On the exterior steady problem for the equations of a viscous isothermal gas

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Abstract. We prove existence and a representation formula for solutions to the equations describing steady flows of an isothermal, viscous, compressible gas having a positive infimum for the density ϱ, moving in an exterior domain, when the speed of the obstacle and the external forces are sufficiently small.

Keywords: compressible flows, existence of steady solutions, exterior domains

Classification: 76N, 35Q

1. Introduction.

In this paper we shall be concerned with well posedness questions for steady flows of an ideal isothermal viscous compressible gas. In order of clarity, we shall investigate, in detail, only the steady flows occurring at the exterior of a compact set, moving at some constant speed. Here, we shall limit ourselves to sketch the full lines of the proof, paying attention in outlining the main difficulties. However, more complete and detailed proofs are provided in Novotný & Padula (forthcoming). Introducing a reference system $\mathscr{R}\ :\ \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, with the origin 0 inside the obstacle and setting $m\mathbf{v}_\infty := \ell e_1$, with $\mathbf{v}_\infty$ velocity of the fluid at infinity and m a given constant value of the density, the above problem is governed by the following system

$$
\begin{aligned}
&\mu \triangle \hat{\mathbf{v}} + (\lambda+\mu)\nabla\nabla\cdot\hat{\mathbf{v}} = \varrho\hat{\mathbf{v}}\cdot\nabla\hat{\mathbf{v}} + \nabla\hat{p} - \varrho\mathbf{f}, && \\
&\nabla\cdot(\varrho\hat{\mathbf{v}}) = 0, && \text{in } \Omega; \\
&\hat{\mathbf{v}}\,|_\Sigma = 0 && \\
&\varrho(x) \to m && \\
&\hat{\mathbf{v}}(x) \to (\ell/m)\mathbf{e}_1 && \text{as } |x| \to \infty.
\end{aligned}
\tag{1.1}
$$

Here, ϱ and $\hat{\mathbf{v}}$ are the density and the velocity, the constants μ and λ denote the shear and the bulk viscosities, furthermore, $p := \mathfrak{h}\varrho$, with $\mathfrak{h}$ the square of the sound speed, is the pressure, and Σ the boundary of Ω.

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In unbounded regions, the two classes of flows:

$$\inf_{x\in\Omega} \varrho \geq a > 0;$$
$$\inf_{x\in\Omega} \varrho \geq 0,$$

behave in a very different manner, cf. Padula (1992-a). The only existence results known in this field have been given for strictly positive densities, i.e. when $m \neq 0$. In this regard, we quote the paper of H. Fujita Yashima (1985) concerning the case $\Omega = \mathbb{R}^4$ for $p = \mathfrak{h}\varrho$, and $(\lambda + 2\mu)$ sufficiently large; that of Matsumura (1986) where a representation for the solution of the linearized problem is given when $\mathbf{v}_\infty \neq 0$, that of Matsumura & Nishida (1989) in which the full problem (1.1) is solved for $\mathbf{v}_\infty = 0$; furthermore, Padula (1992-b), in the whole of $\mathbb{R}^3$ proved existence, when $\mathbf{v}_\infty \neq 0$ or $\mathbf{v}_\infty = 0$, and gave a representation formula, when $\mathbf{v}_\infty = 0$, for the full nonlinear problem. Only very recently, Novotný and Padula solved also the problem of existence of steady flows of viscous gases in domains which are exterior to a compact, either fixed or slowly moving obstacle. This last proof is essentially a consequence of a careful analysis of the difference between a compressible and incompressible fluid. Here, for the same problem, we shall give a somewhat different proof of existence and we shall provide a representation formula for the solutions to (1.1) in $\Omega \subseteq \mathbb{R}^3$. This is achieved supposing $m \neq 0$ in $(1.1)_3$, that is, strictly positive densities, considering both the possibilities $\mathbf{v}_\infty \neq 0$ and $\mathbf{v}_\infty = 0$ in the condition $(1.1)_4$. With the exception of an additional lemma which allow us to shorten the iterative procedure of Novotný & Padula (forthcoming), here we shall strictly follow the lines proposed by these authors and refer to it for the more extended proofs.

The plan of the paper is the following one. After introducing notations and recalling preliminary lemmas (Section 2), in a regular exterior domain Ω, in Section 3, we shall prove existence in Sobolev spaces, when $\mathbf{v}_\infty \neq 0$ and in Section 4 we prove existence in the class of physically reasonable solutions when $\mathbf{v}_\infty = 0$.

In order to state the results explicitly, it is better first to introduce some functional spaces. Let

$$k = 0, 1, 2, \ldots, 1 < q < 3/2, 3 < p.$$

We mean by $\mathbf{W}^{k,q}$ the usual Sobolev space, with the norm

$$\|\varphi\|_{k,q} := \Big(\int \sum_{|\alpha|=0}^{k} |D^\alpha \varphi|^q\Big)^{1/q}, \quad \|\varphi\|_q := \|\varphi\|_{0,q}\,.$$

$\mathbf{W}^{k,q,p} := \mathbf{W}^{k,q} \cap \mathbf{W}^{k,p}$ represents the Banach space with the norm

$$\|\varphi\|_{k,q,p} := \|\varphi\|_{k,q} + \|\varphi\|_{k,p}\,. \tag{1.2}$$

Also, for domains $\Omega \in \mathscr{C}^2$ we denote by $\mathbf{W}^{1-1/q,q}(\Sigma)$ the usual Sobolev space with fractional derivatives at the boundary Σ of Ω, the norm will be denoted by $|\varphi|_{1-1/q,q,\Sigma}$.

Now, $\mathrm{H}_0^{k,q}$ (resp. $\mathrm{H}^{k,q}$) denotes the homogeneous space obtained by completion of C_0^∞ (resp. $\mathrm{C}_0^\infty(\overline{\Omega})$) in the norm

$$|\varphi|_{k,q} := \Big(\int \sum_{|\alpha|=k} |D^\alpha \varphi|^q\Big)^{1/q}, \tag{1.3}$$

furthermore, $\mathrm{H}^{-1,q}$ is the dual space of $\mathrm{H}_0^{1,q'}$, $q' = q/(q-1)$, and its norm is denoted by $|\cdot|_{-1,q}$.

It turns useful, for $s_q = 4q/(4-q)$, to introduce the norm

$$\begin{aligned} \dashv\!\|\varphi\|\!\vdash_{\ell,q} &:= \|\nabla\varphi\|_q + |\ell|^{1/4}\|\varphi\|_{s_q} \\ \dashv\!\|\varphi\|\!\vdash_{\ell,k,q,p} &:= \|\nabla\varphi\|_{k,q,p} + |\ell|^{1/4}\|\varphi\|_{s_q}\,. \end{aligned} \tag{1.4}$$

The subspace of $\mathrm{H}_0^{1,q} \cap \mathrm{H}^{k+1,p}$ constituted by solenoidal functions with the norm $(1.4)_2$ finite is denoted by $\mathbb{V}_\ell^{k,q,p}$.

The subspace of $\mathrm{H}^{1,q} \cap \mathrm{H}^{k+2,p}$ constituted by the functions φ with $\triangle\,\varphi\,|_{\partial\Omega} = 0$ is denoted by $\mathbb{D}_\ell^{k,q,p}$.

For any $\mathbf{w}$ with $\nabla\mathbf{w} \in \mathbf{W}^{k,q,p}$, we set

$$\begin{aligned} \langle\langle\tau\rangle\rangle_p &:= \|\tau\|_p + \|\operatorname{div}(\tau\mathbf{w})\|_p \\ \langle\langle\tau\rangle\rangle_{k,q,p} &:= \|\tau\|_{k,q,p} + \|\operatorname{div}(\tau\mathbf{w})\|_{k,q,p}\,. \end{aligned} \tag{1.5}$$

Also, for $s_q = 4q/(4-q)$, the space $\mathscr{K}_\ell^{k,q,p}$ is defined as the completion of the functions in $\mathrm{C}^\infty \times \mathrm{C}_0^\infty$ in the norm

$$\begin{aligned}](\sigma,\mathbf{v})[_p &:= \|\sigma\|_p + \dashv\!\|\mathbf{v}\|\!\vdash_{\ell,p} \\ ](\sigma,\mathbf{v})[_{\ell,k,q,p} &:= \|\sigma\|_{k,q,p} + \dashv\!\|\mathbf{v}\|\!\vdash_{\ell,k,q,p}\,. \end{aligned} \tag{1.6}$$

For $\mathbf{v}_\infty = 0$ we define by $\mathscr{K}_*^{k,q,p}$ the Banach space completion of $\mathrm{C}^\infty \times \mathrm{C}_0^\infty$ with the norm

$$](\sigma,\mathbf{v})[^*_{k,q,p} :=](\sigma,\mathbf{v})[_{0,k,q,p} + \||x|\sigma(x)\|_q + \||x|\sigma(x)\|_p + \||x|\mathbf{v}(x)\|_\infty \tag{1.7}$$

For the external forces it is useful to introduce the following two spaces. For $1 < q < p$, $\mathscr{L}^{k,q,p} := \mathrm{H}^{-1,q} \cap \mathbf{W}^{k,p} \cap L^q$ is the Banach space equipped with the norm

$$\dashv\!\varphi\!\vdash_{k,q,p} := |\varphi|_{-1,q} + \|\varphi\|_q + \|\varphi\|_{k,p}\,. \tag{1.8}$$

In the sequel we shall take $q \le 6/5$.

