Comment.Math.Univ.Carolin. 34,2 (1993)257–264 

# On zeros and fixed points of multifunctions with non-compact convex domains

Sehie Park, Jong Sook Bae

*Abstract.* Using our own generalization [7] of J.C. Bellenger's theorem [1] on the existence of maximizable u.s.c. quasiconcave functions on convex spaces, we obtain extended versions of the existence theorem of H. Ben-El-Mechaiekh [2] on zeros for multifunctions with non-compact domains, the coincidence theorem of S.H. Kum [5] for upper hemicontinuous multifunctions, and the Ky Fan type fixed point theorems due to E. Tarafdar [13].

*Keywords:* convex space, $c$-compact set, real Hausdorff topological vector space (t.v.s.), linear operator, locally convex, fixed point, coincidence, zero, upper hemicontinuous (u.h.c.) multifunction

*Classification:* Primary 47H10

## 1. Introduction.

In our previous work [7], we removed the paracompactness assumption in Bellenger's theorem [1] on the existence of maximizable u.s.c. quasiconcave functions. This existence theorem extends and unifies earlier results of Ky Fan [3] and S. Simons [12]. Our result is subsequently applied, by the first author [8], [9], to obtain new coincidence and fixed point theorems, the Ky Fan type nonseparation theorems, and the existence of maximizable linear functionals having certain properties.

In the present paper, we apply our existence theorem to obtain extended versions of the following results:

(1) The existence theorem of H. Ben-El-Mechaiekh [2] on zeros for multifunctions with non-compact domains in a locally convex topological vector space.

(2) The coincidence theorem of S.H. Kum [5] on two upper hemicontinuous multifunctions from a convex subset of a locally convex topological vector space into another space.

(3) E. Tarafdar's versions [13] of the Fan-Glicksberg fixed point theorem. We note that Tarafdar's results are simple consequences of known theorems and can be improved in various ways, especially, without using the concept of almost upper semicontinuity.

Note that, in [2], [5], [13], those results are applied in various fields in mathematical sciences including economics.

Supported in part by the Basic Sciences Research Institute Program, Ministry of Education, 1991.

## 2. Preliminaries.

All topological spaces are assumed to be Hausdorff. A real topological vector space is abbreviated as a *t.v.s.*

Following M. Lassonde [6], a *convex space* $X$ is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset $L$ of a convex space $X$ is called a *c-compact set* if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. For an $x \in X$, $[x, L]$ denotes the closed convex hull of $L \cup \{x\}$ in $X$.

Recall that a real-valued function $f : X \to \mathbf{R}$ on a topological space $X$ is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is open for each $r \in \mathbf{R}$. If $X$ is a convex set in a vector space, then $f : X \to \mathbf{R}$ is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is convex for each $r \in \mathbf{R}$.

For a convex space $X$, let $\hat{X}$ be the set of all u.s.c. quasiconcave real functions on $X$.

The following version of Bellenger's theorem [1] is due to the authors [7].

**Theorem 0.** *Let $X$ be a convex space and $S : X \to 2^{\hat{X}}$ a multifunction. Suppose that*

(0.1) *for each $x \in X$, $Sx$ is a nonempty convex subset of $\hat{X}$;*
(0.2) *for each $g \in \hat{X}$, $S^{-1}g$ is compactly open in $X$; and*
(0.3) *there exist a c-compact subset $L$ of $X$ and a nonempty compact subset $K$ of $X$ such that for every $x \in X \backslash K$ and $g \in Sx$, $gx < \max g[x, L]$.*

*Then there exist an $x_0 \in K$ and $g \in Sx_0$ such that $gx_0 = \max g(X)$.*

Let $E$ be a t.v.s., $E^*$ its topological dual, and $X$ a convex space. A multifunction $T : X \to 2^E \backslash \{\emptyset\}$ is said to be *upper demicontinuous* (u.d.c.) if for each $x \in X$ and open half-space $H$ in $E$ containing $Tx$, there exists an open neighborhood $N$ of $x$ in $X$ such that $T(N) \subset H$; and *upper hemicontinuous* (u.h.c.) if for each $f \in E^*$ and each real $\alpha$, the set $\{x \in X : \sup f(Tx) < \alpha\}$ is open in $X$. If $T$ is upper semicontinuous (u.s.c.), then it is u.d.c.; and if $T$ is u.d.c., then it is u.h.c.

