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On the metric dimension of converging sequences

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Abstract. In the paper, some kind of independence between upper metric dimension and natural order of converging sequences is shown — for any sequence converging to zero there is a greater sequence with an arbitrary ($\leqslant 1$) upper dimension. On the other hand there is a relationship to summability of series — the set of elements of any positive summable series must have metric dimension less than or equal to 1/2.

Keywords: metric dimension, converging sequences, summability of series

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Let $(\mathcal{K}, d)$ be a compact metric space. We say that an open covering $\mathfrak{C}$ of $\mathcal{K}$ is an r-covering if for every $C \in \mathfrak{C}$ $\operatorname{diam}(C) < r$. Let us denote by $N(r, \mathcal{K})$ the least cardinality of an r-covering of $\mathcal{K}$. The following characteristic of the "massiveness" of the set $\mathcal{K}$:

$$\underline{\dim}\, \mathcal{K} = \liminf_{r>0} \frac{\log N(r, \mathcal{K})}{-\log r}$$

was defined in [P-S] and it is called the lower metric dimension of the set $\mathcal{K}$. In [K-T] both lower and upper (defined by lim sup) dimensions were investigated.

Let us recall some basic relations of the lower and upper metric dimensions to the well known topological dimension – td and Hausdorff dimension – hd. The inequalities

$$\operatorname{td} \mathcal{K} \leqslant \operatorname{hd} \mathcal{K} \leqslant \underline{\dim}\, \mathcal{K} \leqslant \overline{\dim}\, \mathcal{K}$$

hold (c.f. [V]) for each totally bounded metric space $\mathcal{K}$. For some "nice" spaces $\mathcal{K}$ (e.g., for the subsets of $\mathbb{R}^n$ with nonempty interior) the inequality $\operatorname{td} \mathcal{K} = \overline{\dim}\, \mathcal{K}$ holds. On the other hand, the main difference between the lower (upper) metric dimension and Hausdorff dimension consists in the fact that Hausdorff dimension of any countable set is zero, while the lower (upper) metric dimension can be positive. It is proved in [M-Z, Theorem 4] that in each compact metric space $\mathcal{K}$ there is a converging sequence S of points of $\mathcal{K}$ with $\overline{\dim}\, S = \overline{\dim}\, \mathcal{K}$ (by the upper metric dimension of the sequence we mean the upper metric dimension of the set of points of the sequence), which is not true for lower metric dimension. This fact was the starting point of our interest in the upper metric dimension of sequences. On the other hand, the upper metric dimension gives an important characteristic of sequences — a characteristic of another kind compared with a 'speed of convergence' (see Theorem 2).

Conventions: For simplicity we will speak about dimension instead of the upper metric dimension. For the converging sequence $\{a_n\}_{n=1}^{\infty}$ we will write $A = \{a_n\}$

instead of $A = \bigcup_{n\in\mathbb{N}} \{a_n\} \cup \{\lim_{n\to\infty} a_n\}$ if there is no danger of misunderstanding. For $a_n \in \mathbb{R}$, the symbols $a_n \nearrow a$ ($a_n \searrow a$) will mean that the sequence $\{a_n\}$ is strictly monotonically increasing (decreasing) and converging to a. By $|X|$ we will denote the cardinality of the set X. $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real, positive real and natural numbers, respectively. By minimal r-covering of the set $\mathcal{K}$ we mean any r-covering consisting of $N(r,\mathcal{K})$ sets.

In this paper we will deal with the upper metric dimension of the converging sequences of real numbers. The outstanding position of the real sequences is given by the following fact.

Proposition 1. *Let $(\mathcal{K}, d)$ be a compact metric space and $S = \{s_n\}$ a sequence in $\mathcal{K}$ converging to s_0. Put $d_n = d(s_0, s_n)$ for each $n \in \mathbb{N}$ and $D = \{d_n\}$. Then $\overline{\dim}\, D \leqslant \overline{\dim}\, S$.*

This fact is a consequence of the following

Theorem 1. *Let $(\mathcal{K}, d)$ and $(\mathcal{L}, \varrho)$ be two compact metric spaces. Let $f\colon \mathcal{K} \to \mathcal{L}$ be α-Hölderian for $\alpha \in (0, 1\rangle$, i.e., there is a constant $L > 0$ such that*

$$\rho\big(f(x), f(y)\big) < L\,[d(x,y)]^\alpha \tag{1}$$

for any $x \neq y$. Then $\overline{\dim}\, f(\mathcal{K}) \leqslant \frac{1}{\alpha}\overline{\dim}\,\mathcal{K}$.