For $1 < q_1 \le 3/2 \le q_2 \le 3$, $\mathscr{L}_{q_1,q_2}^{k,p}$ is the Banach space of functions having finite the following norm

$$\dashv\!\varphi\!\vdash^*_{k,q_1,q_2,p} := \||x|\varphi(x)\|_{q_1} + \||x|\varphi(x)\|_{q_2} + \|\varphi\|_{k,p}\,. \tag{1.9}$$

After the introduction of these spaces, we are now in a position to explicit more precisely the content of Sections 3 and 4.

Precisely, in Section 3, under suitable smallness assumptions on the force $\mathbf{f} \in \mathscr{L}^{k,q,p}$, $q < 6/5$, and on ℓ, $1/\mu$ and $1/\mathfrak{h}$, we prove that there exists one and only one solution $\varrho(= m+\sigma), \mathbf{v}$ to (1.1) such that $(\sigma, \mathbf{v}-\mathbf{v}_\infty) \in \mathscr{K}_\ell^{k,q,p}$. Such a solution has the norm controlled by the force and by ℓ.

Next, in Section 4, we consider the case $\ell = 0$. Specifically, we prove that there exists one and only one solution $(\sigma, \mathbf{v} - \mathbf{v}_\infty) \in \mathscr{K}_*^{k,q,p}$ corresponding to suitably small $\mathbf{f} \in \mathscr{L}_{q_1,q_2}^{k,p}$ and $1/\mu$ and $1/\mathfrak{h}$.

The solutions are obtained as limit of a sequence of approximating solutions. The novelty of the approach is due to a new iterative procedure which also allows us to make some considerations on the mathematical structure of the system governing steady flows. In particular, we prove that the local and global regularity and the asymptotic behavior, as well, of the solutions to (1.1) are *exactly the same* as that enjoyed by the solutions $\{\mathbf{v}, p\}$ of the incompressible Navier-Stokes system. Moreover, we provide for our solutions a representation formula which, as a consequence, furnishes also interior regularity result for the solutions to the equations of a compressible fluid. One main tool in our proof is an interior estimate for the pressure field of the Stokes problem which improves those already known, cf. e.g. Galdi (1992-c). The power of such approach finds its validity within the result of Padula & Pileckas (1992) wherein there is proved the existence of a steady solution also in domains with noncompact boundaries.

2. Auxiliary problems.

In this section we shall recall some basic lemmas concerning existence and a priori estimate for solutions to some linear elliptic and symmetric systems. B_l will be any ball of $\mathscr{K}_\ell^{k,q,p}$, $\mathscr{K}_0^{k,q,p}$ respectively, centered at the origin of radius l, and S_R will be a ball of $\mathbb{R}^3$ centered at the origin with radius R.

2.1. Generalized solutions for the Stokes and Oseen problems.

Consider the following nonhomogeneous linear problem, also known as Stokes, for $\ell = 0$, and Oseen, for $\ell \neq 0$, the problem

$$(2.1)\qquad \begin{aligned} &\triangle\, \mathbf{u} - \ell \frac{\partial \mathbf{u}}{\partial x_1} + \nabla p = -\mathbf{F} & \\ &\nabla \cdot \mathbf{u} = 0 & x \in \Omega, \\ &\mathbf{u}_{|\Sigma} = \mathbf{g} & \\ &\mathbf{u}(x) \to 0 & |x| \to \infty. \\ &\varrho(x) \to 0 & \end{aligned}$$

The system (2.2) has been extensively studied in both cases: $\ell = 0$ and $\ell \neq 0$ as well, cf. Galdi (1991), (1992-c), for a clear description of the full problem.

Lemma 2.1. (a) *Let* $\ell = 0$, $k = 0, 1, 2, \ldots$, $3/2 < q < 3$, $p > 3$, $\Omega \in \mathscr{C}^2$ *and*

$$\mathbf{F} \in \mathrm{H}^{-1,q},\ \mathbf{g} \in \mathbf{W}^{1-1/q,q}(\Sigma); \quad \mathbf{F} \in \mathscr{L}^{k,q,p},\ \mathbf{g} \in \mathbf{W}^{k+2-1/p,p}(\Sigma) \ \ \textit{resp.}$$

Then there exists just one solution of the problem (2.1), *with* $\ell = 0$,

$$u \in \mathrm{H}^{1,q},\ p \in \mathrm{L}^q; \quad u \in \mathbb{V}^{2,q} \cap \mathrm{H}^{k+2,p},\ p \in \mathbf{W}^{1,q} \cap \mathbf{W}^{k+1,p} \ \ \textit{resp.}$$

which satisfies the estimates

(2.2)
$$\|\nabla \mathbf{u}\|_q + \|p\|_q \le c\big[|\mathbf{g}|_{1-1/q,q,\Sigma} + |\mathbf{F}|_{-1,q}\big]$$
$$\|\nabla \mathbf{u}\|_q + \|p\|_q + \|\nabla \mathbf{u}\|_{k+1,p} + \|\varrho\|_{k+1,p} \le c\big[|\mathbf{g}|_{k+2-1/p,p,\Sigma} + \dashv\!\mathbf{F}\!\vdash_{k,q,p}\big], \ \ \textit{resp.}$$

(b) *Let* $\ell \neq 0$, $k = 0, 1, 2, \dots$, $3/2 < q < 3$, $3 < p$, $s_q = 4q/(4-q)$, $\Omega \in \mathscr{C}^2$, $\Omega \in \mathscr{C}^{k+2}$ *resp., and let*

$$\mathbf{F} \in \mathrm{H}^{-1,q},\ \mathbf{g} \in \mathbf{W}^{1-1/q,q}(\Sigma); \quad \mathbf{F} \in \mathscr{L}^{k,q,p},\ \mathbf{g} \in \mathbf{W}^{k+2-1/p,p}(\Sigma) \ \ \textit{resp.}$$

Then there exists just one solution of the problem (2.1), *with* $\ell \neq 0$,

$$\mathbf{u} \in \mathrm{L}^{s_q} \cap \mathrm{H}^{1,q},\ p \in \mathrm{L}^q; \quad u \in \mathbb{V}^{2,q} \cap \mathrm{H}^{k+2,p},\ p \in \mathbf{W}^{1,q} \cap \mathbf{W}^{k+1,p} \ \ \textit{resp.}$$

Moreover, $\mathbf{u}$, ϱ *verify the estimates*

(2.3)
$$\|\nabla \mathbf{u}\|_q + \ell^{1/4}\|\mathbf{u}\|_{s_q} + \|p\|_q \le c\big[|\mathbf{g}|_{1-1/q,q,\Sigma} + |\mathbf{F}|_{-1,q}\big]$$
$$\|\nabla \mathbf{u}\|_q + \|\nabla \mathbf{u}\|_{k+1,p} + \ell^{1/4}\|\mathbf{u}\|_{s_q} + \|p\|_q + \|p\|_{k+1,p} \le c\big[|\mathbf{g}|_{k+2-1/p,p,\Sigma} + \dashv\!\mathbf{F}\!\vdash_{k,q,p}\big].$$

The constant c *depends on* k, q, p *and* ℓ. *However, if* $q \in (1, 3/2)$ *and* $\ell \in (0, B]$ *for some* $B > 0$, c *depends solely on* k, q, p *and* B.

The next lemma furnishes an estimate for the negative norm of ϱ in terms of $\mathbf{F}$ only and not of $\mathbf{g}$.

Lemma 2.2. *Let* $\{\mathbf{u}, \varrho\}$ *be a solution to the problem* (2.1) *corresponding to* $\mathbf{F} \in \mathrm{L}^p$, $1 < p < \infty$. *Then, it holds*

$$\begin{aligned} &\langle \nabla p \rangle_{-1,p} := \sup |(\nabla p, \nabla \phi)|/|\phi|_{2,p} \le c\|\mathbf{F}\|_p \\ &| \triangle p|_{k-1,p} \le c\|\mathbf{F}\|_{k,p} \end{aligned} \tag{2.4}$$

where the supremum is taken over all $\phi \in \mathrm{H}^{2,p}_0$.

