# The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations

E. Liz*, Juan J. Nieto*

*Abstract.* In this paper, we develop monotone iterative technique to obtain the extremal solutions of a second order periodic boundary value problem (PBVP) with impulsive effects. We present a maximum principle for "impulsive functions" and then we use it to develop the monotone iterative method. Finally, we consider the monotone iterates as orbits of a (discrete) dynamical system.

*Keywords:* impulsive differential equations, periodic boundary value problem, monotone iterative technique

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## 1. Introduction.

Impulsive differential equations are experiencing an important development because they are a framework richer than ordinary differential equations in order to model several real world processes.

In this paper, we develop monotone iterative technique to obtain the minimal and maximal solutions of a periodic boundary value problem (PBVP) with impulsive effects, namely

$$-u''(t) = f(t, u(t)),\ t \in I,\ t \neq t_k, 0, 2\pi \tag{1.1}$$

$$u(t_k^+) = I_k(u(t_k)) \tag{1.2}$$

$$u'(t_k^+) = N_k(u'(t_k)) \tag{1.3}$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi) \tag{1.4}$$

We shall refer to system (1.1)–(1.4) as problem (P).

Existence of solutions for this problem has already been proved for continuous functions [4] and for discontinuous (Carathéodory) functions [7]. Our result generalizes the analogous problem without impulses [1], [5].

In Section 2 we present a new maximum principle for discontinuous functions. It is, to the best of our knowledge, the first maximum principle for "impulsive functions" in the literature, and it is of fundamental importance to develop the monotone method. Section 3 is devoted to demonstrate our main result and, finally, in Section 4 we consider the monotone iterates as orbits of a dynamical system and we obtain a global attractor, following the ideas of [1].

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## 2. A maximum principle.

The following lemma is essential to develop the monotone iterative technique in our problem:

**Lemma 2.1.** *Let* $I = [0, 2\pi]$; $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 2\pi$; $A = \{0, 1, \ldots, p+1\}$. *Let* $\phi \in C^2(t_{k-1}, t_k) \cap C(t_{k-1}, t_k]$, *for each* $k = 1, \ldots, p+1$. *Suppose that the following five conditions are satisfied:*

(i) *There exist* $\phi(t_k^+)$, $\phi'(t_k^+)$ *for* $k = 0, 1, \ldots, p$, *and* $\phi'(t_k^-)$ *for* $k = 1, \ldots, p+1$.
(ii) *There exists a real constant* $M > 0$ *such that* $\phi''(t) \geq M\phi(t)$, $\forall\, t \in I - \{t_k : k \in A\}$.
(iii) $\phi(0) = \phi(2\pi)$, $\phi'(0^+) \geq \phi'(2\pi^-)$.
(iv) $\phi(t_k) \geq 0 \Rightarrow \phi(t_k^+) \geq \phi(t_k)$; $\phi'(t_k^-) \geq 0 \Rightarrow \phi'(t_k^+) \geq 0$, $\forall\, k \in A$.
(v) *If* $\phi(t_k) \leq 0$, *then* $\phi(t_k^+) \leq 0$, $\forall\, k \in A$.

*Then,* $\phi(t) \leq 0$, $\forall\, t \in I$.

Proof: Suppose that this result is false. Thus there exists a point $\tau \in I$ such that $\phi(\tau) = \sup_{t \in I} \phi(t) = C > 0$.

We shall distinguish four cases:

**Case 1.** $\exists\, k \in A - \{p+1\}$ such that $\tau \in (t_k, t_{k+1})$.

Then $\phi \in C^2(t_k, t_{k+1})$, $\phi'(\tau) = 0$ and $\phi''(\tau) \leq 0$. But $\phi''(\tau) \geq M\phi(\tau) = MC > 0$, which is a contradiction.