Let $cc(E)$ denote the set of nonempty closed convex subsets of $E$ and $kc(E)$ the set of nonempty compact convex subsets of $E$. Bd, Int, and — will denote the boundary, interior, and closure, resp., with respect to $E$.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of $X$ at $x$, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

A function $T : X \to 2^E$ is said to be *weakly inward* (*outward*, resp.) if

$$Tx \cap \overline{I}_X(x) \neq \emptyset \ [Tx \cap \overline{O}_X(x) \neq \emptyset, \text{ resp.}] \quad \text{for each} \quad x \in \operatorname{Bd} X \backslash Tx.$$

For $f \in E^*$ and $U, V \subset E$, let

$$d_f(U, V) = \inf\{|f(u - v)| : u \in U, \ v \in V\}.$$

## 3. Main results.

The following can be deduced from Theorem 0 as in [9].

**Theorem 1.** *Let $X$ be a convex space, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, $F$ a t.v.s. with topological dual $F^*$, $B : F^* \to 2^{\hat{X}}\backslash\{\emptyset\}$ a multifunction with convex graph, and $P, Q : X \to 2^F\backslash\{\emptyset\}$. Suppose that for each $f \in F^*$,*

(1.1) $X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ *is compactly closed;*
(1.2) *for each $x \in K$ and $g \in Bf$, $gx = \max g(X)$ implies $x \in X_f$; and*
(1.3) *for each $x \in X\backslash K$ and $g \in Bf$, $gx = \max g[x, L]$ implies $x \in X_f$.*

*Then there exists an $x \in \bigcap\{X_f : f \in F^*\}$.*

**Remark.** Theorem 1 is exactly the same as Park [9, Theorem 3]. Some of its particular forms are due to Simons [12, Theorem 2.2] and Park and Bae [7, Theorem 3].

Moreover, we have the following:

**Corollary 1.** *Let $X$ be a nonempty convex subset of a t.v.s. $E$, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, $F$ a t.v.s., and $l \in \mathcal{L}(E, F)$ a bounded linear operator. Let $S, T : X \to 2^F\backslash\{\emptyset\}$ be u.h.c. multifunctions such that, for each $x \in X\backslash K$ and $f \in F^*$,*

$$(l^*f)x \geq \sup\,(l^*f)(L) \quad \textit{implies} \quad \sup\, f(Sx) \geq \inf\, f(Tx).$$

*Then one of the following properties holds:*

(1) *There exists an $x \in X$ such that*

$$\sup\, f(Sx) \geq \inf\, f(Tx) \quad \textit{for all} \quad f \in F^*.$$

(2) *There exist an $x \in K$ and an $f \in F^*$ such that*

$$(l^*f)x = \max(l^*f)(X) \quad \textit{and} \quad \sup\, f(Sx) < \inf f(Tx).$$

Proof: Put $P = S$, $Q = T$, and $Bf = \{(l^*f)|_X\}$ for each $f \in F^*$ in Theorem 1. Then $B : F^* \to 2^{\hat{X}}\backslash\{\emptyset\}$ has the convex graph. Since $S$ and $T$ are u.h.c., (1.1) holds as shown in [9], but not conversely. Note that the negation of (2) implies (1.2) and that (1.3) follows from the hypothesis. Therefore, by Theorem 1, the property (1) follows. □

**Remarks.** 1. Corollary 1 generalizes Ben-El-Mechaiekh [2, Proposition 5], where $X$ is assumed to be paracompact and $L$ a compact convex subset of $X$. Note that, in Corollary 1, $(l^*f)x \geq \sup\,(l^*f)(L)$ is equivalent to $(l^*f)x = \max\,(l^*f)[x, L]$.

2. If $S(X) \cup T(X) \subset l(E)$ in Corollary 1, then (2) is equivalent to the following:

(2)′ There exist an $x \in K \cap \operatorname{Bd} X$ and an $f \in F^*$ such that $(l^*f)x = \max\,(l^*f)(X)$ and $\sup\, f(Sx) < \inf f(Tx)$.