Proof: From the weak formulation of the problem (2.1), choosing the test function as the gradient of a function ϕ in C^∞_0 yields

$$(\nabla p, \nabla \psi) = (\mathbf{F}, \nabla \psi).$$

□

Remark 2.1. The seminorm defined by $(2.4)_1$ is equivalent to the usual L^p-norm only for the functions which have zero value on the boundary. In fact, set $\triangle\,\psi = h$, $h\,|_\Sigma = 0$, for $h \in \mathrm{L}^{p'}$ it holds

$$\|p\|_p = \sup \frac{|(p,h)|}{|\|h\|_{p'}|} = \sup \frac{|(p,\triangle\,\psi)|}{|\|\nabla\nabla\psi\|_{p'}|} = \sup \frac{|(\nabla p,\nabla\psi)|}{|\|\nabla\nabla\psi\|_{p'}|}\,.$$

Corollary. *Let* $\mathbf{w} \in \mathrm{H}_0^{1,p} \cap \mathrm{H}^{2,p}$, $\sigma \in \mathbf{W}^{1,p}$, $\mathbf{w}\cdot\nabla\sigma \in \mathbf{W}^{1,p}$, $p > 3$, $\operatorname{div}\mathbf{w} = 0$ *on* Σ, *thus*

$$\operatorname{div}(\sigma\mathbf{w})\,|_\Sigma = 0.$$

PROOF: Here we just give the proof for a plane, bounded boundary $\Sigma = \pi(0)$, with z normal direction. We put $\psi_\varepsilon(z) = 0$, if $|z| > 2\varepsilon$, $\psi_\varepsilon(z) = 1$, if $|z| < \varepsilon$, $|\nabla\psi_\varepsilon(z)| < \varepsilon^{-1}$. We now observe that $|\mathbf{w}(x)| < \varepsilon$ for $|x| < \varepsilon$, because $\mathbf{w}$ on Σ is lipschitzian. From the regularity properties of $\mathbf{w}$ it follows that $|\nabla\psi_\varepsilon\mathbf{w}|$ is bounded in ε. Therefore, from the identity

$$\int_\Sigma |\mathbf{w}\cdot\nabla\sigma|^p = \int_\Sigma |\psi_\varepsilon\mathbf{w}\cdot\nabla\sigma|^p = \int_0^\varepsilon\int_{\pi(t)} p(\psi_t\mathbf{w}\cdot\nabla\sigma)^{p-1}\Big(\frac{\partial}{\partial t}(\psi_t\mathbf{w}\cdot\nabla\sigma)\Big)$$

using Poincaré inequality we easily recover the lemma. □

In particular, under the assumption of Corollary, it results $\operatorname{div}(\sigma\mathbf{w}) = 0$ on Σ, and

$$\|\operatorname{div}(\sigma\mathbf{w})\|_p \le \mathrm{C}\langle\nabla\operatorname{div}(\sigma\mathbf{w})\rangle_{-1,p}\,. \tag{2.5}$$

2.2. Generalized solutions for symmetric equations.

Next, in order to prove an estimate for the density ϱ, we are led to solve the following equation which is symmetric positive in the sense of Friedrichs (1958)

$$\begin{aligned} &\mathfrak{h}\omega + (\lambda+2\mu)\operatorname{div}(\omega\mathbf{w}) = \varrho,\\ &\mathbf{w}\cdot\nu\,|_\Sigma = 0. \end{aligned} \tag{2.6}$$

This problem was studied by several authors, here we shall report only a recent result proved in exterior domains, cf. Padula (1992), Novotný & Padula (forthcoming), Novotný (in preparation).

Lemma 2.3. *Let* $1 < q < 3$, $3 < p$, $k = 0,1,2,\dots$, $\Omega \in \mathscr{C}^{k+3}$. *Assume* $p \in \mathbf{W}^{k+1,p} \cap \mathbf{W}^{1,q}$, $\mathbf{w} \in \mathrm{H}_0^{1,q} \cap \mathrm{H}^{k+2,p}$, *and let there exist suitable* $\gamma > 0$ *such that* $\mathbf{w}$ *satisfies the following condition*

$$\|\mathbf{w}\|_{k+2,q,p} < \gamma. \tag{2.7}$$

Then there exists a unique solution ω *to the equation* (2.6) *in the space* $\mathbb{W}^{k+1,p} \cap \mathbb{W}^{1,q}$ *satisfying the following estimate*

$$\begin{aligned} &\langle\langle\omega\rangle\rangle_{k,q,p} \le (\mathrm{C}/\mathfrak{h})[\|p\|_{1,q} + \|p\|_{k,q,p}]\\ &\langle\nabla\operatorname{div}(\mathbf{w}\omega)\rangle_{-1,p} \le c\{\langle\nabla p\rangle_{-1,p} + \|\nabla\mathbf{w}\|_{1,q,p}\|\omega\|_{1,p}\}\\ &|\triangle\operatorname{div}(\mathbf{w}\omega)|_{k-1,p} \le c\{|\triangle\,p|_{k-1,p} + \|\nabla\mathbf{w}\|_{k+1,p}\|\omega\|_{1,p}\} \end{aligned} \tag{2.8}$$

with C *positive constant.*

PROOF: From known results on the symmetric positive equations, cf. Friedrichs (1958), da Veiga (1987), we can establish a one to one correspondence between ω and p in the space in which ω exists for bounded domains having the boundary as characteristic surface. Moreover, in the general case it has been studied for the first time by Padula (1992-b) and then systematically by Novotný (in preparation), therefore we shall omit the proof. □

2.3. On the estimates for the Dirichlet and Neumann problems.

As the last two linear problems we shall consider the Neumann and Dirichlet problems. Precisely, we start with the following Neumann problem

$$\begin{gathered} \triangle\,\varphi = G \\ \nabla\varphi\cdot\nu\,|_\Sigma = \phi\cdot\nu\,|_\Sigma \\ \varphi(x)\to 0 \qquad \text{as } |x|\to\infty \\ \int_\Omega G\,dx = \int_\Sigma \phi\cdot\nu\,d\Sigma. \end{gathered} \tag{2.9}$$

The result below provides some existence, uniqueness and estimates which will be used in the sequel.

Lemma 2.4. *Let* $\Omega\in\mathscr{C}^1$, $1<q<3$, $1<p<\infty$, *and* $k=0,1,2,\dots$, $\Omega\in\mathscr{C}^{k+1}$, $G\in(\mathrm{H}^{1,q'})^*$, $G\in(\mathrm{H}^{1,q'})^*\cap\mathrm{L}^q\cap\mathbf{W}^{k,p}$, *resp.*, $\phi=$ const. *Then there exists only one solution to* (2.9)

$$\varphi\in\mathrm{H}^{1,q};\quad \varphi\in\mathrm{H}^{1,q}\cap\mathbf{W}^{k+3,p}\ \ \textit{resp.}$$

which satisfies the estimate

$$\begin{gathered} |\phi|_{1,q}\le\{|G|_{-1,q}+|\phi|\} \\ |\nabla\phi|_{1,q}\le c\{|\nabla G|_{-1,q}+|\phi|\} \\ |\phi|_{1,q}+\|\nabla\varphi\|_{k+2,p}\le c\{|G|_{-1,q}+|\phi|\}. \end{gathered} \tag{2.10}$$

Next, consider the Dirichlet problem

$$\begin{gathered} \triangle\,\theta = G \\ \theta\,|_\Sigma = 0 \\ \theta(x)\to 0 \qquad \text{as } |x|\to\infty. \end{gathered} \tag{2.11}$$

Lemma 2.5. *Let* $3/2<q<3$, $\Omega\in\mathscr{C}^1$, $G\in\mathrm{H}^{-1,q}$. *Then there exists only one weak solution to* (2.9) $\varphi\in\mathrm{H}^{1,q}$, *which satisfies the estimate*

$$|\varphi|_{1,q}\le c|G|_{-1,q}\,. \tag{2.12}$$

As a consequence of Lemmas 2.4, 2.5 we can now state the following existence and uniqueness result for the Neumann problem below, cf. also Galdi (1992).

$$\begin{gathered} \triangle\,\varphi = G \\ (\nabla\varphi+\mathbf{v}_\infty)\cdot\nu\,|_\Sigma = 0 \\ \varphi(x)\to 0 \qquad \text{as } |x|\to\infty. \end{gathered} \tag{2.13}$$