**Case 2.** $\tau = 0$ or $\tau = 2\pi$.

In this case:

$$\phi(0) = \phi(2\pi) = C > 0; \quad \phi(t) \leq C \quad \text{on} \quad I. \tag{2.1}$$

Then we have:

$$\phi'(0^+) \leq 0; \quad \phi'(2\pi^-) \geq 0.$$

Thus $\phi'(0^+) = \phi'(2\pi^-) = 0$ since $\phi'(0^+) \geq \phi'(2\pi^-)$. Now, $\exists\, \delta > 0$ such that $\phi''(t) \geq M\phi(t) > 0$, for every $t \in (o, \delta)$. Hence $\phi'$ is strictly increasing on $(o, \delta)$. Taking into account that $\phi'(0^+) = 0$, we see that $\phi$ is strictly increasing on $(o, \delta)$, which contradicts (2.1).

**Case 3.** $\tau = t_k$, for some $k \in \{1, 2, \ldots, p\}$.

We have:

$$\phi(t_k) = C = \sup_{t \in I} \phi(t) > 0.$$

As $\phi(t_k^+) \geq \phi(t_k)$, it must be $\phi(t_k^+) = \phi(t_k) = C > 0$. (Note that $\phi$ is continuous at $t_k$.)

Now

$$\phi'(t_k^-) \geq 0 \Rightarrow \phi'(t_k^+) \geq 0. \tag{2.2}$$

However, there exists a $\delta > 0$ such that $\phi''(t) \geq M\phi(t) > 0$, $\forall\, t \in (t_k, t_k + \delta)$ and then $\phi'$ is strictly increasing on $(t_k, t_k + \delta)$.

By (2.2), $\phi'(t) > 0$, $\forall\, t \in (t_k, t_k + \delta)$ and $\phi$ is strictly increasing on $(t_k, t_k + \delta)$, which is a contradiction with the fact that $\phi$ attains its maximum at $t = t_k$.

**Case 4.** $\phi(t) < C$, $\forall\, t \in I$, $\exists\, k \in \{1, \dots, p\}$ such that $\phi(t_k^+) = C$.

$\phi(t_k^+) > 0$ implies $\phi(t_k) > 0$, in view of condition (v). Then $\sup\{\phi(t) : t_{k-1} < t \le t_k\} = C_k > 0$. We have two possibilities:

**4.1.** $\phi(\tau) = C_k$, for some $\tau \in (t_{k-1}, t_k]$.
– If $\tau \in (t_{k-1}, t_k)$, then $\phi$ has a relative maximum in $\tau$ and it leads to a contradiction as in the first one.
– If $\tau = t_k$, then $\phi'(t_k^-) \ge 0$ and $\phi'(t_k^+) \ge 0$. Repeating the arguments used in Case 3, we can show that this is impossible.

**4.2.** $C_k = \phi(t_{k-1}^+)$.

Employing the same procedures inductively, we either get a contradiction on the way or we have:

$$\phi(t_1^+) = \sup\{\phi(t) : t_1 < t \le t_2\} = C_2 > 0; \quad \phi(t_1) > 0. \tag{2.3}$$

Then, $\sup\{\phi(t) : 0 < t \le t_1\} = C_1 > 0$.
– If $\phi(0) = C_1$, we have a contradiction as in Case 2.
– If $\phi(t) = C_1$ for some $t \in (0, t_1)$, this implies a contradiction as in Case 3.
– Finally, if $\phi(t_1) = C_1$, then $\phi'(t_1^+) \ge 0$ and, using (2.3), it leads to a contradiction as in Case 3.

Now, the proof is complete. $\square$

## 3. Monotone iterative technique.

In this section, we develop monotone iterative technique, which provides a constructive method to approach the extremal solutions of the problem (P).

