \\ \text{tion } \alpha : [0,1] \to [0,\infty) \text{ such that } |f(t,u,p)| \le \alpha(t)\psi(|p|) \text{ for } (t,|u|) \in \\ [0,1] \times [0,M_0], \text{ and } \int_c^\infty \frac{dx}{\psi(x)} > \int_0^1 \alpha(t)\eta(t)\,dt. \text{ Here } c \text{ and } M_0 \text{ are as in} \\ \text{Theorem 3.1.} \end{cases}$$

*Then* (3.5) *has a solution $y \in W^{2,1}[0,1]$ (in fact $C^1[0,1] \cap C^2(0,1)$).*

Proof: The differential operator $\Lambda : C^2_{\mathcal{B}_O} \to C$, $\Lambda y = y''$ is easily seen to be invertible and $(*)$ holds because $f$ is completely continuous. Therefore existence of a solution to (3.5) in $W^{2,1}[0,1]$ will follow if there is an *a priori* bound $K$ of $\lambda$ in $(0,1)$ on $|y|_1$ for all $y \in W^{2,1}[0,1]$ which satisfy

$$y'' = \lambda\eta(t) f(t,y,y'), \quad y \in \mathcal{B}. \tag{3.6$_\lambda$}$$

Let $y = y(t) \in W^{2,1}[0,1]$ solve $(3.6)_\lambda$ for some $\lambda \in (0,1)$. Then essentially the argument in Lemma 3.1 of [8] yields $|y|_0 \le M_0$ where $M_0$ is as in Theorem 3.1.

Also as in Lemma 4.1 of [8] there exists $\tau \in [0,1]$ with $|y'(\tau)| \le c$. In addition we have $|y'|' \le |y''|$ whenever $y''(t)$ exists and $y'(t) \ne 0$. We now obtain from $(3.6)_\lambda$ that

$$|y'(t)|' \le \eta(t)\alpha(t)\psi(|y'(t)|) \tag{3.7}$$

almost everywhere on $[0,1]$. Now, suppose $|y'(t)| > c$ for some $t \in [0,1]$. Since $|y'(\tau)| \le c$ and $y'$ is continuous, there is an interval $d \le s \le t$ (or $t \le s \le d$) such that $|y'(t)| > 0$ and $|y'(d)| = c$. To be definite, suppose the interval is $d \le s \le t$. Then (3.7) and Theorem 1.3 yields

$$\int_c^{|y'(t)|} \frac{du}{\psi(u)} = \int_d^t \frac{|y'(s)|'}{\psi(|y'(s)|)}\,ds \le \int_d^t \alpha(s)\eta(s)\,ds \le \int_0^1 \alpha(s)\eta(s)\,ds < \int_c^\infty \frac{du}{\psi(u)}$$

and so there exists a constant $M_1$ (independent of $\lambda$) such that $|y'(t)| \le M_1$. We conclude $|y'|_0 \le \max\{c, M_1\}$ and the existence of a solution to (3.5) in the required class follows. □

The next results extend the ideas in [6], [13], [14] about systems in $\Re^n$ to a Hilbert space setting. Suppose $a_0, b_0, a_1, b_1 \ge 0$ with $b_0, b_1 > 0$. Let $\mathcal{B}$ denote either the boundary conditions

$$y(0) = 0, \quad a_1 y(1) + b_1 y'(1) = r_1, \tag{3.8}$$

or

$$a_0 y(0) - b_0 y'(0) = r_0, \quad y(1) = 0, \tag{3.9}$$

where $r_0, r_1 \in H$.