Lemma 2.6. *Let* $3/2 < q < 3$, $3 < p$, $k = 0, 1, 2, \dots$, $\Omega \in \mathscr{C}^{k+3}$, *and, for any* $\mathbf{w}$ *with* $\nabla \mathbf{w} \in \mathbf{W}^{k+1,q,p}$, *let* $G \in \mathbf{W}^{k+1,q,p}$. *Then there exists only one solution to* (2.13) *with* $\nabla\varphi \in \mathrm{L}^r \cap \mathbf{W}^{1,q} \cap \mathbf{W}^{k+1,p}$, $r = 3q/(3-q)$ *such that*

$$|\varphi|_{1,r} + \|\nabla\varphi\|^{1,q} + \|\nabla\varphi\|_{k+1,p} \le c\big[\|G\|_{k,q,p} + \|G\|_{k+1,q,p}\|\mathbf{w}\|_{k+2,p} + \ell\big]. \tag{2.14}$$

2.4. The fundamental tensors for the Oseen and the Stokes problems.

As it is known, the fundamental tensors for the Oseen and Stokes problems admit the following compact and elegant form

$$\begin{aligned} \mathbf{U}_{ij}(x,y) &= \big(\delta_{ij} \,\triangle - \frac{\partial^2}{\partial x_i \partial x_j}\big)\mathscr{O}(x,y), \\ q_j(x,y) &= \frac{\partial}{\partial x_j}\big(\triangle + 2\ell\frac{\partial}{\partial x_1}\big)\mathscr{O}(x,y), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \mathscr{O}(x,y) &:= -\frac{1}{4\pi\ell}\int_0^s \frac{(1-e^{-\alpha})}{\alpha}\,d\alpha, \quad &\text{(Oseen)} \\ \mathscr{O}(x,y) &:= (|x-y|/8\pi). &\text{(Stokes)} \end{aligned}$$

Put

$$\mathbf{U}^i := \mathbf{U}\cdot\mathbf{e}_i, \quad q^i := \mathbf{q}\cdot\mathbf{e}_i\,.$$

If $\mathbf{u}$, ϱ is a solution to the problem (2.1) we have the following result, cf. Finn (1965), Galdi (1992-c).

Lemma 2.7. *Suppose* $\mathbf{F} \in \mathscr{C}^{\alpha}$, $\mathbf{g} \in \mathscr{C}^{\alpha}$. *Let* $\mathbf{u}, p$ *be the solution to* (2.1) given by Lemma 2.1. *Then the following representation in the large holds:*

$$\begin{aligned} u^i(x) &= \int_\Omega \mathbf{U}^i(x-y)\cdot\mathbf{F}(y)\,dy + \int_\Sigma \{\mathbf{u}\cdot\mathrm{T}\mathbf{U}^i - \mathbf{U}^i\cdot\mathrm{T}\mathbf{u} + (\mathbf{U}^i\cdot\mathbf{u})\mathbf{v}_\infty\}\,d\Sigma_y\,, \\ p(x) &= \int_\Omega \mathbf{q}(x-y)\cdot\mathbf{F}(y)\,dy + \int_\Sigma \{\mathbf{u}\cdot\mathrm{T}\mathbf{q} - \mathbf{q}\cdot\mathrm{T}\mathbf{u} + (\mathbf{q}\cdot\mathbf{u})\mathbf{v}_\infty\}\,d\Sigma_y\,. \end{aligned} \tag{2.16}$$

Here, corresponding to any vector fields $\mathbf{w}$ and scalar field p, the stress tensor $\mathrm{T}\mathbf{w}$ is defined by the relation

$$(\mathrm{T}\mathbf{w})_{i,j} := -p\delta_{i,j} + \Big(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\Big)\,.$$

For $\mathbf{F}$ with compact support it can be stated that the behavior of the generalized solution $\mathbf{u}, p$ at infinity can be controlled by that of the fundamental tensor $\mathbf{U}$, $\mathbf{q}$, cf. Finn (1965).

One of the main tools in the proof of the existence theorem in Section 4 will be the following basic estimate proved in Finn (1965).

Lemma 2.8. *There is a constant* $\hbar$*, uniform in each interval of* ℓ *such that*

$$|x| \int_\Omega |y|^{-2} |\nabla_y \mathbf{U}(x, y, \ell)| \, dy < \hbar. \tag{2.17}$$

Let $\mathrm{K}(x, y)$ denote any of the quantities in (2.15), we now set

$$\varphi(x) := \mathrm{T}(\phi) := \int_\Omega \mathrm{K}(x, y) \phi(y) \, dy. \tag{2.18}$$

The following results of potential theory can be proved, cf. Galdi (1992-c), Novotný & Padula (forthcoming):

Lemma 2.9. *Let* T *be given by* (2.18). *Then,* T *defines a linear continuous integral transform from* $\mathscr{A}$ *in* $\mathscr{B}$*, defined below.*

(i) *Let* $\mathrm{K}(x, y) \le c|x - y|^{-2}$, *and* $\mathscr{A} := \{\phi : |x|^2 |\phi(x)| \le c\}$, *then* $\mathscr{B} \equiv \mathrm{L}^\infty$ *and*

$$\| |x| \varphi \|_\infty \le c \| |x|^2 \phi \|_\infty .$$

(ii) *Let* $1 < q_1 \le 3/2 \le q_2 < 3$, $1 < p < \infty$, $k = 0, 1, \dots$, *let* $\mathrm{K}(x, y) \le c|x - y|^{-1}$, *and* $\mathscr{A} := \{\phi : |x| \phi \in \mathrm{L}^{q_1, q_2}\}$ *then* $\mathscr{B} \equiv \mathrm{L}^\infty$ *and*

$$\| |x| \varphi \|_\infty \le c \{ \| |x| \phi \|_{q_1} + \| |x| \phi \|_{q_2} \} .$$

(iii) *Let* $3/2 < q < 3 < p < \infty$, $\mathrm{K}(x, y) \le c|x - y|^{-2}$, *and* $\mathscr{A} := \{\phi : |x| \phi \in \mathrm{L}^{q_1, q_2}\}$. *Then* $\mathscr{B} \equiv \mathrm{L}^\infty$ *and*

$$\| |x| \varphi \|_\infty \le c \{ \| |x| \phi \|_q + \| |x| \phi \|_p \} .$$

3. Existence of steady flows when $\mathbf{v}_\infty \neq 0$.

We shall prove that, once the external forces are suitable, there exists a solution $\varrho := m + \sigma, \mathbf{v}$, with $(\sigma, \mathbf{v} - \mathbf{v}_\infty) \in \mathscr{K}_\ell^{k,q,p}$ to the problem (1.1), $m\mathbf{v}_\infty = \ell \mathbf{e}_1$, $\ell \neq 0$. Precisely, the main purpose of this section is the proof of the following

Theorem 3.1. *Let* $1 < r \le 6/5$, $3 < p$, $k = 0, 1, 2, \dots$, $\Omega \in \mathscr{C}^{k+2}$ *and* $\mathbf{f} \in \mathrm{L}^r \cap \mathbf{W}^{k,p}$. *Then there exist positive constants* ℓ^*, $\mathfrak{f}^*$ *functions of* r, p, k *only such that for*

$$0 \neq \ell < \ell^* \quad \dashv \mathbf{f} \vdash_{k,q,p} \le \mathfrak{f}^* \tag{3.1}$$

there exists one and only one solution $(\sigma \mathbf{v})$ *in the ball* $\mathscr{B}_R \subseteq \mathscr{K}_\ell^{k,q,p}$, $q = 3r/(3 - r)$ *to* (1.1) *satisfying the estimate*

$$](\sigma, \mathbf{v})[_{\ \ell,k,q,p} \le c^* [\|\mathbf{f}\|_r + \|\mathbf{f}\|_{k,p} + \ell] \tag{3.2}$$

where also c^* depends on ℓ, r, k, p.