In fact, if $x \in K \cap \operatorname{Int} X$, there exists an open neighborhood $U$ of the origin of $E$ such that $x + U \subset X$; and if $(l^*f)x = \max(l^*f)(X)$, then $(l^*f)x \geq (l^*f)(x+U)$, or $(l^*f)u \leq 0$ for all $u \in U$. Therefore, $l^*f$ is the zero functional and, hence, $f$ is an annihilator of $l(E)$. Since $S(X) \cup T(X) \subset l(E)$, for any $x \in K \cap \operatorname{Int} X$, the condition (2) cannot be satisfied.

This observation is due to Kum [5], and originates from Ky Fan [3] for the simplest case $l = 1_E$ and $E = F$. In this case, a particular form of Corollary 1 is given by Ben-El-Mechaiekh [2, Proposition 6].

3. Ben-El-Mechaiekh [2, Proposition 7] obtained a Ky Fan type matching theorem, which is a simple consequence of Park [10, Theorem 5].

From Theorem 1, we obtain the following coincidence theorem:

**Theorem 2.** *Suppose that, in addition to the hypothesis of Theorem* 1, *either*

(A) *$F^*$ separates points of $F$ and, for each $x \in X$, $\overline{\mathrm{co}}\, Px$ and $\overline{\mathrm{co}}\, Qx$ are compact; or*

(B) *$F$ is locally convex and, for each $x \in X$, $\overline{\mathrm{co}}\, Px$ or $\overline{\mathrm{co}}\, Qx$ is compact.*

*Then there exists an $x \in X$ such that $(\overline{\mathrm{co}}\, Px) \cap (\overline{\mathrm{co}}\, Qx) \neq \emptyset$.*

Proof: Suppose that $(\overline{\mathrm{co}}\, Px) \cap (\overline{\mathrm{co}}\, Qx) = \emptyset$ for all $x \in X$. Then, by the standard separation theorem for a t.v.s., (A) or (B) assures that, for each $x \in X$, there is an $f \in F^*$ such that $\inf f(\overline{\mathrm{co}}\, Qx) > \sup f(\overline{\mathrm{co}}\, Px)$; that is, $\inf f(Qx) > \sup f(Px)$. This contradicts the conclusion of Theorem 1. □

**Remark.** Theorem 2 is equivalent to Park [9, Theorem 4], which was stated for convex-valued multifunctions $P$ and $Q$. For some particular forms of Theorem 2, see also [9]. Moreover, we have the following:

**Corollary 2.** *Let $X$ be a nonempty convex subset of a t.v.s. $E$, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, $F$ a t.v.s., and $l \in \mathcal{L}(E,F)$ a bounded linear operator. Let $G$ and $H$ be u.h.c. multifunctions such that either*

(A) *$F^*$ separates points of $F$ and $G, H : X \to kc(F)$; or*

(B) *$F$ is locally convex, $G, H : X \to cc(F)$, and $Gx$ or $Hx$ is compact for each $x \in X$.*

*Suppose that*

(1) *for each $x \in K$ and $f \in F^*$,*

$$(l^*f)x = \max\,(l^*f)(X) \quad \textit{implies} \quad \sup f(Gx) \geq \inf f(Hx); \ \textit{and}$$

(2) *for each $x \in X \backslash K$ and $f \in F^*$,*

$$(l^*f)x \geq \sup\,(l^*f)(L) \quad \textit{implies} \quad \sup f(Gx) \geq \inf f(Hx).$$

*Then $G$ and $H$ have a coincidence.*

Proof: Put $P = G$, $Q = H$, and $Bf = \{(l^*f)\,|_X\}$ for $f \in F^*$ in Theorem 2. Since $G$ and $H$ are u.h.c., the condition (1.1) follows. Further, (1.2) and (1.3) follow from (1) and (2), respectively. Other requirements of Theorem 2 are clearly satisfied. Therefore, there exists an $x \in X$ such that $(\overline{\mathrm{co}}\, Gx) \cap (\overline{\mathrm{co}}\, Hx) = Gx \cap Hx \neq \emptyset$. This completes our proof. □

**Remarks.** 1. Corollary 2(B) is due to Kum [5]. As we noted in Remark 2 following Corollary 1, if $G(X) \cup H(X) \subset l(E)$, then the condition (1) is equivalent to the following:

(1)′ for each $x \in K \cap \operatorname{Bd} X$ and $f \in F^*$,

$$(l^* f)x = \max (l^* f)(X) \quad \text{implies} \quad \sup f(Gx) \geq \inf f(Hx).$$

2. Kum [5] further applied Corollary 2(B) to several existence problems of zeros (or critical points) of multifunctions.

From Theorem 2, we obtain the following existence theorem on zeros:

**Theorem 3.** *Let $X$ be a convex space, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, $F$ a t.v.s., and $B : F^* \to 2^{\hat{X}}\backslash\{\emptyset\}$ a multifunction with convex graph. Let $R : X \to 2^F\backslash\{\emptyset\}$ be a multifunction such that either*

(A) *$F^*$ separates points of $F$ and $\overline{\mathrm{co}}\, Rx$ is compact for each $x \in X$; or*

(B) *$F$ is locally convex.*

*Suppose that for each $f \in F^*$,*

(3.1) *$X_f = \{x \in X : \inf f(Rx) \leq 0\}$ is compactly closed;*

(3.2) *for each $x \in K$ and $g \in Bf$, $gx = \max g(X)$ implies $x \in X_f$; and*

(3.3) *for each $x \in X\backslash K$ and $g \in Bf$, $gx = \max g[x, L]$ implies $x \in X_f$.*

*Then there exists an $x \in X$ such that $0 \in \overline{\mathrm{co}}\, Rx$.*

Proof: Put $Px = \{0\}$ and $Qx = Rx$ for all $x \in X$, in Theorem 2. □

**Remark.** Theorem 3 is equivalent to Park [9, Theorem 5], which was stated for a closed convex-valued multifunction $R$. Some particular forms of Theorem 3 are also given in [9]. Moreover, we have the following:

**Corollary 3.** *Let $X$ be a nonempty convex subset of a t.v.s. $E$, $F$ a t.v.s., $l \in \mathcal{L}(E, F)$, and $S : X \to cc(F)$ an u.h.c. multifunction such that either*

(A) *$F^*$ separates points of $F$ and $Sx$ is compact for each $x \in X$; or*

(B) *$F$ is locally convex.*

*Suppose that there exist a c-compact subset $L$ of $X$ and a nonempty compact subset $K$ of $X$ such that*

(i) *for each $x \in K$ and each $f \in F^*$,*

$$(l^* f)(x) = \max (l^* f)(X) \quad \textit{implies} \quad \sup f(Sx) \geq 0;$$

*and*

(ii) *for each $x \in X\backslash K$ and each $f \in F^*$,*

$$(l^* f)(x) \geq \sup (l^* f)(L) \quad \textit{implies} \quad \sup f(Sx) \geq 0.$$

*Then $S$ has a zero and $(l + S)(X) \supset l(X)$.*

Proof: Put $R = -S$ and $Bf = \{(l^* f)|_X\}$ for each $f \in F^*$ in Theorem 3. Since $S$ is u.h.c., (3.1) holds. Note that (i) and (ii) imply (3.2) and (3.3), respectively.

Therefore, by Theorem 3, there exists an $x \in X$ such that $0 \in \overline{\text{co}}\,(-S)x$; that is, $0 \in Sx$.

In order to show $(l+S)(X) \supset l(X)$, let $v \in l(X)$ and say $v = lx_0$ for some $x_0 \in X$. The multifunction $R : X \to cc(F)$ defined by $Rx = Sx + lx - v$ is u.h.c. Since $\sup f(Rx) = \sup f(Sx) + (l^*f)(x - x_0)$ for any $f \in F^*$. $R$ satisfies the conditions (i) and (ii) with replacing $S$ by $R$ and $L$ by $[x_0, L]$. Therefore, by the first part, there exists an $x \in R$ such that $0 \in Rx$; that is, $v \in (l+S)x$. This completes our proof. □

**Remarks.** 1. Corollary 3(B) is actually due to Kum [5], and improves the main result of Ben-El-Mechaiekh [2, Theorem 1], where $X$ is assumed to be paracompact.