**Theorem 3.3.** *Let $f : [0,1] \times H \times H \to H$ be an $L^p$-Carathéodory (respectively, continuous) function which is completely continuous and consider the problem*

$$y'' = f(t, y, y'), \quad y \in \mathcal{B} \tag{3.10}$$

*where $\mathcal{B}$ denotes either (3.8) or (3.9) and $f$ has the decomposition $f(t,u,p) = g(t,u,p) + h(t,u,p)$ such that*

$$\begin{cases} \langle u, g(t,u,p)\rangle \ge a|u|^2 + b|u||p| \text{ for certain constants} \\ a \text{ and } b \text{ and } |g(t,u,p)| \le A(t,u)|p|^2 + B(t,u) \text{ where} \\ A, B \text{ are bounded on bounded sets} \end{cases} \tag{3.11}$$

*and*

$$|h(t,u,p)| \le M(|u|^\alpha + |p|^\beta) \text{ for } 0 \le \alpha,\ \beta < 1 \text{ and some constant } M. \tag{3.12}$$

*Then the problem (3.10) has a solution $y \in W^{2,p}[0,1]$ (respectively, $C^2[0,1]$) in each of the following cases:*

(i) *$a \ge 0$ and $|b| < \frac{\pi}{2}$;*
(ii) *$a < 0$ and $|b| + \frac{2|a|}{\pi} < \frac{\pi}{2}$.*

Proof: Just as in Theorem 3.2 existence of a solution to (3.10) in $W^{2,p}[0,1]$ (respectively, $C^2[0,1]$) will follow if there is an *a priori* bound independent of $\lambda$ in $(0,1)$ on $|y|_1$ for all $y \in W^{2,p}[0,1]$ (respectively, $C^2[0,1]$) which satisfy

$$y'' = \lambda f(t,y,y'), \quad y \in \mathcal{B}. \tag{3.13$_\lambda$}$$

Let $y$ be such a solution. Then $\langle y, y'\rangle$ is absolutely continuous and $\langle y,y'\rangle' = \langle y, y''\rangle + \langle y', y'\rangle$ a.e. is integrable from $(3.13)_\lambda$ because $f$ is a Carathéodory function. So Theorem 1.1 gives

$$\int_0^1 \langle y, y''\rangle\, dt = \langle y, y'\rangle\big]_0^1 - \int_0^1 |y'|^2\, dt, \tag{3.14}$$

and use of the boundary conditions yields

$$\langle y, y'\rangle\big]_0^1 \le \frac{\langle y(i), r_i\rangle}{b_i}$$

where $i = 0$ or $1$ according as the boundary conditions are (3.9) or (3.8). In either case the boundary conditions also give $|y(i)| = |\int_0^1 y'(t)|\, dt \le \|y'\|$ where $\|\cdot\|$ denotes the $L^2$ norm on $[0,1]$. Consequently,

$$\langle y, y'\rangle\big]_0^1 \le r\|y'\|, \quad r = \frac{\max\{|r_0|, |r_1|\}}{\min\{b_0, b_1\}} \ge 0. \tag{3.15}$$

Use of $(3.13)_\lambda$, (3.14), and (3.15) gives

$$\begin{aligned}\|y'\|^2 &= \int_0^1 |y'|^2\, dt = \langle y, y'\rangle\big]_0^1 - \int_0^1 \langle y, y''\rangle\, dt \\ &\le r\|y'\| - \lambda \int_0^1 \langle y, g(t,y,y')\rangle\, dt - \lambda \int_0^1 \langle y, h(t,y,y')\rangle\, dt.\end{aligned}$$

From (3.11) and (3.12), $-\langle y, g(t,y,y')\rangle \le -a|y|^2 - b|y||y'| \le -a|y|^2 + |b||y||y'|$ and

$$\begin{aligned}|\langle y, h(t,y,y')\rangle| &\le |y||h(y,y,y')| \le \frac{\varepsilon}{2}|y|^2 + \frac{1}{2\varepsilon}|h(t,y,y')|^2 \\ &\le \frac{\varepsilon}{2}|y|^2 + \frac{M^2}{\varepsilon}(|y|^{2\alpha} + |y'|^{2\beta})\end{aligned}$$

where $\varepsilon > 0$ will be fixed shortly. Therefore

$$\|y'\|^2 \le r\|y'\| - \lambda a\|y\|^2 + |b|\|y\|\|y'\| + \frac{\varepsilon}{2}\|y\|^2 + \frac{M^2}{\varepsilon}(\|y\|^{2\alpha} + \|y'\|^{2\beta}) \tag{3.16}$$