PROOF: Let us first choose a suitable linearization of (1.1). To this end, we follow the lines of Novotný & Padula (forthcoming), precisely, using the velocity $\mathbf{v} := \hat{\mathbf{v}} - \mathbf{v}_\infty$, we first rewrite the system (1.1) in the more suitable form

$$
(3.3)\qquad
\begin{aligned}
&-\mu \triangle \mathbf{v} - (\lambda+\mu)\nabla \operatorname{div} \mathbf{v} + \frac{\ell}{m}\frac{\partial(\varrho \mathbf{v})}{\partial x_1} + \mathfrak{h}\nabla\varrho = \mathscr{F}'(\varrho,\mathbf{v}), \\
&\operatorname{div}(\varrho(\mathbf{v}+\mathbf{v}_\infty)) = 0 && \text{in } \Omega; \\
&\mathbf{v}\,|_\Sigma = -\mathbf{v}_\infty \\
&\mathbf{v}(x) \to 0 && \text{as } |x| \to \infty. \\
&\varrho(x) \to m
\end{aligned}
$$

with

$$\mathscr{F}'(\varrho,\mathbf{v}) := \varrho\mathbf{f} - \operatorname{div}(\varrho\mathbf{v}\oplus\mathbf{v}).$$

Next, we decompose the density and the velocity $(\varrho, \mathbf{v})$ into an incompressible $(m, \mathbf{u})$ and compressible $(\sigma, \nabla\varphi)$ part following the Helmholtz decomposition, by writing

$$\varrho := m + \sigma, \quad \mathbf{v} := \mathbf{u} + \nabla\varphi, \quad \operatorname{div}\mathbf{u} = 0.$$

Let us observe that, while the density m is exactly constant, we take $m = 1$ for simplicity, the component $\mathbf{u}$ of the velocity even though solenoidal (does not change the volume of the fluid), still remember the compressibility of the gas at the boundary Σ, in fact for the Helmholtz decomposition it holds

$$(3.4)\qquad \triangle\varphi = \operatorname{div}\mathbf{v}, \quad \frac{\partial\varphi}{\partial\nu}\,|_\Sigma = -\mathbf{v}_\infty\cdot\nu; \qquad \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}\,|_\Sigma = -\mathbf{v}_\infty - \nabla\varphi\,|_\Sigma\ .$$

In this way, the equation $(3.3)_2$ becomes

$$\operatorname{div}\mathbf{v} = -\operatorname{div}(\sigma(\mathbf{v}+\mathbf{v}_\infty)).$$

Therefore, integrating over Ω $(3.3)_2$, by Gauss lemma and by $(3.4)_2$, we deduce that the following compatibility must be satisfied

$$(3.5)\qquad \int_\Sigma (\nabla\varphi + \mathbf{v}_\infty)\cdot\nu\, d\Sigma = -\int_\Sigma \sigma(\nabla\varphi+\mathbf{v}_\infty)\cdot\nu\, d\Sigma = 0.$$

Thus, to prove existence we reduce ourselves to study the following three more simple linear problems:

(i) Neumann problem

$$
(3.6)\qquad
\begin{aligned}
&\triangle\varphi = G(\sigma, p, \nabla\varphi), \\
&\nabla\varphi\cdot\nu\,|_\Sigma = -\mathbf{v}_\infty\cdot\nu, \\
&\varphi(x) \to 0 \qquad \text{as } |x| \to \infty.
\end{aligned}
$$

(ii) Oseen problem

$$
(3.7) \qquad \begin{aligned}
&\triangle\, \mathbf{u} - \ell \frac{\partial \mathbf{u}}{\partial x_1} - \nabla p = -\mathbf{F}(\sigma, \mathbf{v}), \\
&\nabla \cdot \mathbf{u} = 0 && x \in \Omega, \\
&\mathbf{u}_{|\Sigma} = \mathbf{g} \\
&\mathbf{u}(x) \to 0 && |x| \to \infty. \\
&p(x) \to 0
\end{aligned}
$$

(iii) Symmetric system

$$
(3.8) \qquad \begin{gathered}
\mathfrak{h}\sigma + (\lambda + 2\mu)\,\mathrm{div}(\sigma(\mathbf{v} + \mathbf{v}_\infty)) = p - \lambda \frac{\partial \varphi}{\partial x_1}, \\
(\mathbf{v} + \mathbf{v}_\infty)\,|_\Sigma = 0.
\end{gathered}
$$

Here it is

$$
(3.9) \qquad \begin{aligned}
&G(\sigma, p, \nabla\varphi) := -\frac{1}{(\lambda + 2\mu)}\left[p - \mathfrak{h}\sigma - \ell \frac{\partial \varphi}{\partial x_1}\right] \\
&\mathbf{F}(\sigma, \mathbf{u}, \nabla\varphi) := +\,\mathrm{div}((1+\sigma)\mathbf{v} \oplus \mathbf{v} - (1+\sigma)\mathbf{f}) + \ell \frac{\partial(\sigma \mathbf{v})}{\partial x_1} \\
&\mathbf{g}(\nabla\varphi) := -\mathbf{v}_\infty - \nabla\varphi.
\end{aligned}
$$

The iterative procedure is at once naturally suggested from the three problems written above. In fact, in order to solve the full problem (3.6–3.9), it is enough to find a sequence $(\varphi_n, \mathbf{u}_n, p_n, \sigma_n)$ of solutions to the linearized system (3.6–3.8) when the quantities $G, \mathbf{F}, \mathbf{g}$ are given by

$$
(3.10) \qquad \begin{gathered}
\mathbf{v}_{n-1} := \mathbf{u}_{n-1} + \nabla\varphi_{n-1}; \qquad G = G(\sigma_{n-1}, p_{n-1}, \nabla\varphi_{n-1}) \\
\mathbf{F}(\sigma_{n-1}, \mathbf{v}_{n-1}) := +(1 + \sigma_{n-1})[\mathbf{v}_{n-1} \cdot \nabla \mathbf{v}_{n-1} - \mathbf{f}] + \ell \frac{\partial(\sigma_{n-1}, \mathbf{v}_{n-1})}{\partial x_1} \\
\mathbf{g} = \mathbf{g}(\nabla\varphi_{n-1}).
\end{gathered}
$$

Now, the iterative procedure starts by fixing at the step $n = 1$

$$
\mathbf{u}_0 = p_0 = \sigma_0 = 0, \quad \varphi_0 = \mathrm{const}\,.
$$

The existence of φ_1 follows from Lemma 2.4, when we take $G = 0$, $\phi = \mathbf{v}_\infty \cdot \nu$ and states that there exists only one $\varphi_1 \in \mathrm{H}^{1,q} \cap \mathbf{W}^{k+3,p}$ satisfying the estimate

$$
(3.11) \qquad |\varphi|_{1,q} + \|\nabla\varphi\|_{k+2,p} \leq c|\ell|.
$$

Moreover, the existence of a unique $\mathbf{u}_1, p_1$ follows from Lemma 2.1 when we put $\mathbf{F} = \mathbf{f}$, $\mathbf{g} = -\mathbf{v}_\infty - \nabla\varphi_1$ and assures that $\mathbf{u}_1 \in \mathbb{V}_q^{k+1,p,q}$, $p_1 \in \mathbf{W}^{1,q} \cap \mathbf{W}^{k+1,p}$,

$$
(3.12) \quad \|\nabla \mathbf{u}_1\|_q + \|\nabla \mathbf{u}_1\|_{k+1,p} + |\ell|^{1/4}\|\mathbf{u}_1\|_{s_q} + \|p_1\|_q + \|p_1\|_{k+1,p} \leq c[|\ell| + \dashv \mathbf{f} \vdash_{k,q,p}].
$$

Finally, Lemma 2.3 can be applied with $\mathbf{w} := \mathbf{v}_1 + \mathbf{v}_\infty$, $p := p_1 - \ell(\partial\varphi_1/\partial x_1)$. It provides the existence and uniqueness of $\sigma_1 \in \mathbb{W}^{k+1,p} \cap \mathbb{W}^{1,q}$ once the smallness assumption $\|\mathbf{v}_1 + \mathbf{v}_\infty\|_{k+2,q,p} < \gamma$ holds true. We are so led to assume *even at the first step* the assumptions (3.1) of Theorem 3.1

$$c\{\ell^* + \mathfrak{f}^*\} < \gamma.$$

Under the assumption (3.1), Lemma 2.3 holds and the solution σ_1 satisfies

$$\langle\langle\sigma_1\rangle\rangle_{k+1,q,p} \leq \frac{\mathrm{C}}{\mathfrak{h}}\left[\|p\|_{1,q} + \|p\|_{k+1,q,p}\right]. \tag{3.13}$$

The proof of the existence of $\varphi_1, \mathbf{u}_1, p_1, \sigma_1$ together with the estimates (3.11), (3.12), (3.13) is so complete.

By recurrency, now, we prove the existence of $\varphi_n, \mathbf{u}_n, p_n, \sigma_n$. To this end, assume what follows

$$\begin{gathered}\varphi_{n-1} \in \mathrm{H}^{1,q} \cap \mathbf{W}^{k+3,p},\ \mathbf{u}_{n-1} \in \mathbb{V}^{k+1,p,q},\\ p_{n-1}\mathbf{W}^{1,q} \cap \mathbf{W}^{k+1,p},\ \sigma_{n-1} \in \mathbb{W}^{k+1,p} \cap \mathbb{W}^{1,q}\end{gathered} \tag{3.14}$$

together with

$$\begin{aligned}|\varphi_{n-1}|_{1,q} + \|\nabla\varphi_{n-1}\|_{k+2,p} + \langle\langle\sigma_{n-1}\rangle\rangle_{k+1,q,p} + \|\nabla\mathbf{u}_{n-1}\|_q +&\\ + \|\nabla\mathbf{u}_{n-1}\|_{k+1,p} + |\ell|^{1/4}\|\mathbf{u}_{n-1}\|_{s_q} + \|\varrho_{n-1}\|_q + \|\varrho_{n-1}\|_{k+1,p} \leq \mathrm{R}.&\end{aligned} \tag{3.15}$$

We must prove that also $\varphi_n, \mathbf{u}_n, p_n, \sigma_n$ exist in the same space and verify the estimate (3.15) with the *same* constant R. Let us consider the three problems separately.