Proof: Take $r > 0$ and let $\mathfrak{C}(r)$ be a minimal r-covering of $\mathcal{K}$. Then by (1) we have $\operatorname{diam} f(C) < L \cdot r^\alpha$ for every $C \in \mathfrak{C}(r)$ and, moreover, all sets $f(C)$ cover $f(\mathcal{K})$. Each $f(C)$ has a neighbourhood with diameter less than Lr^α. Therefore $N\big(Lr^\alpha, f(\mathcal{K})\big) \leqslant N(r, \mathcal{K})$ and

$$\begin{aligned}\overline{\dim}\, f(\mathcal{K}) &= \limsup_{r\to 0^+} \frac{\log N\big(r, f(\mathcal{K})\big)}{-\log r} = \limsup_{r\to 0^+} \frac{\log N\big(Lr^\alpha, f(\mathcal{K})\big)}{-\log Lr^\alpha} \leqslant \\ &\leqslant \limsup_{r\to 0^+} \frac{\log N(r,\mathcal{K})}{-\alpha\log r - \log L} = \frac{1}{\alpha}\limsup_{r\to 0^+} \frac{\log N(r,\mathcal{K})}{-\log r} = \tfrac{1}{\alpha}\overline{\dim}\,\mathcal{K}\,.\end{aligned}$$

□

Remark 1. We will use the previous theorem for a Lipschitzian ($\alpha = 1$) surjective mapping $f\colon \mathcal{K} \to \mathcal{L}$, in this case we have $\overline{\dim}\,\mathcal{L} \leqslant \overline{\dim}\,\mathcal{K}$.

Proof of Proposition 1: Define $\pi\colon S \to D$ by $\pi(s_0) = 0$ and $\pi(s_n) = d_n$ for $n \in \mathbb{N}$. It is easy to see that for each $x, y \in S$ we have the inequality $|\pi(x) - \pi(y)| \leqslant d(x, y)$. Now it is sufficient to apply Theorem 1 to get the result. □

We can define the algebraic operations of addition and multiplication of two converging sequences as well as the scalar multiplication of a converging sequence — this can be considered as a particular case of multiplication with a constant sequence. Hence, if $A = \{a_n\}$ and $B = \{b_n\}$ are two converging sequences, then

$$A + B = \{a_n + b_n\}\,,$$
$$A \cdot B = \{a_n \cdot b_n\}\,,$$

and, for $\alpha \in \mathbb{R}$,

$$\alpha \cdot A = \{\alpha \cdot a_n\}\,.$$

First we will consider the behaviour of the dimension with respect to these operations.

Proposition 2. *Let $A = \{a_n\}$, $B = \{b_n\}$ be two converging sequences of real numbers and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then the relations*

$$\overline{\dim}\,(A+B) \leqslant \overline{\dim}\,A + \overline{\dim}\,B\,,$$
$$\overline{\dim}\,(A\cdot B) \leqslant \overline{\dim}\,A + \overline{\dim}\,B\,,$$

and

$$\overline{\dim}\,A = \overline{\dim}\,\alpha A$$

hold.

Proof: First note that the inequality $\overline{\dim}\,(A\times B) \leqslant \overline{\dim}\,A + \overline{\dim}\,B$ holds for compact $A, B \subset \mathbb{R}$. Indeed, if $\mathfrak{A}(r)$ and $\mathfrak{B}(r)$ are minimal r-coverings of the sets A and B, respectively, then $\mathfrak{C} = \{\mathcal{A}\times\mathcal{B};\ \mathcal{A}\in\mathfrak{A}(r),\ \mathcal{B}\in\mathfrak{B}(r)\}$ is a $(\sqrt{2}\cdot r)$-covering of $A\times B \subset \mathbb{R}^2$. Hence $N(\sqrt{2}\cdot r, A\times B) \leqslant N(r,A)\cdot N(r,B)$ and by the properties of lim sup the statement is obtained.

Moreover, $\overline{\dim}\,A \leqslant \overline{\dim}\,B$ if $A\subset B$ and A, B are compact (see [M-Z, Proposition 3 (ii)]). Put $D = \{(a_n, b_n)\} \subset A\times B$ and apply the above facts to get $\overline{\dim}\,D \leqslant \overline{\dim}\,A\times B \leqslant \overline{\dim}\,A + \overline{\dim}\,B$. Define $\pi\colon D \to A+B$ by $\pi\big((a_n,b_n)\big) = a_n + b_n$ and note that

$$|(a_n+b_n)-(a_m+b_m)| \leqslant |a_n-a_m| + |b_n-b_m| \leqslant \sqrt{2}\sqrt{(a_n-a_m)^2+(b_n-b_m)^2}\,.$$

Applying Theorem 1 we get the first inequality.