In order to define the concept of solution, lower solution and upper solution for (P) we introduce the following space:

$$\begin{aligned}\Omega = \{&u : I \to \mathbb{R} : u\,|_{(t_k, t_{k+1}]} \in E_k,\\ &\text{there exist } \; u(t_k^+),\; u'(t_k^+) \; \text{ for every } \; k = 0, 1, \dots, p\},\end{aligned}$$

where $E_k = C^2(t_k, t_{k+1}) \cap C^1(t_k, t_{k+1}]$.

By a solution of (P), we mean a function $u \in \Omega$ satisfying (1.1)–(1.4).

A function $\alpha \in \Omega$ is said to be a lower solution for (P) if:

$$-\alpha''(t) \le f(t, \alpha(t)),\; t \in I,\; t \ne t_k, 0, 2\pi \tag{3.1}$$

$$\alpha(t_k^+) = I_k(\alpha(t_k)) \tag{3.2}$$

$$\alpha'(t_k^+) \ge N_k(\alpha'(t_k)) \tag{3.3}$$

$$\alpha(0) = \alpha(2\pi),\; \alpha'(0) \ge \alpha'(2\pi) \tag{3.4}$$

An upper solution is defined analogously by reversing the inequalities.

Let $\alpha, \beta$ be lower and upper solutions of (P) respectively, with $\alpha \le \beta$ on $I$. Let us list the following assumptions:

($A_0$) $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $t \in I$, $t \ne t_k$, $k \in A$.

(A$_1$) $I_k, N_k : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing functions, $k \in \{1, 2, \ldots, p\}$.
(A$_2$) $f(t,u) - f(t,v) \geq -M(u-v)$, $M > 0$, whenever $t \in I$, $\alpha(t) \leq v \leq u \leq \beta(t)$.
(A$_3$) $I_k(u) - I_k(v) \leq u - v$, whenever $\alpha(t_k) \leq u \leq v \leq \beta(t_k)$, $k \in \{1, 2, \ldots, p\}$.

Now, we shall prove our main result:

**Theorem 3.1.** *Assume that* (A$_0$)–(A$_3$) *hold. Then there exist monotone sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *such that* $\alpha_0 = \alpha \leq \alpha_n \leq \beta_n \leq \beta_0 = \beta$ *for every* $n \in \mathbb{N}$ *which converge piecewise uniformly to the minimal and maximal solutions of (P) on* $I$ *respectively.*

PROOF: Let $[\alpha, \beta] = \{u \in PC(I) : \alpha \leq u \leq \beta \text{ on } I\}$, where

$PC(I) = \{u : I \to \mathbb{R} : u(t)$ continuous for $t \neq t_k$, $u(t_k^-)$, $u(t_k^+)$ exist and $u(t_k^-) = u(t_k)$, $k = 0, 1, \ldots, p\}$.

For $\eta \in [\alpha, \beta]$, we shall consider the following linear problem:

$$-u''(t) + Mu(t) = f(t, \eta(t)) + M\eta(t), \ t \in I, \ t \neq t_k, 0, 2\pi \tag{3.5}$$

$$u(t_k^+) = I_k(u(t_k)) \tag{3.6}$$

$$u'(t_k^+) = N_k(u'(t_k)) \tag{3.7}$$

$$u(0) = u(2\pi), \ u'(0) = u'(2\pi) \tag{3.8}$$

where $k \in \{1, 2, \ldots, p\}$.

We shall refer to problem (3.5)–(3.8) as (PL).

Using the variation of parameters formula (with impulsive effects), we can see that (PL) possesses a unique solution:

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A} g(s)\, ds + \sum_{0<t_k<t} e^{(t-t_k)A} \overline{I}_k(x(t_k))$$

with

$$x(0) = (I_2 - e^{2\pi A})^{-1} \Big[ \int_0^{2\pi} e^{(2\pi - s)A} g(s)\, ds + \sum_{0<t_k<2\pi} e^{(2\pi - t_k)A} \overline{I}_k(x(t_k)) \Big]$$

where

$$A = \begin{pmatrix} 0 & 1 \\ M & 0 \end{pmatrix}; \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad x(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$$

$$g(t) = \begin{pmatrix} 0 \\ -\sigma(t) \end{pmatrix}; \quad \sigma(t) = f(t, \eta(t)) + M\eta(t)$$

$$\overline{I}_k(x(t_k)) = \begin{pmatrix} I_k(u(t_k)) - u(t_k) \\ N_k(u'(t_k)) - u'(t_k) \end{pmatrix}$$

Now, we define the mapping

$$A : [\alpha, \beta] \longrightarrow \Omega$$

by $A\eta = u =$ unique solution of (PL).