(i) In view of (3.14) and (3.8), it is easy to check that

$$G = \operatorname{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty)) \in (\mathrm{H}^{1,q'})^* \cap \mathrm{L}^q \cap \mathbf{W}^{k,p}.$$

Moreover, by assuming (3.15) true at the step $n-1$, it naturally follows that the compatibility condition $(2.9)_4$ holds true. Therefore, Lemma 2.4 allows us to state that there exists $\varphi_n \in \mathrm{H}^{1,q} \cap \mathbf{W}^{k+3,p}$ which satisfy the estimate

$$\begin{gathered}|\nabla\varphi_n|_{1,q} + \|\nabla\varphi_n\|_{k+2,p} \leq\\ c|\operatorname{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty))|_{-1,q} + \|\operatorname{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty))\|_{k,p}.\end{gathered}$$

Furthermore, a straightforward calculation shows that

$$\begin{gathered}|\nabla\varphi_n|_{1,q} + \|\nabla\varphi_n\|_{k+2,p} \leq\\ \mathrm{C}\left[|\triangle\operatorname{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty))|_{k-1,p} + \|\sigma_{n-1}\|_{k+1,q,p}(\|\mathbf{v}_{n-1}\|_{k+2,p} + \ell)\right].\end{gathered} \tag{3.16}$$

(ii) Next, we verify if conditions of Lemma 2.1 are satisfied. To this end, from $(3.10)_3$ we observe that the main difficulty consists in the estimate of the convective nonlinearity which contains as the worst term the term $\mathbf{v}_{n-1} \cdot \nabla \mathbf{v}_{n-1}$. However, this term coincides with the usual convective term in the incompressible case and thus we know how to deal with it easily. Therefore, in the wake of Galdi (1992-a, b), see also Novotný & Padula (forthcoming), it results

$$\begin{aligned} \dashv \mathbf{F}(\sigma_{n-1}, \mathbf{v}_{n-1}) \vdash_{k,q,p} &= \dashv -(1+\sigma_{n-1})[\mathbf{v}_{n-1} \cdot \nabla \mathbf{v}_{n-1} + \mathbf{f}] - \ell \frac{\partial(\sigma_{n-1}, \mathbf{v}_{n-1})}{\partial x_1} \vdash_{k,q,p} \\ &\le c \rbrack(\sigma_{n-1}, \mathbf{v}_{n-1})\lbrack^{2}_{\ell,k+1,q,p}(1+\ell^{-1/2})(1+\|\sigma_{n-1}\|_{k+1,q,p}) + \\ &\quad + c(1+\|\sigma_{n-1}\|_{k+1,q,p})\|\mathbf{f}\|_{k,r,p} \,. \end{aligned}$$

From (3.15), $\nabla\varphi_n \in \mathbf{W}^{k+2,p}$, hence the assumptions of Lemma 2.1 are satisfied and we can state also the existence of $\mathbf{u}_n \in \mathbb{V}^{2,q} \cap \mathrm{H}^{k+2,p}$, $p_n \in \mathbf{W}^{k+1,q,p}$ solution to (3.7) satisfying the estimate

$$\begin{aligned} \dashv\|\mathbf{u}_n\|\vdash_{\ell,k,q,p} + \|\varrho\|_{k+1,q,p} &\le c[|\mathbf{g}|_{k+2-1/p,p} + \dashv \mathbf{F} \vdash_{k,q,p}] \\ &\le c \rbrack(\sigma_{n-1}, \mathbf{v}_{n-1})\lbrack^{2}_{\ell,k+1,q,p}(1+\ell^{-1/2})(1+\|\sigma_{n-1}\|_{k+1,q,p}) + \\ &\quad + c(1+\|\sigma_{n-1}\|_{k+1,q,p})\|\mathbf{f}\|_{k,r,p} \,. \end{aligned} \tag{3.17}$$

The constant c depends on k, q, p and ℓ. However, if $q \in (1, 3/2)$ and $\ell \in (0, \mathrm{B}]$ for some $\mathrm{B} > 0$, c depends solely on k, q, p and B.

(iii) Finally, concerning the problem (3.8), we observe that from (i) and (ii) it follows that $\varphi_n \in \mathrm{H}^{1,q} \cap \mathbf{W}^{k+3,p}$ and $\mathbf{u}_n \in \mathbb{V}^{2,q} \cap \mathrm{H}^{k+2,p}$, $p \in \mathbf{W}^{k+1,q,p}$. Therefore, also $(\nabla\varphi_n + \mathbf{u}_n + \mathbf{v}_\infty) \in \mathbf{W}^{k+2,q,p} \cap \mathbf{W}^{1,q}_0$, $p_n - \ell(\partial\varphi_n/\partial x_1) \in \mathbf{W}^{k+1,q,p}$, next we must check that the condition (2.7) is satisfied. To this end, from (3.16) and (3.17) we have

$$\begin{aligned} \|\nabla\varphi_n + \mathbf{u}_n + \mathbf{v}_\infty\|_{k+2,q,p} &\le c\big[\ell + (1+\|\sigma_{n-1}\|_{k+1,q,p})\|\mathbf{f}\|_{k,r,p} + \\ &+ \rbrack(\sigma_{n-1}, \mathbf{v}_{n-1})\lbrack^{2}_{\ell,k+1,q,p}(1+\ell^{-1/2})(1+\|\sigma_{n-1}\|_{k+1,q,p})\big] + \\ &+ \|\sigma_{n-1}\|_{k+1,q,p}([\mathbf{v}_{n-1}]_{\ell,k,q,p} + \ell) + |\triangle\, \mathrm{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty))|_{k-1,p}. \end{aligned} \tag{3.18}$$

In (3.18) all terms at the right hand side but the last can be taken arbitrarily small through a suitable choice of $\mathbf{f}$ and R. We next consider the term

$$\mathrm{A} := |\triangle\, \mathrm{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty))|_{k-1,p} \,. \tag{3.19}$$

In order to bound A with the *given* constant γ, we should find a new estimate. This is essentially the purpose of the inequality $(2.8)_3$ of Lemma 2.3 which provides a sharp estimate for the negative norm of the $\triangle\, \mathrm{div}(\sigma_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_\infty))$ in terms of the same negative norm of the $\triangle\, [p_n - \ell(\partial\varphi_n/\partial x_1)]$. Next, the inequality $(2.4)_2$ provides an estimate for this last term in the function of F only which can be taken as small as we like. In doing so, we can state that for $\ell, \mathbf{f}, \mathrm{R}$ suitably small, we can bound the convective term $(\nabla\varphi_n + \mathbf{u}_n + \mathbf{v}_\infty)$ by a given positive constant γ. Then there exists a unique solution σ_n to the equation (3.8) in the space $\mathbf{W}^{k+1,p} \cap \mathbf{W}^{1,q}$ satisfying the following estimate

$$\langle\langle \sigma_n \rangle\rangle_{k+1,q,p} \leq (\mathrm{C}/\mathfrak{h}) \|p_n\|_{k+1,q,p}$$

which together with (3.17) furnishes

$$\begin{aligned}\langle\langle \sigma_n \rangle\rangle_{k+1,q,p} \leq c] (\sigma_{n-1}, \mathbf{v}_{n-1}) [\,^2_{\ell,k+1,q,p} (1 + \ell^{-1/2})(1 + \|\sigma_{n-1}\|_{k+1,q,p}) + \\ + c(1 + \|\sigma_{n-1}\|_{k+1,q,p}) \|\mathbf{f}\|_{k,r,p}\,. \end{aligned} \tag{3.20}$$

From (i), (ii), (iii) we can only conclude that there exists a solution to the n-th step and that, for suitably small $\mathbf{f}, \ell$, its norm satisfies the estimate (3.15) with the *same* constant R. To complete the existence proof we must prove now the convergence of the full sequence in a suitably weak norm. For this last step of proof, as it will be clear in a moment, it becomes fundamental estimate $(2.4)_1$. In particular, the intent of the last part of this section is to prove that for any positive constant ε there exists an integer $\overline{n}$ such that for all $n > \overline{n}$ it occurs

$$|\mathbf{v}_{n+1} - \mathbf{v}_n|_{1,p} + \|\sigma_{n+1} - \sigma_n\|_p \leq \varepsilon. \tag{3.21}$$

Inequality (3.21) allows us to pass to the limit in (3.21) and in (3.6) so we obtain the system (3.3).

