On the k-Baire property

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Abstract. In this note we show the following theorem: "Let X be an almost k-discrete space, where k is a regular cardinal. Then X is k^+-Baire iff it is a k-Baire space and every point-k open cover $\mathcal{U}$ of X such that $\operatorname{card}(\mathcal{U}) \le k$ is locally-k at a dense set of points." For $k = \aleph_0$ we obtain a well-known characterization of Baire spaces. The case $k = \aleph_1$ is also discussed.

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Let k be an infinite cardinal number. A space is almost k-discrete if every non-empty intersection of fewer than k open sets has non-empty interior. Almost $\aleph_1$-discrete spaces are called almost P-spaces [4]. A k-Baire space is a space in which the intersection of fewer than k dense open sets is dense ([3], [5]). Thus the usual Baire spaces are $\aleph_1$-Baire spaces. A collection $\mathcal{U}$ of subsets of a space X is said to be point-k if each point $x \in X$ is in fewer than k members of $\mathcal{U}$. Point-$\aleph_0$ collections are called point-finite, point-$\aleph_1$ collections are called point-countable. A collection $\mathcal{U}$ is locally-k at a point x if there is an open neighborhood of x meeting fewer than k members of $\mathcal{U}$. Locally-$\aleph_0$ (locally-$\aleph_1$) collections are called locally finite (locally countable). The least cardinal strictly greater than k is denoted by k^+.

Theorem 1. *Let X be an almost k-discrete space, where k is a regular cardinal. Then X is k^+-Baire iff it is a k-Baire space and every point-k open cover $\mathcal{U}$ of X such that* $\operatorname{card}(\mathcal{U}) \le k$ *is locally-k at a dense set of points.*

Proof: Let k be a regular cardinal. The hypothesis that X is an almost k-discrete space is used only for the sufficiency. The proof of the necessity is essentially similar as the one showing that every k^+-Baire space satisfying the countable chain condition has caliber λ, for each regular cardinal $\lambda \le k$ ([5, Theorem 3.6]). So let X be a k^+-Baire space and let $\mathcal{U} = \{U_\alpha\}_{\alpha<k}$ be a point-k open cover of X. Suppose that the set $A = \{x \in X : \mathcal{U} \text{ is locally-}k \text{ at } x\}$ is not dense, then there is a non-empty open set V such that $V \cap A = \emptyset$. For each $\beta < k$ let $C_\beta = V - \bigcup\{U_\alpha : \beta \le \alpha < k\}$. Since $\mathcal{U}$ is point-k (and k is regular), $V = \bigcup\{C_\beta : \beta < k\}$. But each C_β is nowhere dense, for if $W_\beta = \operatorname{int}_X(\operatorname{cl}_X C_\beta) \ne \emptyset$ then $G_\beta = W_\beta \cap V \ne \emptyset$ and $G_\beta \cap (\bigcup\{U_\alpha : \beta \le \alpha < k\}) \subseteq \operatorname{cl}_X(C_\beta) \cap (\bigcup\{U_\alpha : \beta \le \alpha < k\}) = \emptyset$, so