Now define $\pi'\colon D\to A\cdot B$ by $\pi'\big((a_n,b_n)\big) = a_n\cdot b_n$ and choose some $\alpha,\beta>0$ with $\alpha > a_n$ and $\beta > b_n$ for each $n\in\mathbb{N}$. Now the following inequalities are true for each $m,n\in\mathbb{N}$:

$$\begin{aligned}
&|a_nb_n - a_mb_m| \leqslant\\
&\quad\leqslant |a_nb_n - a_mb_m - (a_nb_m - a_mb_n)| + |a_nb_n - a_mb_m + (a_nb_m - a_mb_n)| =\\
&= |(a_n+a_m)(b_n-b_m)| + |(a_n-a_m)(b_n+b_m)| \leqslant 2\alpha|b_n-b_m| + 2\beta|a_n-a_m| \leqslant\\
&\qquad\qquad\leqslant 2\sqrt{2}(\alpha+\beta)\sqrt{(a_n-a_m)^2+(b_n-b_m)^2}\,.
\end{aligned}$$

Applying Theorem 1 we get the second inequality.

For the proof of the third equality define $B=\{b_n\}$ by $b_n=\alpha$ for each $n\in\mathbb{N}$ and note that dimension of finite space is 0. This follows $\overline{\dim}\,\alpha A = \overline{\dim}\,B\cdot A \leqslant \overline{\dim}\,A$. Now define $C=\{c_n\}$ by $c_n=\frac{1}{\alpha}$ for each $n\in\mathbb{N}$. Thus $\overline{\dim}\,A = \overline{\dim}\,C\cdot(\alpha A) \leqslant \overline{\dim}\,\alpha A$ and the proof is finished. ☐

The following Corollary is a straightforward application of Proposition 2.

Corollary 1. *Let $A = \{a_n\}$, $B = \{b_n\}$ be converging real sequences and $\overline{\dim}\, B = 0$. Then $\overline{\dim}\,(A + B) = \overline{\dim}\, A$ and $\overline{\dim}\,(A \cdot B) \leqslant \overline{\dim}\, A$.*

Theorem 1 implies that the dimension is invariant under Lipschitzian homeomorphisms. That is the reason why in the rest of the paper we will consider only the sequences converging to zero. Note that $\overline{\dim}\, A = \overline{\dim}\ - A$ and $\overline{\dim}\,(A \cup B) = \max\{\overline{\dim}\, A, \overline{\dim}\, B\}$, so we can restrict to positive sequences. Moreover, the dimension of the sequence is the dimension of the set of the values of the sequence. That is why we will consider only the set $\mathcal{S}^+$ of positive strictly decreasing sequences converging to zero, where we do not distinguish sequences which differ on the finite set of indexes. Define a partial order on $\mathcal{S}^+$ in the usual way:

$$\{a_n\} \preccurlyeq \{b_n\} \iff \{n;\, a_n > b_n\} \text{ is finite}.$$

The relations between dimension and this partial order will be considered in the next part of the paper.

Denote $A_k = \{\frac{1}{n^k}\}_{n=1}^{\infty}$. By [Y] $\overline{\dim}\, A_k = \frac{1}{k+1}$ and $A_k \preccurlyeq A_l$ for $k \geqslant l$, so that smaller sequence has smaller dimension. The sequence $\{\frac{1}{2^n}\}$ has $\overline{\dim}$ zero and is smaller than any A_n above. On the contrary the sequence $\{\frac{1}{\log n}\}$ is greater than any previous sequence and its $\overline{\dim}$ is 1 (see, e.g., [Z]). It might seem that greater sequences have greater dimensions, but the relations are much more complicated.

Lemma 1. *For each $\{a_n\} \in \mathcal{S}^+$ there is a $\{b_n\} \in \mathcal{S}^+$ with $\{b_n\} \succcurlyeq \{a_n\}$ and $\overline{\dim}\,\{b_n\} = 0$.*