Let us take the equations for the differences

$$\varphi_n' = (\varphi_{n+1} - \varphi_n), \quad \mathbf{u}_n' = \mathbf{u}_{n+1} - \mathbf{u}_n, \quad p_n' = p_{n+1} - p_n, \quad \sigma_n' = \sigma_{n+1} - \sigma_n \tag{3.22}$$

in (3.6)–(3.8) then we receive

$$\begin{gathered} \triangle\, \varphi_n' = -\operatorname{div}(\sigma_{n-1}'(\mathbf{v}_n + \mathbf{v}_\infty)) + \operatorname{div}(\sigma_{n-1}\mathbf{v}_{n-1}') \\ \triangle\, \mathbf{u}_n' - \ell \frac{\partial \mathbf{u}_n'}{\partial x_1} - \nabla p_n' = -\mathbf{F}_{n-1}'(\sigma, \mathbf{v}), \qquad \nabla \cdot \mathbf{u}_n' = 0, \\ \mathfrak{h}\sigma_n' + (\lambda + 2\mu) \operatorname{div}(\sigma_n'(\mathbf{v}_{n+1} + \mathbf{v}_\infty)) = p_n' - \ell \frac{\partial \varphi_n'}{\partial x_1} - (\lambda + 2\mu) \operatorname{div}(\sigma_n \mathbf{v}_n') \\ \mathbf{u}_n' \,|_\Sigma = -\nabla\varphi_n', \quad \mathbf{v}_n' \,|_\Sigma = 0, \quad \nabla\varphi_n' \cdot \nu \,|_\Sigma = 0, \\ \mathbf{u}_n'(x) \to 0, \quad p_n'(x) \to 0, \quad \varphi_n'(x) \to 0, \quad \text{as } |x| \to \infty \end{gathered} \tag{3.23}$$

with

$$\mathbf{F}_{n-1}'(\sigma, \mathbf{v}) := \mathbf{F}(\sigma_n, \mathbf{v}_n) - \mathbf{F}(\sigma_{n-1}, \mathbf{v}_{n-1}).$$

Now, it can be easily checked that $\mathbf{F}_{n-1}'(\sigma, \mathbf{v})$ admits the following estimate

$$\begin{aligned} |\mathbf{F}_{n-1}'(\sigma, \mathbf{v})_{-1,p} \leq c\{](\sigma_n, \mathbf{v}_n)[\,_{\ell,q,p} +](\sigma_{n-1}, \mathbf{v}_{n-1})[\,_{\ell,q,p} \} \times \\ \times [\|\nabla \mathbf{v}_{n-1}'\|_p + \|\sigma_{n-1}'\|_p]\,. \end{aligned}$$

Thus, using $(2.3)_1$ we find

$$(3.24)\quad |\mathbf{u}'_n|_{1,q} + \ell^{1/4}\|\mathbf{u}'_n\|_{s_q} + \|p'_n\|_q \le c|\nabla\varphi'_n|_{1,q} + [\|\nabla\mathbf{v}'_{n-1}\|_q + \|\sigma'_{n-1}\|_q]\times \\ \times\{](\sigma_n,\mathbf{v}_n)[_{\ \ell,q,p} +](\sigma_{n-1},\mathbf{v}_{n-1})[_{\ \ell,q,p}\}\,.$$

Furthermore, using $(2.10)_2$ into $(3.23)_1$ we receive

$$(3.25)\qquad |\nabla\varphi'_n|_{1,q} \le c\{|\nabla\operatorname{div}(\sigma'_{n-1}(\mathbf{v}_n+\mathbf{v}_\infty))|_{-1,q} + \|\operatorname{div}(\sigma_{n-1}\mathbf{v}'_{n-1})\|_q\}\,.$$

Next, we notice that $(2.8)_{1,2}$ imply

$$(3.26)\qquad \begin{aligned} \langle\langle\sigma'_n\rangle\rangle_p &\le (\mathrm{C}/\mathfrak{h})[\|p'_n\|_q] \\ \langle\nabla\operatorname{div}(\sigma'_n(\mathbf{v}_{n+1}+\mathbf{v}_\infty))\rangle_{-1,p} &\le c\langle\nabla p'_n\rangle_{-1,p} + \|\nabla\mathbf{v}'_n\|_p\|\sigma'_{n-1}\|_{1,p}\,. \end{aligned}$$

We put

$$\mathscr{X}_n :=](\sigma'_n,\mathbf{u}'_n)[_{\ 1,q} + \|p'_n\|_q + |\nabla\varphi'_n|_{1,q}\,.$$

Employing (3.15) which states the boundedness of our sequence through the constant R, the inequalities (3.24), (3.25), $(3.26)_1$ infer

$$(3.27)\qquad \mathscr{X}_n \le \mathrm{CR}\mathscr{X}_{n-1} + \mathrm{C}\langle\nabla\operatorname{div}(\sigma'_{n-1}(\mathbf{v}_n+\mathbf{v}_\infty))\rangle_{-1,p}\,.$$

Here, a new problem arises again, as in (3.19), due to the term

$$\mathrm{A}' := \langle\nabla\operatorname{div}(\sigma'_{n-1}(\mathbf{v}_n+\mathbf{v}_\infty))\rangle_{-1,p}\,.$$

In fact, such a term can be considered as nonlinear, however, in such a case, it is increased through a norm of σ'_{n-1} stronger than the one at left hand side of (3.27). On the other side, we can consider A'_n as linear, including it in the norm $\mathscr{X}_n$. Unfortunately, using this approach, the constant which multiplies A'_{n-1} cannot be taken arbitrarily small. Therefore, we are led to consider the sharper inequality $(2.4)_1$, (2.7), $(2.8)_2$ and Remark 2.1, thus delivering

$$(3.28)\quad \|\operatorname{div}(\sigma'_{n-1}\mathbf{w}_n)\|_p \le \mathrm{C}\langle\nabla\operatorname{div}(\sigma'_{n-1}\mathbf{w}_n)\rangle_{-1,p} \le \mathrm{C}\{\langle\mathrm{R}\mathscr{X}'_{n-1} + \|\mathbf{F}'_{n-1}\|\rangle\}$$

we set $\mathbf{w}_{n+1} = \mathbf{v}_{n+1} + \mathbf{v}_\infty$. From (3.27) and (3.28) we conclude that

$$\mathscr{X}_n \le \mathrm{CR}\mathscr{X}_{n-1} \le (\mathrm{CR})^n\,.$$

Thus, the statement (3.21) results completely proved. □

Remark 3.1. To prove the existence for the linear scheme, it is equivalent to prove existence for the first step. With this in mind, we stress the fact that in order to obtain an estimate for the density we assumed the validity of the smallness condition (3.1). Precisely, if we interpret it from a physical point of view we deduce that we are in a subsonic regime, see Padula (1992-b).

Remark 3.2. The iterative procedure used here is new and seems more fruitful than all others already introduced because it presents a technique allowing us to deduce the properties of the solution of the full nonlinear problem from those that can be proved for the easiest linearized one. This fact allows us to claim that our linearized systems (i), (ii), (iii) for the system governing the compressible nonlinear flows play the *same* role as that played by the linearized Stokes or Oseen systems for the full nonlinear Navier-Stokes system.

4. Existence and asymptotic behavior of the solution when $\mathbf{v}_\infty = 0$.

The aim of this section is to prove that, once the external forces in $\mathscr{L}^{k,p}_{q_1,q_2}$ are small and $\mathfrak{h}$ in (3.8) is large, there exists at least one solution to (1.1). Precisely, we prove

Theorem 4.2. *Let $p > 3$, $1 < q_1 < 3/2 < q_2 < 3$, $k = 0, 1, \dots$, $\Omega \in \mathscr{C}^{k+2}$ and $\mathbf{f} \in \mathbb{L}^{k,p}_{q_1,q_2}$. Then there exists a positive constant $\mathfrak{f}^*$ such that if*

$$\dashv\mathbf{f}\vdash^*_{k,q_1,q_2,p} \le \mathfrak{f}^*$$

then there exists at least one solution $\mathbf{v}$, ϱ to (1.1), $(\sigma, \mathbf{v})$ in the ball $\mathscr{B}_{\mathrm{R}} \subseteq \mathscr{K}^{k,q,p}_*$. Moreover, it satisfies the estimate

$$](\sigma, \mathbf{v})[^{\,*}_{k,q,p} \le c\{\dashv\mathbf{f}\vdash^*_{k,q_1,q_2,p}\}\,. \tag{4.1}$$