2. Kum [5] noted that, if $S(X) \subset l(E)$, then the condition (i) is equivalent to the following:

(i)′ for each $x \in K \cap \operatorname{Bd} X$ and $f \in F^*$,

$$(l^*f)x = \max\,(l^*f)(X) \quad \text{implies} \quad \sup f(Sx) \geq 0.$$

From Theorem 3, we have the following fixed point and surjectivity theorem:

**Theorem 4.** *Let $X$ be a convex space, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, and $E$ a t.v.s. containing $X$ as a subset. Let $R : X \to 2^E \backslash \{\emptyset\}$ be a multifunction such that either*

(A) *$E^*$ separates points of $E$ and, for each $x \in X$, $\overline{\text{co}}\,Rx$ is compact; or*
(B) *$E$ is locally convex.*

(I) *Suppose that, for each $f \in E^*$,*

(4.0) *$f|_X$ is continuous on $X$;*
(4.1) *$\{x \in X : fx \geq \inf f(Rx)\}$ is compactly closed;*
(4.2) *$d_f(Rx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \operatorname{Bd} X$; and*
(4.3) *$d_f(Rx, \overline{I}_L(x)) = 0$ for every $x \in X \backslash K$.*

*Then $\overline{\text{co}}\,R$ has a fixed point.*

(II) *Suppose that, for each $f \in E^*$,*

(4.0) *$f|_X$ is continuous on $X$;*
(4.1)′ *$\{x \in X : \sup f(Rx) \geq fx\}$ is compactly closed;*
(4.2)′ *$d_f(Rx, \overline{O}_X(x)) = 0$ for every $x \in K \cap \operatorname{Bd} X$; and*
(4.3)′ *$d_f(Rx, \overline{O}_L(x)) = 0$ for every $x \in X \backslash K$.*

*Then $\overline{\text{co}}\,R$ has a fixed point. Further, if $R$ is u.h.c., then $(\overline{\text{co}}\,R)(X) \supset X$.*

Proof: We use Theorem 3 with $E = F$ and $Bf = \{f|_X\}$ for each $f \in E^*$. Then $Bf \in 2^{\hat{X}} \backslash \{\phi\}$ because of (4.0).

(I) Considering $Rx - x$ instead of $Rx$ in Theorem 3, (4.1) implies (3.1). Since $I_X(x) = O_X(x) = E$ for $x \in \operatorname{Int} X$, (4.2) is actually equivalent to $d_f(Rx, \overline{I}_X(x)) = 0$ for all $x \in K$. Note that, by J. Jiang [4, Lemma 2.1] (see also [9, Lemma 1]), (4.2) and (4.3) imply (3.2) and (3.3), respectively. Therefore, by Theorem 3, we have the conclusion.

(II) Considering $x - Rx$ instead of $Rx$ in Theorem 3, $(4.1)'$ implies (3.1). Note that, by the lemma of Jiang, $(4.2)'$ and $(4.3)'$ imply (3.2) and (3.3), respectively. Therefore, we have a fixed point. For the surjectivity result, let $y \in X$. Consider $Rx - y$ instead of $Rx$ and $[y, L]$ instead of $L$ in Theorem 3. Then there exists an $x \in X$ such that $0 \in \overline{\mathrm{co}}\,(Rx - y)$; that is, $y \in \overline{\mathrm{co}}\,Rx$. This completes our proof. □

**Remarks.** 1. Theorem 4 extends Park [9, Theorem 6], which was given for a multifunction $R : X \to cc(E)$. As the first author noted in [9], Theorem 4 includes a lot of the well-known fixed point theorems on weakly inward (outward) u.h.c. multifunctions defined on convex subsets of a t.v.s. $E$ on which $E^*$ separates points.

2. In Theorem 4, we do not require any concrete connection between topologies of $X$ and $E$ except (4.0). Therefore, it is sufficient to assume that

(i) as a convex space, $X$ has any topology finer than the relative weak topology with respect to $E$, and

(ii) $E$ has a topology finer than its weak topology.