Proof: By induction we will construct two sequences $\{r_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ such that $r_k \searrow 0$, $n_k \nearrow \infty$:
Put $r_1 = \frac{1}{3}$ and choose an n_1 such that $a_{n_1} < r_1$. Take an $r_2 < r_1$ for which $\left(\frac{1}{r_2}\right)^{\frac{1}{2}} > n_1 + 2$ and an n_2 such that $a_{n_2} < r_2$.
Suppose we have chosen all r_{k-1} and n_{k-1}, for some $k \geqslant 3$. Then we can find r_k with

$$\left(\frac{1}{r_k}\right)^{\frac{1}{k}} > n_{k-1} + 2\,, \tag{1}$$

and, moreover, satisfying the inequality

$$r_k < r_{k-2} - a_{n_{k-2}}\,. \tag{2}$$

Finally, there exists also an n_k such that

$$a_{n_k} < r_k\,. \tag{3}$$

Choose an arbitrary sequence $b_n \searrow 0$, for which

$$b_{n_{k+1}-1}, \dots, b_{n_k} \in (a_{n_k}, a_{n_k} + r_{k+2}). \tag{4}$$

Let $r \in \langle r_{k+1}, r_k)$. Then

$$N\left(r_{k+1}, \bigcup_{i=n_{k+1}}^{\infty} \{b_i\}\right) = 1\,,$$

because from (4) and (2) we have $b_{n_{k+1}} < a_{n_{k+1}} + r_{k+3} < r_{k+1}$;

$$N\left(r_{k+1}, \bigcup_{i=n_k}^{n_{k+1}-1} \{b_i\}\right) = 1\,,$$

as $a_{n_k} < b_{n_{k+1}-1} < \cdots < b_{n_k} < a_{n_k} + r_{k+2}$ and $r_{k+2} < r_{k+1}$;

$$N\left(r_{k+1}, \bigcup_{i=n_{k-1}}^{n_k-1} \{b_i\}\right) = 1$$

from the same reason and

$$N\left(r_{k+1}, \bigcup_{i=1}^{n_{k-1}-1} \{b_i\}\right) \leqslant n_{k-1} - 1\,.$$

Hence by (1) and the choice of r,

$$N(r, \{b_i\}) \leqslant 1 + 1 + 1 + n_{k-1} - 1 = n_{k-1} + 2 < \left(\frac{1}{r_k}\right)^{\frac{1}{k}} < \left(\frac{1}{r}\right)^{\frac{1}{k}}.$$

Therefore $\overline{\dim}\left(\bigcup_{i=1}^{\infty}\{b_i\}\right) = 0.$ □

Theorem 2. *For each $\{a_n\} \in \mathcal{S}^+$ and for each $\alpha \in \langle 0, 1\rangle$ there is $\{b_n\} \in \mathcal{S}^+$ with $\{b_n\} \succcurlyeq \{a_n\}$ and $\overline{\dim}\{b_n\} = \alpha$.*

Proof: Let $\alpha \in \langle 0, 1\rangle$ and $A = \{a_n\}$ be given. Lemma 1 states the existence of a greater sequence D with dimension zero and [M-Z, Theorem 4] guarantees the existence of a sequence $C = \{c_n\}$ with $\{c_n\} \in \mathcal{S}^+$ and $\overline{\dim}\, C = \alpha$. Then $B \equiv D + C \succcurlyeq D \succcurlyeq A$ and by Corollary 1 $\overline{\dim}\, B = \overline{\dim}\, C = \alpha$. □

The above fact implies that there are arbitrary big sequences with prescribed dimension. This is not true for smaller sequences (see Theorem 4 and Remark 2 below). On the other hand, if two sequences are in some sense mutually uniformly distributed, then their dimensions are equal. This is the content of the following theorem.

Theorem 3. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be two sequences from $\mathcal{S}^+$ and let there be a constant $\ell \in \mathbb{N}$ such that for every $k > 0$*

$$\bigl|\{i;\ a_{k+1} < b_i < a_k\}\bigr| < \ell\,.$$

Then $\overline{\dim}\, A \geqslant \overline{\dim}\, B$.

Proof: For $r > 0$ let $\mathfrak{A}(r)$ be a covering of A by $N(r, A)$ sets of diameter less than r. Denote $m = N(r, A) - 1$. Then there are numbers $n_1, n_2, \dots, n_m$ such that each interval $\langle a_{n_1}, a_1\rangle$, $\langle a_{n_2}, a_{n_1+1}\rangle$, $\langle a_{n_m}, a_{n_{m_1}+1}\rangle, \dots, (0, a_{n_m+1}\rangle$ is a subset of one set from $\mathfrak{A}(r)$. This covering covers also all points of the sequence B except for points in the intervals $(a_{n_1+1}, a_{n_1}), \dots, (a_{n_m+1}, a_{n_m})$. Their number is at most $(\ell - 1)m$, therefore $N(r, B) \leqslant m + 1 + (\ell - 1)m < \ell(m + 1) = \ell \cdot N(r, A)$. From this we derive $\overline{\dim}\, B \leqslant \overline{\dim}\, A$. □

Example. Let $\{a_n\}$ be a sequence in $\mathbb{R}^+$, monotonically converging to zero, and $\overline{\dim}\, \{a_n\} = \alpha$. Let $\{b_n\}$ be a subsequence of $\{a_n\}$ with $b_n = a_{7n}$. What we can say about $\overline{\dim}\, \{b_n\}$?