It is not difficult to see that $A$ is a continuous operator and that $u \in [\alpha, \beta]$ is a solution of (P) if and only if $Au = u$.

Now we shall prove that:

$$\text{(a)}\ \alpha \le A\alpha;\ \beta \ge A\beta \tag{3.9}$$

$$\text{(b)}\ \alpha \le \eta_1 \le \eta_2 \le \beta \Rightarrow A\eta_1 \le A\eta_2;\ \eta_1, \eta_2 \in [\alpha, \beta] \tag{3.10}$$

with the help of Lemma 2.1.

Proof of (a): Set $\alpha_1 = A\alpha$; $\phi = \alpha - \alpha_1$. We have:

(i) $\phi''(t) = \alpha''(t) - \alpha_1''(t) \ge f(t, \alpha) - f(t, \alpha) + M[\alpha(t) - \alpha_1(t)] = M\phi(t)$, $t \ne t_k$.

(ii) $\phi(0) = \alpha(0) - \alpha_1(0) = \alpha(2\pi) - \alpha_1(2\pi) = \phi(2\pi)$.

(iii) $\phi'(0) = \alpha'(0) - \alpha_1'(0) \ge \alpha'(2\pi) - \alpha_1'(2\pi) = \phi'(2\pi)$.

(iv) $\phi(t_k) \ge 0 \Rightarrow \alpha(t_k) \ge \alpha_1(t_k) \Rightarrow \phi(t_k^+) = \alpha(t_k^+) - \alpha_1(t_k^+) = I_k(\alpha(t_k)) - I_k(\alpha_1(t_k)) \ge \alpha(t_k) - \alpha_1(t_k) = \phi(t_k)$.

(v) $\phi(t_k) \le 0 \Rightarrow \alpha(t_k) \le \alpha_1(t_k) \Rightarrow \alpha(t_k^+) \le \alpha_1(t_k^+) \Rightarrow \phi(t_k^+) \le 0$, in view of the nondecreasing character of $I_k$.

(vi) $\phi'(t_k) \ge 0 \Rightarrow \alpha'(t_k) \ge \alpha_1'(t_k) \Rightarrow \alpha(t_k^+) \ge \alpha_1(t_k^+) \Rightarrow \phi'(t_k^+) \ge 0$, in view of the nondecreasing character of $N_k$.

By Lemma 2.1, $\phi(t) \le 0$ on $I$ and then $\alpha \le A\alpha$. Similarly, we can show that $\beta \ge A\beta$.

Proof of (b): Set $\phi = A\eta_1 - A\eta_2 = v_1 - v_2$.

(i) $\phi''(t) = v_1''(t) - v_2''(t) = f(t, \eta_2(t)) - f(t, \eta_1(t)) + M[v_1(t) - \eta_1(t)] - M[v_2(t) - \eta_2(t)] \ge -M[\eta_2(t) - \eta_1(t)] + M[v_1(t) - \eta_1(t)] - M[v_2(t) - \eta_2(t)] = M[v_1(t) - v_2(t)] = M\phi(t)$, $t \ne t_k$.

(ii) $\phi(0) = v_1(0) - v_2(0) = v_1(2\pi) - v_2(2\pi) = \phi(2\pi)$.

(iii) $\phi'(0) = v_1'(0) - v_2'(0) = v_1'(2\pi) - v_2'(2\pi) = \phi'(2\pi)$.