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$\emptyset \neq G_\beta \subseteq V \cap A$, a contradiction. Hence V is the union of less than k^+ nowhere dense sets, contradicting the hypothesis that X is k^+-Baire (note that a space is k-Baire iff no non-empty open set is the union of fewer than k nowhere dense sets). Finally we show the sufficiency. Let X be a (non-empty) k-Baire almost k-discrete space such that every its point-k open cover of cardinality $\leq k$ is locally-k at a dense set of points. Let $\{D_\alpha\}_{\alpha<k}$ be a family of dense open subsets of X. For each $\alpha < k$ let $H_\alpha = \bigcap\{D_\beta : \beta \leq \alpha\}$. From our hypothesis it follows that H_α is dense in X, so H_α is a non-empty intersection of fewer than k open sets, for each $\alpha < k$. Now let $G_\alpha = \mathrm{int}_X(H_\alpha)$, X is almost k-discrete so $G_\alpha \neq \emptyset$. We claim that G_α is dense in X for each $\alpha < k$. Let us suppose that there is a non-empty open set G such that $G \cap G_\alpha = \emptyset$, then $G \cap \mathrm{cl}_X(G_\alpha) = \emptyset$. Take $y \in G \cap H_\alpha$ (H_α is dense in X). Let V be an open neighborhood of y such that $V \cap G_\alpha = \emptyset$. $\bigcap\{D_\beta \cap V : \beta \leq \alpha\}$ is non-empty (it contains y) and X is almost k-discrete so $W = \mathrm{int}_X(\bigcap\{D_\beta \cap V : \beta \leq \alpha\}) \neq \emptyset$. Therefore $\emptyset \neq W \subseteq V \cap G_\alpha$, a contradiction. Hence $\mathrm{cl}_X(G_\alpha) = X$ for each $\alpha < k$. Therefore $\mathcal{G} = \{G_\alpha : \alpha < k\}$ is a decreasing family of dense open subsets of X. Without loss of generality we may assume that $\alpha \neq \beta \to G_\alpha \neq G_\beta$. Since $\bigcap\{G_\alpha : \alpha < k\} \subseteq \bigcap\{D_\alpha : \alpha < k\}$ then it is enough to show that $\bigcap\{G_\alpha : \alpha < k\}$ is dense in X. Let $C = \mathrm{cl}_X(\bigcap\{G_\alpha : \alpha < k\})$. If $C \neq X$ consider the open cover $\{X\} \cup \{G_\alpha - C : \alpha < k\}$. This cover is point-$k$ and by hypothesis, there is some $y \in X - C$ such that this cover is locally-k at y. Hence there is an open neighborhood V of y, and a $A \subset k$, $\mathrm{card}\,(A) < k$, such that $V \cap (G_\alpha - C) \neq \emptyset$ iff $\alpha \in A$. Since each G_α is dense, we have a contradiction. Therefore $C = X$ and X is k^+-Baire. □

For $k = \aleph_0$ we obtain the following well-known characterization of Baire spaces ([1], [2]).

Corollary 2. *X is a Baire space iff every countable point-finite open cover of X is locally finite at a dense set of points.*

The class of $\aleph_2$-Baire spaces is also interesting (see, for instance, Chapter 4 of [6]). Two well-known results about this class of spaces are: (1) $(MA + \neg CH)$ Every Čech-complete space satisfying the countable chain condition is $\aleph_2$-Baire [5]; (2) Every Hausdorff locally compact almost P-space is $\aleph_2$-Baire [6]. The following corollary gives a characterization, in the realm of almost P-spaces, of $\aleph_2$-Baire spaces.

Corollary 3. *Let X be an almost P-space. X is $\aleph_2$-Baire iff it is a Baire space and every point-countable open cover $\mathcal{U}$ of X such that $\mathrm{card}\,(\mathcal{U}) \leq \aleph_1$ is locally countable at a dense set of points.*

Remark 4. In the above corollary the assumption that X is an almost P-space cannot be omitted, as the following example shows ([5], [7]). Let $2^{\aleph_1}$ be the topological product of $\aleph_1$ copies of the two-point discrete space $\{0,1\}$. Let X be the subspace of $2^{\aleph_1}$ consisting of all functions $f : \omega_1 \to \{0,1\}$ such that $\{\alpha \in \omega_1 : f(\alpha) = 1\}$ is countable. X is a Baire space which is the union of $\aleph_1$ nowhere dense sets. Let D be a countable dense subset of $2^{\aleph_1}$ and let Y be the subspace $D \cup X$ of $2^{\aleph_1}$. Y is a Baire space (X is a dense Baire subspace of Y), it is separable (hence every point-countable open cover of Y is countable) but it is not $\aleph_2$-Baire (it is union of $\aleph_1$ nowhere dense sets).

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