First note that if we denote $A = \{a_n\}$ and $B = \{b_n\}$, then $B \subset A$ and by [M-Z, Proposition 3] we obtain the inequality $\overline{\dim}\, B \leqslant \overline{\dim}\, A$.

Now, for any natural $k \geqslant 1$ there are exactly seven elements $a_{7(k+1)}, a_{7(k+1)-1}, \dots$ $\dots, a_{7k}$ of A between b_{k+1} and b_k. By the above theorem we have then $\overline{\dim}\, A \leqslant \overline{\dim}\, B$, the opposite inequality.

The answer is: $\overline{\dim}\, \{b_n\} = \alpha$.

Theorem 4. *Let for a set A of elements of a series $\sum_{n=1}^{\infty} a_n$, $a_n > 0$, the inequality $\overline{\dim}\, A > 1/2$ hold. Then the series is divergent.*

Proof: To simplify the proof we will use the equivalent definition (see [K-T] and [M-Z] for more details) of the upper dimension of a given compact $\mathcal{K}$:

$$\overline{\dim}\, \mathcal{K} = \limsup_{r \to 0^+} \frac{\log M(r, \mathcal{K})}{-\log r}\,, \tag{4}$$

where $M(r, \mathcal{K})$ for $r > 0$ means the maximal cardinality of r-discrete subsets of $\mathcal{K}$. $X \subset \mathcal{K}$ is called r-discrete iff $\inf\{d(x, y);\ x, y \in X, x \neq y\} \geqslant r$.

By the assumption of the theorem $\overline{\dim}\, A > 1/2 + \alpha$, where $0 < \alpha < 1/2$. Then by (4) for every positive $r_0 \in \mathbb{R}$ there exists a number r, $0 < r < r_0$, such that

$$M(r, A) \geqslant \left(\frac{1}{r}\right)^{1/2 + \alpha}. \tag{5}$$

Fix this r for a moment and denote $m = M(r, A)$. Let $A(r)$ be a subset of A corresponding to $M(r, A)$, so that if $x, y \in A(r)$ and $x > y$, then $x - y \geqslant r$. This set is finite, hence it can be ordered as an increasing sequence

$$0 \leqslant a_1(r) < a_2(r) < \cdots < a_m(r)\,,$$

where $a_i(r) - a_{i-1}(r) \geqslant r$, $i = 2, \dots, m$. By this property we obtain inequalities

$$a_j(r) \geqslant (j-1)r\,, \qquad j = 1, 2, \dots, m\,.$$

Further, using these inequalities and (5),

$$a_1(r) + a_2(r) + \dots + a_m(r) \geqslant r\big(1 + 2 + \dots + (m-1)\big) = r\frac{m(m-1)}{2} \geqslant$$

$$\geqslant r\frac{(\frac{1}{r})^{1/2+\alpha}\left((\frac{1}{r})^{1/2+\alpha} - 1\right)}{2} = \frac{1}{2}\left(r^{-2\alpha} - r^{(1/2-\alpha)}\right)\,.$$

As $1/2 - \alpha > 0$, the last expression tends to infinity for $r \to 0^+$. Therefore for any $\ell > 0$, there is a sufficiently small $r > 0$ such that $\sum\limits_{k=1}^{s} a_k > \ell$, where $a_s \equiv a_1(r)$ for this r. This means that the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent. $\square$

Remark 2. Theorem 2 states the existence of a greater sequence to a given sequence from $\mathcal{S}^+$ with an arbitrary dimension between 0 and 1. Theorem 4 in [M-Z] guarantees the existence of a smaller sequence with dimension less than or equal to a dimension of a given sequence. But the situation is not wholly symmetrical — Theorem 4 in contrast to Theorem 2 prevents the existence, in general case, of a smaller sequence with an arbitrary dimension near to 1. Indeed, let $A = \{a_k\}_{k=1}^{\infty}$ be a sequence such that $\sum\limits_{k=1}^{\infty} a_k < \infty$. Then for every sequence $B = \{b_k\}_{k=1}^{\infty}$, $B \preccurlyeq A$, we have $\sum\limits_{k=1}^{\infty} b_k < \infty$ and by Theorem 4 we have $\overline{\dim}\, B \leqslant 1/2$.

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