(iv) $\phi(t_k) \ge 0 \Rightarrow v_1(t_k) \ge v_2(t_k) \Rightarrow \phi(t_k^+) = v_1(t_k^+) - v_2(t_k^+) = I_k(v_1(t_k)) - I_k(v_2(t_k)) \ge v_1(t_k) - v_2(t_k) = \phi(t_k)$.

(v) $\phi(t_k) \le 0 \Rightarrow v_1(t_k) \le v_2(t_k) \Rightarrow v_1(t_k^+) \le v_2(t_k^+) \Rightarrow \phi(t_k^+) \le 0$.

(vi) $\phi'(t_k) \ge 0 \Rightarrow v_1'(t_k) \ge v_2'(t_k) \Rightarrow v_1'(t_k^+) \ge v_2'(t_k^+) \Rightarrow \phi'(t_k^+) \ge 0$.

Thus Lemma 2.1 assures that $\phi(t) \le 0$ on $I$ and then $A\eta_1 \le A\eta_2$.

Now we can define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ by $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$, $\alpha_0 = \alpha$, $\beta_0 = \beta$. Then we have: $\alpha_0 = \alpha \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0 = \beta$, for every $n \in \mathbb{N}$. It follows, by using standard arguments, that:

(1) There exist $\lim_{n\to\infty} \alpha_n = \varrho$, $\lim_{n\to\infty} \beta_n = \gamma$, piecewise uniformly on $I$.

(2) $\varrho(t)$ and $\gamma(t)$ are solutions of (P).

We next prove that $\varrho$ and $\gamma$ are the minimal and maximal solutions of (P) on $[\alpha, \beta]$ respectively. For it, let $u(t)$ be a solution of (P) with $\alpha(t) \le u(t) \le \beta(t)$, $t \in I$. We shall show below that:

$$\alpha_n(t) \le u(t) \le \beta_n \Rightarrow \alpha_{n+1}(t) \le u(t) \le \beta_{n+1}(t), \quad \text{for every } n \in \mathbb{N}. \tag{3.11}$$

Thus, since $\alpha_1(t) \le u(t) \le \beta_1(t)$, by induction and passing to the limit when $n \to \infty$, we obtain that $\varrho(t) \le u(t) \le \gamma(t)$ on $I$.

To prove (3.11), let us consider $\phi(t) = \alpha_{n+1}(t) - u(t)$. Then we have:

(i) $\phi''(t) = \alpha''_{n+1}(t) - u''(t) = f(t, u(t)) - f(t, \alpha_n(t)) - M\alpha_n(t) + M\alpha_{n+1}(t) \ge -M[u(t) - \alpha_n(t)] - M\alpha_n(t) + M\alpha_{n+1}(t) = M[\alpha_{n+1}(t) - u(t)] = M\phi(t)$.

(ii) $\phi(0) = \alpha_{n+1}(0) - u(0) = \alpha_{n+1}(2\pi) - u(2\pi) = \phi(2\pi)$.

(iii) $\phi'(0) = \alpha'_{n+1}(0) - u'(0) = \alpha'_{n+1}(2\pi) - u'(2\pi) = \phi'(2\pi)$.

(iv) $\phi(t_k) \ge 0 \Rightarrow \alpha_{n+1}(t_k) \ge u(t_k) \Rightarrow \phi(t_k^+) = \alpha_{n+1}(t_k^+) - u(t_k^+) = I_k(\alpha_{n+1}(t_k)) - I_k(u(t_k)) \ge \alpha_{n+1}(t_k) - u(t_k) = \phi(t_k)$.

(v) $\phi(t_k) \le 0 \Rightarrow \alpha_{n+1}(t_k) \le u(t_k) \Rightarrow \alpha_{n+1}(t_k^+) \le u(t_k^+) \Rightarrow \phi(t_k^+) \le 0$.