where Hölder's inequality was used to obtain $\int_0^1 |y||y'|\, dt \le \|y\|\|y'\|$ and $\int_0^1 |y|^{2\gamma}\, dt \le \|y\|^{2\gamma}$ valid for any $\gamma$ in $[0,1]$. Since $y(t)$ vanishes either at $t = 0$ or $1$ we may apply Theorem 1.4 (ii) and this together with (3.16) yields

$$\Big(1 - \frac{2|b|}{\pi} - \frac{2\varepsilon}{\pi^2}\Big)\|y'\|^2 \le -\lambda a\|y\|^2 + r\|y'\| + \frac{M^2}{\varepsilon}\Big(\frac{2^{2\alpha}}{\pi^{2\alpha}}\|y'\|^{2\alpha} + \|y'\|^{2\beta}\Big). \tag{3.17}$$

Now, assume $a \geq 0$ and $|b| < \frac{\pi}{2}$ as in (i) of the theorem. Then $-\lambda a\|y\|^2$ can be dropped from the right member of (3.17) and $\varepsilon > 0$ can be fixed close enough to zero so that the coefficient of $\|y'\|^2$ in (3.17) is positive. Since $2\alpha, 2\beta < 2$ these observations and (3.17) yield an *a priori bound*

$$\frac{\pi}{2}\|y\| \leq \|y'\| \leq M_1 \tag{3.18}$$

for some constant $M_1$ independent of $\lambda$ in $(0,1)$. Next, assume $a < 0$ and $|b| + \frac{2|a|}{\pi} < \frac{\pi}{2}$ as in (ii) of the theorem. Since $a < 0$, we have $-a\lambda < -a$ and Theorem 1.4 (ii) gives $-\lambda a\|y\|^2 \leq -a\|y\|^2 \leq -a(\frac{4}{\pi^2})\|y'\|^2$. Then (3.17) leads to

$$\Big(1 - \frac{2|b|}{\pi} + \frac{4a}{\pi^2} - \frac{2\varepsilon}{\pi^2}\Big)\|y'\|^2 \leq r\|y'\| + \frac{M^2}{\varepsilon}\Big(\frac{2^{2\alpha}}{\pi^{2\alpha}}\|y'\|^{2\alpha} + \|y'\|^{2\beta}\Big). \tag{3.19}$$

Under the conditions in (ii) we can fix $\varepsilon > 0$ so that the coefficient of $\|y'\|^2$ in (3.19) is positive and as above this leads again to an *a priori* bound (3.18) for $\|y\|$ and $\|y'\|$.

From (3.18) and the fact that $y(t)$ vanishes at $i = 0$ or $1$ we obtain $|y(t)| = |\int_i^t y'(s)\,ds| \leq \|y'\| \leq M_1$, and so

$$|y|_0 \leq M_1. \tag{3.20}$$

Now the assumption (3.11) and (3.12) reveal that there are constants $E, F$ such that $|f(t,y,p)| \leq E|p|^2 + F$ provided $(t,u) \in [0,1] \times [-M_1, M_1]$. Then $(3.13)_\lambda$, (3.20) and (3.18) yield

$$\int_0^1 |y''|\,dt \leq E\int_0^1 |y'|^2\,dt + F \leq EM_1^2 + F = M_2. \tag{3.21}$$

Fix $z$ in $H$ with norm 1 and set $\phi(t) = \langle y(t), z\rangle$. Clearly $|\phi(t)| \leq |y(t)| \leq M_1$ and hence there exists $t_0$ (dependent on $y$ and $z$) in $[0,1]$ such that $|\phi'(t_0)| = |\phi(1) - \phi(0)| \leq 2M_1$. Then

$$\begin{aligned}|\phi'(t)| &\leq |\phi'(t_0)| + \Big|\int_{t_0}^t |\phi''(s)|\,ds\Big| \\ &\leq 2M_1 + \int_0^1 |\langle y''(s), z\rangle|\,ds \leq 2M_1 + M_2 = M_3.\end{aligned}$$

