Non-commutative Gelfand-Naimark theorem

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Abstract. We show that if Y is the Hausdorffization of the primitive spectrum of a C^*-algebra A then A is $*$-isomorphic to the C^*-algebra of sections vanishing at infinity of the canonical C^*-bundle over Y.

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Terminology and notations.

A function $f : X \to \mathbb{R}$ of a topological space X is called vanishing at infinity if for every $\varepsilon > 0$ there is quasicompact $K \subset X$ with $|f(y)| < \varepsilon$ for every $y \notin K$. By an H-family $\varphi : A \to \xi$ of a C^*-algebra A we mean a family $\varphi = \{\varphi_x\}_X$ of $*$-epimorphisms $\varphi_x : A \to \xi_x$ where X is a topological space, $\xi = \{\xi_x\}_X$ is a family of C^*-algebras and for every $s \in A$ the function $x \mapsto \|\varphi_x(s)\|$ is upper semicontinuous and vanishing at infinity (or equivalently for every $s \in A$ and $\varepsilon > 0$ the set $\{x \in X \mid \|\varphi_x(s)\| \geq \varepsilon\}$ is quasicompact and closed in X). If $\varphi : A \to \xi$ is an H-family and $\xi = \{\xi_x\}_X$ we denote by $b(\varphi)$ the triple $(p, \coprod\xi, X)$ where $p : \coprod\xi \to X$ is the canonical projection of disjoint sum, and $\coprod\xi$ is equipped with the topology generated by all tubes $T(V, s, \varepsilon) = \coprod_{x\in V} B(\varphi_x(s), \varepsilon)$ (disjoint sum of open balls), V open in X, $s \in A$, $\varepsilon > 0$. By the same argument as in [1], [5], $b(\varphi)$ is a C^*-bundle, by which we mean an (H) C^*-bundle defined as in [3]. It is easy to see that for any C^*-bundle η the set $\Gamma_0(\eta)$ of sections vanishing at infinity is a C^*-algebra. For every H-family $\varphi : A \to \xi$ the formula $\tilde{\varphi}(s)(x) = \varphi_x(s)$ gives a $*$-homomorphism $\tilde{\varphi} : A \to \Gamma_0(b(\varphi))$.

Example 1. Let $c : \check{A} \to X$ be a continuous map of the primitive spectrum $\check{A}$ of a C^*-algebra A onto a Hausdorff space X. Let $\overline{c}_x : A \to A/\bigcap c^{-1}(x)$ be the quotient map for every $x \in X$. If W is a closed subset of $\check{A}$ and $s \in A$ then there is $w_0 \in W$ such that $\|s + \bigcap W\| = \sup\{\|s + w\| \mid w \in W\} = \|s + w_0\|$. Indeed the first equality is well known (cf. e.g. [4, 1.9]) and the existence of w_0 is an easy consequence of [2, 3.3.6]. Using this we see that for every $s \in A$ and $\varepsilon > 0$ we have $c(\{w \in \check{A} \mid \|s + w\| \geq \varepsilon\}) = \{x \in X \mid \|\overline{c}_x(s)\| \geq \varepsilon\}$, whence we obtain an H-family $\overline{c}$.

Example 2. For every C^*-bundle η the family of evaluations is an H-family of the C^*-algebra $\Gamma_0(\eta)$.

Theorem 1 (Stone-Weierstrass theorem for H-families)**.** *Let $\varphi : A \to \xi$ be an H-family, and B a C^*-subalgebra of A. Assume that $B + (\ker \varphi_x \cap \ker \varphi_y) = A$ for all $x, y \in X$. Then $B + \bigcap_X \ker \varphi_x = A$.*