(vi) $\phi'(t_k) \ge 0 \Rightarrow \alpha'_{n+1}(t_k) \ge u'(t_k) \Rightarrow \alpha'_{n+1}(t_k^+) \ge u'(t_k^+) \Rightarrow \phi'(t_k^+) \ge 0$.

In view of Lemma 2.1, $\phi(t) \le 0$ on $I$ and thus $\alpha_{n+1}(t) \le u(t)$, $\forall\, t \in I$. Similarly, we get $u(t) \le \beta_{n+1}(t)$ on $I$, and the proof is now complete. □

## 4. Monotone iterates as orbits of a dynamical system.

Let $H$ be the metric space $([\alpha,\beta], d)$, where $[\alpha,\beta]$ is the sector considered in Theorem 3.1 and $d$ is the piecewise uniform distance. By using (3.9) and (3.10), we see that the operator $A$ maps continuously $H$ into itself.

The purpose of this section is to study the discrete dynamical system $(H, A)$.

For $u \in H$, let $u_n = A^n u$, $n = 0, 1, 2, \dots$ .

We shall consider here the definitions about attractors given in [2]. For $u \in H$, the orbit starting at $u$ is $G_u = \{u_n = A^n u,\ n \in \mathbb{N}\}$. We say that a subset $\Gamma$ of $H$ is an invariant set if $A^n\Gamma = \Gamma$ for every $n \in \mathbb{N}$. An invariant set $\Gamma$ is an attractor if it possesses an open neighborhood $U$ such that

$$d(u_n, \Gamma) = \inf\{d(u_n, v) : v \in \Gamma\} \to 0 \ \text{ as } \ n \to \infty \ \text{ for every } \ u \in U.$$

We say that $\Gamma$ is a global attractor if it is a compact set and $d(u_n, \Gamma) \to 0$ as $n \to \infty$ for every $u \in H$. For $\Gamma$, the $\omega$-limit of $\Gamma$ is defined by $\omega(\Gamma) = \bigcap_{j>0} \overline{\bigcup_{n \ge j} A^n\Gamma}$. We shall prove the following result:

**Theorem 4.1.** *Under the assumptions of Theorem* 3.1, *let* $\varrho$ *and* $\gamma$ *be the extremal solutions of* (P) *on* $[\alpha,\beta]$ *and let* $\Gamma = [\varrho,\gamma] \cap \Omega$. *Then* $\omega(\Gamma)$ *is a global attractor for the dynamical system* $(H, A)$. *Moreover,* $\omega(\Gamma)$ *is uniformly asymptotically stable.*

Proof: $\omega(\Gamma)$ is a compact subset of $H$. Since $\lim_{n\to\infty} \alpha_n = \varrho$, $\lim_{n\to\infty} \beta_n = \gamma$ and $\alpha_n \le u_n \le \beta_n$ for $u \in [\alpha,\beta]$, we have that $\Gamma$ attracts $[\alpha,\beta]$.

Now, Theorem 3.1 of [2] assures that $\omega(\Gamma)$ is a nonempty compact invariant set, a global attractor for $(H, A)$ and it is uniformly asymptotically stable. □

Note that if $u$ is a solution of (P) in $[\alpha,\beta]$, then $u \in \omega(\Gamma)$ since $u \in \Gamma$ and $G_u = \{u\}$. If (P) has a unique solution $\varrho$ in $[\alpha,\beta]$, then $\omega(\Gamma) = \Gamma = \{\varrho\}$ and thus we have the following corollary:

**Corollary 4.1.** *Under the assumptions of Theorem* 3.1, *if* (P) *has a unique solution* $\varrho \in [\alpha,\beta]$, *then the set* $\{\varrho\}$ *is a global attractor for* $(H, A)$.

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Departamento de Matemática Aplicada, E.T.S.I. Industriales, Universidad de Vigo, Spain

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Spain

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