That is $|\langle y'(t), z\rangle| \leq M_3$ for all $z \in H$ of norm 1. If $y'(t) \neq 0$ set $z = \frac{y'(t)}{|y'(t)|}$ to obtain $|y'(t)| \leq M_3$, which also holds if $y'(t) = 0$. Thus, $|y'|_0 \leq M_3$ and with (3.20) this implies the required *a priori* bound in the $C^1[0,1]$ norm. □

The reasoning above, with minor simplifications, also works when the boundary conditions are homogeneous Dirichlet conditions $y(0) = y(1) = 0$. In this case, the boundary term in (3.14) vanishes and we arrive at (3.16) with $r = 0$. The reasoning following (3.16) only changes in the way that Wirtinger's inequality (Theorem 1.4 (i)) takes the stronger form $\|y\| \leq \frac{1}{\pi}\|y'\|$. The argument now proves:

**Theorem 3.4.** *Let $f$ be as in Theorem 3.3 and let $\mathcal{B}$ in (3.10) be the homogeneous Dirichlet conditions $y(0) = y(1) = 0$. Then (3.10) has a solution $y \in W^{2,p}[0,1]$ (respectively, $C^2[0,1]$) in each of the following cases:*

(i) $a \geq 0$ *and* $|b| < \pi$;

(ii) $a < 0$ *and* $|b| + \frac{|a|}{\pi} < \pi$.

**Remarks.** (i) Theorem 3.4 with $H = \Re^n$ and $a = b = 0$ is the result established in [6] for systems.

(ii) Further modifications permit corresponding results involving inhomogeneous Dirichlet data (and inhomogeneous Sturm-Liouville data). See [14] for details when $H = \Re^n$. Such results in the context of differential delay equations will be forthcoming [10].

(iii) The ideas of this paper together with those in [13], [14] provide existence results for higher order singular and nonsingular problems in a real Hilbert space. Since the extensions are immediate we will omit the details.

Finally we discuss briefly a singular second order boundary value problem in a Banach space. Specifically consider

$$y'' = \eta(t) f(t, y, y'), \quad y \in \mathcal{B} \tag{3.22}$$

with $f : [0,1] \times B \times B \to B$, $\eta : [0,1] \to [0, \infty)$, $B$ is a Banach space. Here the boundary conditions $\mathcal{B}$ denote either

$$y'(0) = r, \quad ay(1) + by'(1) = s, \tag{3.23}$$

or

$$-\alpha y(0) + \beta y'(0) = s, \quad y'(1) = r, \tag{3.24}$$

where $r, s \in B$, $\beta, b \geq 0$, and $a, \alpha > 0$.

**Theorem 3.5.** *Let $\frac{1}{p} + \frac{1}{q} = 1$, $f : [0,1] \times B \times B \to B$ be an $L^p$-Carathéodory (respectively, continuous) function which is completely continuous, $\eta \in L^q([0,1], \Re)$ (respectively, continuous) be nonnegative. Assume that $\psi : [0, \infty) \to (0, \infty)$ is a nondecreasing Borel function, $\alpha \in L^p([0,1], \Re)$, and that*

$$|f(t, y, p)| \leq \alpha(t) \psi(|p|)$$

*for almost all $t \in [0,1]$, $y \in B$ and $p \in B$. Then (3.22) has a solution $y$ in $W^{2,p}[0,1]$ (respectively, $C^2[0,1]$) provided*

$$\int_0^1 \alpha(t) \eta(t)\, dt < \int_{|r|}^{\infty} \frac{du}{\psi(u)}\,.$$

Proof: Essentially the same argument as in Theorem 2.1 yields the result. The only major difference is that $\varrho(t) = |r| + \int_0^t |y''(s)|\, ds$ if (3.23) holds whereas $\varrho(t) = |r| + \int_t^1 |y''(s)|\, ds$ if (3.24) is satisfied. $\square$

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