PROOF: Taking the quotient $A/\bigcap_X \ker \varphi_x$ and factorizations of all of φ_x we may assume that $\bigcap_X \ker \varphi_x = 0$. Let $\mathrm{hull}\,(\ker \varphi_x)$ denote the set $\{w \in \check{A} \mid \ker \varphi_x \subset w\}$. Then $\bigcup_X \mathrm{hull}\,(\ker \varphi_x)$ is a dense subset of $\check{A}$, whence, by the openness of the canonical map $P(A) \to \check{A}$, $\bigcup_X \mathrm{im}\; P(\varphi_x)$ is dense in the weak closure $\overline{P(A)}$ of the pure state space $P(A)$, here $P(\varphi_x) : P(\xi_x) \to P(A)$ is the canonical map induced by φ_x. We shall show that for any $f \in \overline{P(A)}$ there are $x \in X$ and a map $g : \xi_x \to \mathbb{C}$ with $f = g \circ \varphi_x$. Choose a net $\{f_i\}_I \subset \bigcup_X \mathrm{im}\; P(\varphi_x)$ such that $f_i \to f$. For every $i \in I$ there are $x_i \in X$ and $g_i \in P(\xi_{x_i})$ with $f_i = g_i \circ \varphi_{x_i}$. Let $x_i \to x$ and $a \in \ker \varphi_x$. If $|f(a)| = 2\delta > 0$ then there is $i_1 \in I$ such that $|f_i(A)| > \delta$ for every $i \geq i_1$. Then $\|\varphi_{x_i}(a)\| \geq |g_i(\varphi_{x_i}(a))| = |f_i(a)| > \delta$ for every $i \geq i_1$. Since the function $y \mapsto \|\varphi_y(a)\|$ is upper semicontinuous, the set $U = \{y \in X \mid \|\varphi_y(a)\| < \delta\}$ is a neighborhood of x. Hence, there is $i_2 \in I$ such that $x_i \in U$ for every $i \geq i_2$. Suppose now that $i \geq i_1$ and $i \geq i_2$. Then we obtain $\delta > \|\varphi_{x_i}(a)\| > \delta$ and this contradiction shows that $f(a) = 0$. Hence $\ker \varphi_x \subset \ker f$ and this shows the existence of g. Taking a subnet if necessary, we see that if x is an accumulation point of $\{x_i\}_I$ then there is a map $g : \xi_x \to \mathbb{C}$ such that $f = g \circ \varphi_x$. Suppose that the set of accumulation points of $\{x_i\}_I$ is empty. Let $s \in A$ and $\varepsilon > 0$. Choose a quasicompact $K \subset X$ with $\|\varphi_x(s)\| < \varepsilon$ for $x \notin K$. Then for sufficiently large $i \in I$

$$|f(s)| \leq |f(s) - f_i(s)| + |f_i(s)| < \varepsilon + |g_i(\varphi_{x_i}(s))| < 2\varepsilon.$$

Hence $f = 0$ and the existence of g (for every $x \in X$) is obvious. Now, let $f_1, f_2 \in \overline{P(A)} \cup \{0\}$ and $f_1 \neq f_2$. Take $s \in A$ such that $f_1(s) \neq f_2(s)$, choose $x_1, x_2 \in X$ and maps g_1, g_2 with $f_i = g_i \circ \varphi_{x_i}$, $i = 1, 2$. Since $A = B + (\ker \varphi_{x_1} \cap \ker \varphi_{x_2})$, there are $t \in B$ and $t' \in (\ker \varphi_{x_1} \cap \ker \varphi_{x_2})$ such that $s = t + t'$. We obtain $f_1(t) = f_1(s) \neq f_2(s) = f_2(t)$. Thus $B = A$ by Stone-Weierstrass-Glimm theorem [2, 11.5.2]. □

Corollary 1. *Let η be a C^*-bundle over X, B and A C^*-subalgebras of $\Gamma_0(\eta)$ and $B \subset A$. Assume that for all $x, y \in X$ and $s \in A$ there is $t \in B$ with $t(x) = s(x)$ and $t(y) = s(y)$. Then $B = A$.*