PROOF: Here we follow the lines of Section 3, namely, we restrict ourselves to the study of the systems (3.6), (3.7), (3.8) for the step $n = 0$, and then employ the recurrency procedure. Repeating the same argument of Section 3 we can still state the existence and uniqueness of the sequences φ_n, $\mathbf{u}_n$, p_n, σ_n such that $(\sigma_n, \mathbf{v}_n)$ is in the space $\mathscr{K}^{k,q,p}_0$ once we prove that $\mathbf{F} \in \mathscr{L}^{k,q,p}$, $g \in \mathbf{W}^{k+2-1/p,p}(\Sigma)$. Remark that, now, it is $\mathscr{L} = 0$, in this circumstance we are not able any more to prove $\mathbf{u} \in \mathrm{L}^{s_q}$. Therefore, to prove $\mathbf{F} \in \mathscr{L}^{k,q,p}$ we need some further regularity. Assume that the sequence at the $(n-1)$-th step verifies

$$\begin{aligned}
&|\varphi_{n-1}|_{1,q} + \|\varphi_{n-1}\|_{k+1,p} + \langle\langle\sigma_{n-1}\rangle\rangle_{k+1,q,p} + \\
&\quad + \|\nabla\mathbf{u}_{n-1}\|_{k+1,p} + \|\mathbf{u}_{n-1}\|_{s_q} + \|p_{n-1}\|_q + \|p_{n-1}\|_{k+1,p} \le \mathrm{R}, \\
&\||x|\mathbf{v}_{n-1}\|_\infty \le \mathrm{R}, \qquad \||x|^\alpha\sigma_{n-1}\|_q + \||x|^\alpha\sigma_{n-1}\|_p \le \mathrm{R}.
\end{aligned} \tag{4.2}$$

The use of the decay properties $(4.2)_{2,3}$ of the solutions together with the summability hypotheses $(4.2)_1$ enables us again to prove that $\mathbf{F} \in \mathscr{L}^{k,q,p}$; furthermore, for small $\mathbf{f}$ and R it is possible to prove that all regularity and compatibility conditions requested in Lemmas 2.1–2.6 are satisfied, cf. Lemma 7.1 Novotný & Padula (forthcoming). Thus, existence of a solution φ_n, $\mathbf{u}_n$, p_n, σ_n to the problems (i), (ii), (iii) with $(\sigma_n, \mathbf{v}_n) \in \mathscr{K}^{k,q,p}_0$ is ensured. Next we prove that the same estimates with the same constant R continue to hold for the sequence at the n-th step.

Specifically, we must prove boundedness for the "physically reasonable" norm of the known sequence φ_n, $\mathbf{u}_n$, p_n, σ_n, i.e.

$$](\sigma_n, \mathbf{v}_n)[\,^*_{k,q,p} \leq \mathrm{R}.$$

From existence results stated in Lemmas 2.1–2.6, we already know the belonging of φ_n, $\mathbf{u}_n$, p_n, σ_n to the classical Sobolev spaces, it remains to prove the decay properties $(4.2)_{2,3}$. To this end, the solution to the problem (ii) will be sought in a slightly different way. Precisely, let $\mathbf{u}(x)$ be such that $\mathbf{u}(x) < \mathrm{C}|x|^{-1}$ at infinity and having the same regularity used in Section 3 in any compact region. Using the Gauss-Green identity, we begin by representing a generic solution $\mathbf{u}(x)$ of (3.7) in Ω through the fundamental tensors (2.15), see Finn (1965). For an annular region $\mathscr{A}_\mathrm{R}$ bounded by Σ and by a sphere Σ_R of large radius R, we find

$$\begin{aligned}\mathbf{u}(x) = &\int_{\mathscr{A}_\mathrm{R}} \mathbf{U}(x;y;\ell)\cdot\mathbf{F}''(y)\,dy \\ &+\int_{\Sigma_\mathrm{R}} \{\mathbf{u}\cdot\mathrm{T}\mathbf{U} - \mathbf{U}\cdot\mathrm{T}\mathbf{u} + (\mathbf{U}\cdot\mathbf{u})\mathbf{v}_\infty\}\,d\Sigma_y \\ &+\int_{\mathscr{A}_\mathrm{R}} \varrho\mathbf{v}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\,dy + \int_\Sigma [\{-\nabla\varphi - \mathbf{v}_\infty\}\cdot\mathrm{T}\mathbf{U} - \mathbf{U}\cdot\mathrm{T}\mathbf{u}]\,d\Sigma_y,\end{aligned} \tag{4.3}$$

where $\mathbf{F}'' = (1+\sigma)\mathbf{f} + \ell\frac{\partial(\sigma,\mathbf{v})}{\partial x_1}$.

If we apply the results of §5, Corollary 2.8, Finn (1965) to $\mathbf{U}(x;y;\ell)$ for $y \to \infty$, then we get that the outer surface integral vanishes in the limit and we obtain the representation, valid whenever $|x|^\beta\mathbf{F}'' \in \mathrm{L}^2$, $\beta > 1/2$,

$$\mathbf{u}_n(x) = \int_\Omega \mathbf{U}(x;y;\ell)\cdot\mathbf{F}''(y)\,dy + \int_\Omega \varrho\mathbf{v}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\,dy + \int_\Sigma \nabla\varphi\cdot\mathrm{T}\mathbf{U}\,d\Sigma_y\,. \tag{4.4}$$

Then any solution of (3.7) with specified decay at infinity admits the representation

$$\mathbf{u}_n(x) = \mathbf{W}(x) + \int_\Omega \varrho_{n-1}\mathbf{v}_{n-1}\cdot\nabla\mathbf{U}\cdot\mathbf{v}_{n-1}\,dy \tag{4.5}$$

where

$$\mathbf{W}(x) := \mathbf{U}_1(x) + \mathbf{U}_2(x) := \int_\Omega \mathbf{U}(x;y;\ell)\cdot\mathbf{F}''(y)\,dy + \int_\Sigma \nabla\varphi_n\cdot\mathrm{T}\mathbf{U}\,d\Sigma_y\,. \tag{4.6}$$

Concerning the behavior at infinity, the integral $\mathbf{U}_2(x)$, since Σ is a compact, it coincides with that of $\mathrm{T}\mathbf{U}$ i.e.

$$\mathbf{U}_2(x) \leq c|x|^{-1}, \quad \ell = 0.$$

Thus employing (ii) of Lemma 2.9 we receive

$$\mathbf{U}_1(x) \leq c|x|^{-1}, \quad \ell = 0.$$

Finally, by use of Lemma 2.8 together with the hypothesis $\||x|\mathbf{u}_{n-1}\|_\infty \le \mathrm{R}$, we obtain

$$\||x|\mathbf{u}_n\|_\infty \le c\mathrm{R}^2 h + \|\nabla\varphi_n\|_{2,p} \tag{4.7}$$

and from (3.16) with $k=0$, $(2.4)_2$ we can render $\|\varphi_n\|_{2,p}$ less than R^2.

Moreover, recalling (3.6)

$$\triangle\, \varphi_n = -\operatorname{div}(\sigma_{n-1}(\mathbf{v}_{n-1}))$$

by Lemma 2.4 and by the induction assumption (4.2), we find $q \le \infty$,

$$\||x| \triangle \varphi_n\|_q \le c[\||x|(\mathbf{v}_{n-1})\|_\infty + \|\operatorname{div}(\mathbf{v}_{n-1})\|_\infty]\|\sigma_{n-1}\|_{1,q}\,. \tag{4.8}$$

The estimates (4.7), (4.8) allow us to state the boundedness of our sequence in the ball $\mathscr{B}_\mathrm{R} \subseteq \mathscr{K}_*^{k,q,p}$. Concerning the convergence, we do not need to add a word to the reasoning done in Section 3. Therefore, the unique limit $(\sigma, \mathbf{v})$ is still in $\mathscr{B}_\mathrm{R}$ and the theorem is proved. □

Remark 4.1. In order to obtain more information on the order of decay of the sequence we use the analogous representation formula for the pressure that after a careful analysis can be written

$$p(x) = \int_\Omega \mathbf{q}(x;y;\ell)\cdot \mathbf{F}''(y)\,dy + \int_\Sigma \{\nabla\varphi\cdot \mathrm{T}(\mathbf{U},\mathbf{q}) - \mathbf{q}\cdot\mathrm{T}(\mathbf{v},p)\}_y\,d\Sigma + \\ + \int_\Omega \operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v})\mathbf{q}(x;y;\ell)\,dy$$

and $|\mathbf{q}| \le c|x|^{-2}$. In the line of the proof of (4.6), we can safely deduce from the equation (4.8) with $1 \le \alpha < 2$

$$\||x|^\alpha p\|_\infty \le c\mathrm{R}^2\,.$$

From the symmetric system we can thus deduce the same decay for σ, $\operatorname{div}(\sigma(\mathbf{v}))$ and hence for $\triangle\,\varphi$. In particular, it follows that the compressible part $\nabla\varphi$ of the kinetic field has a better decay compared with that of the incompressible part $\mathbf{u}$. Finally, the density possesses the same behavior of the pressure and decays as the compressible part.

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