Moreover, we have the following:

**Corollary 4.** *Let $K$ be a nonempty compact convex subset of a t.v.s. $E$ on which $E^*$ separates points and $R : K \to 2^K \backslash \{\emptyset\}$ a multifunction satisfying*

$(*)$ *for each $f \in E^*$, $\{x \in K : fx \geq \inf f(Rx)\}$ is closed.*

*Then there exists an $x \in K$ such that $x \in \overline{\mathrm{co}}\,Rx$.*

Proof: Note that $\overline{\mathrm{co}}\,Rx$ is compact for each $x \in K$ and the conditions (4.1)-(4.3) clearly hold with $X = K$. Therefore, by Theorem 4, we have the conclusion. □

**Remarks.** 1. In [9], some known equivalent form or simple consequences of Corollary 4 can be found.

2. E. Tarafdar [13, Corollary 2.1] obtained Corollary 4 for a locally convex t.v.s. $E$ and a u.s.c. multifunction $R$ with closed values.

**Corollary 5.** *Let $K$, $E$, and $R$ be the same as in Corollary 4. If $S : K \to 2^K$ is a multifunction such that, for each $x \in K$, $Sx$ is closed and $\mathrm{co}\,Rx \subset Sx$, then $S$ has a fixed point.*

**Remarks.** 1. Tarafdar [13, Theorem 2.1] obtained Corollary 5 for a locally convex t.v.s. $E$ and a u.s.c. closed-valued multifunction $R$. Moreover, he assumed the "almost upper semicontinuity" of $S$. However, this concept for closed-valued multifunction is same as the upper semicontinuity if the range is normal.

2. Therefore, in view of Tarafdar [13, Lemma 2.2], if $G : K \to 2^K$ is a u.s.c. closed-valued multifunction, where $K$ is a nonempty compact convex subset of a locally convex t.v.s., then the multifunction $H : K \to 2^K$ defined by $Hx = \overline{\mathrm{co}}\,Gx$ for $x \in K$ is also u.s.c.

Moreover, if $G$ is u.h.c., so is $H$, because we have $\sup f(Hx) = \sup f(Gx)$ for each $f \in E^*$.

Furthermore, note that if $T : X \to 2^Y \backslash \{\emptyset\}$ is a convex-valued multifunction, where $X$ is a topological space and $Y$ a nonempty compact subset of a locally convex t.v.s., then $T$ is u.s.c. iff $T$ is u.d.c. iff $T$ is u.h.c. See M.-H. Shih and K.-K. Tan [11, Proposition 2].

3. Tarafdar [13] applied his version of Corollary 5 to establish the existence of an equilibrium point of an abstract economy given by preferences and an economy given by utility functions.

## References


[1] Bellenger J.C., *Existence of maximizable quasiconcave functions on paracompact convex spaces*, J. Math. Anal. Appl. **123** (1987), 333–338.

[2] Ben-El-Mechaiekh H., *Zeros for set-valued maps with non-compact domains*, C. R. Math. Rep. Acad. Sci. Canada **12** (1990), 125–130.

[3] Ky Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.

[4] Jiang J., *Fixed point theorems for paracompact convex sets*, Acta Math. Sinica **4** (1988), 64–71.

[5] Kum S.H., Ph.D. Dissertation, Seoul National University, 1991.

[6] Lassonde M., *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.

[7] Park S., Bae J.S., *Existence of maximizable quasiconcave functions on convex spaces*, J. Korean Math. Soc. **28** (1991), 285–292.

[8] Park S., *Applications of maximizable linear functionals on convex sets*, "Proc. in Honor of C. N. Lee", 537–548, 1991.

[9] ———, *Fixed point theory of multifunctions in topological vector spaces*, J. Korean Math. Soc. **29** (1992), 191–208.

[10] ———, *Generalized matching theorems for closed coverings of convex sets*, Numer. Funct. Anal. and Optimiz. **11** (1990), 101–110.

[11] Shih M.-H., Tan K.-K., *Covering theorems of convex sets related to fixed-point theorem*, in "Nonlinear and Convex Analysis (Proc. in Honor of Ky Fan)" (B.-L. Lin and S. Simons, eds.), 235–244, Marcel Dekker, Inc., New York, 1987.

[12] Simons S., *On existence theorem for quasiconcave functions with applications*, Nonlinear Anal. TMA **10** (1986), 1133–1152.

[13] Tarafdar E., *An extension of Fan's fixed point theorem and equilibrium point of an abstract economy*, Comment. Math. Univ. Carolinae **31** (1990), 723–730.



Seoul National University and MSRI-Korea, Seoul 151–742, Korea

Myongji University and MSRI-Korea, Yong-In 449–800, Korea


(*Received* November 25, 1991)