PROOF: Let $e_x : \Gamma_0(\eta) \to \eta_x$, $e_x(s) = s(x)$ be the evaluation map for every $x \in X$. Let $\xi_x = e_x(A)$ and $\varphi_x : A \to \xi_x$ denote the restriction of e_x for every $x \in X$, we obtain an H-family $\varphi : A \to \xi$. It is obvious that by our assumption we have $B + (\ker \varphi_x \cap \ker \varphi_y) = A$ for every $x, y \in X$. Now, the result follows immediately from Theorem 1. □

Corollary 2. *Let $\varphi : A \to \xi$ be an H-family. Assume that* $\ker \varphi_x + \ker \varphi_y = A$ *whenever $x, y \in X$, $x \neq y$. Then $\tilde{\varphi} : A \to \Gamma_0(b(\varphi))$ is a $*$-epimorphism.*

PROOF: Let $x, y \in X$, $x \neq y$. If $w \in \xi_x$, $v \in \xi_y$ then by the condition $\ker \varphi_x + \ker \varphi_y = A$ there is $t \in A$ such that $\varphi_x(t) = w$ and $\varphi_y(t) = v$. This implies that for every $s \in \Gamma_0(b(\varphi))$ there is $t \in A$ such that $\tilde{\varphi}(t)(x) = s(x)$ and $\tilde{\varphi}(t)(y) = s(y)$. Now, applying Corollary 1 to C^*-algebras $\Gamma_0(b(\varphi))$ and $\tilde{\varphi}(A)$ we obtain $\tilde{\varphi}(A) = \Gamma_0(b(\varphi))$. $\square$

Corollary 3. *Let $c : \check{A} \to X$ be a continuous map onto a Hausdorff space X. Then $\tilde{\overline{c}} : A \to \Gamma_0(b(\overline{c}))$ is a $*$-isomorphism.*

Proof: Obviously $\ker \tilde{\overline{c}} = \bigcap_X \ker \overline{c}_x = \bigcap_X \bigcap c^{-1}(x) = \bigcap \check{A} = \{0\}$. If $x, y \in X$, $x \neq y$, then $c^{-1}(x)$, $c^{-1}(y)$ are closed disjoint subsets of $\check{A}$. Assume $p \in \check{A}$ is a primitive ideal such that $(\ker \overline{c}_x + \ker \overline{c}_y) \subset p$. Then $\bigcap c^{-1}(x) \subset p$, hence $p \in c^{-1}(x)$. Similarly $p \in c^{-1}(y)$ and this contradiction shows that the closed ideal $\ker \overline{c}_x + \ker \overline{c}_y$ is equal to A. Now the result follows from Corollary 2. $\square$

The next theorem is our main result and it is an immediate consequence of Corollary 3.

Theorem 2 (Non-commutative Gelfand-Naimark theorem)**.** *Let $h : \check{A} \to h(\check{A})$ be the Hausdorffization map of the primitive spectrum $\check{A}$ of a C^*-algebra A. Then $\tilde{\overline{h}}$ is a $*$-isomorphism.*

Remarks. Corollary 1 generalizes Theorem 4.1 of [4], Corollary 3 is an analogue of Theorem 3.1 in [6]. If $h(\check{A}) = \check{A}$ then Theorem 2 coincides with Non-commutative Gelfand-Naimark theorem obtained by Fell in [4] and Tomiyama in [6]. If A is a C^*-algebra with identity then Theorem 2 coincides with Non-commutative Gelfand-Naimark theorem obtained by Dauns and Hofmann in [1].

References

[1] Dauns J., Hofmann K.H., *Representation of rings by sections*, Mem. Amer. Math. Soc. **83** (1968).

[2] Dixmier J., *Les C^*-algèbres et leurs representations*, Gauthier-Villars, Paris, 1969.

[3] Dupre M.J., Gillette M.R., *Banach bundles, Banach modules and automorphisms of C^*-algebras*, Research Notes in Math. 92, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1983.

[4] Fell J.M.G., *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233–280.

[5] ———, *An extension of Macley's method to Banach $*$-algebraic bundles*, Mem. Amer. Math. Soc. **90** (1969).

[6] Tomiyama J., *Topological representations of C^*-algebras*, Tohôku Math. J. **14** (1962), 187–204.

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