Some conditions under which a uniform space is fine

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Abstract. Let X be a uniform space of uniform weight μ. It is shown that if every open covering, of power at most μ, is uniform, then X is fine. Furthermore, an ω_μ-metric space is fine, provided that every finite open covering is uniform.

Keywords: uniform space, uniform weight, fine uniformity, uniformly locally finite, ω_μ-additive space, ω_μ-metric space

Classification: Primary 54E15; Secondary 54A25, 54A35

0. A recurrent problem about uniform spaces is to see whether a uniformity is the finest one compatible with the topology. Isiwata and Atsuji solved this problem in metric spaces [2], [6]. The following theorem summarizes some equivalent conditions of Theorem 1 of [2].

Theorem 1. *The following conditions on a metric space X are equivalent:*

(1) *every open covering is uniform;*
(2) *every countable open covering is uniform;*
(3) *every open covering consisting of two elements is uniform;*
(4) *the subset K of limit points is compact and, for every uniform covering $\mathcal{U}$, the subspace $X \setminus \mathrm{St}(K,\mathcal{U})$ is uniformly discrete.*

(*The star* $\mathrm{St}(K,\mathcal{U})$ *of K with respect to $\mathcal{U}$ is the union of all elements of $\mathcal{U}$ which have a non-empty intersection with K*).

It is interesting to see if a suitable version of Theorem 1 also holds in uniform spaces. Metric spaces are uniform spaces with a countable base for the uniformity; this base can be assumed to be well ordered by star-refinement. In uniform spaces, the existence of a well ordered base is a very strong property.

We will prove that an equivalence analogous to $1 \Leftrightarrow 2$ holds for general uniform spaces and depends only on cardinal properties, while the equivalence $1 \Leftrightarrow 3$ can be generalized to ω_μ-metric spaces (= uniform spaces which admit a base of uniform coverings well ordered by a regular cardinal ω_μ). In ω_μ-metric spaces we will provide a suitable formulation of the condition 4 (obviously in uniform spaces $3 \not\Rightarrow 1$ and $1 \not\Rightarrow 4$).

1. Unless otherwise specified, the space X is a uniform space, with the uniform topology.

The following lemma is useful for working with cardinal properties of locally finite families. The proof is easy to check (for example, see [7]).

Lemma 1. *Let $\mathcal{G}$ be a locally finite family of subsets of X. If the power of $\mathcal{G}$ is at most μ, then there exists an open covering $\mathcal{B}$, of power at most μ, such that every element of $\mathcal{B}$ meets only finitely many elements of $\mathcal{G}$.*

We recall that the uniform weight of X is the smallest cardinal number of a base for the uniformity.

The following theorem is the uniform analogue of $2 \Rightarrow 1$ in Theorem 1.

Theorem 2. *Let μ be the uniform weight of X. If every open covering of power at most μ is uniform, then every open covering is uniform.*

We shall prove Theorem 2 in three steps (the statements of each step hold under the hypotheses of Theorem 2).

Let $\{\mathcal{U}_\alpha : \alpha < \mu\}$ be a base for the uniformity, consisting of open uniform coverings.

Step 1. *X is a paracompact topological space.*

PROOF: Let $\mathcal{A}$ be an open covering. For every $x \in X$, choose $\alpha(x) < \mu$ such that $\mathrm{St}(x, \mathcal{U}_{\alpha(x)})$ is contained in some element of $\mathcal{A}$. For every α, let

$$A_\alpha = \bigcup_{\alpha(x)=\alpha} \mathrm{St}(x, \mathcal{U}_{\alpha(x)})\,.$$

By assumption, the covering $\{A_\alpha : \alpha < \mu\}$ is uniform.

The covering

$$\mathcal{B} = \{A_\alpha \cap \mathrm{St}(x, \mathcal{U}_{\alpha(x)}) : \alpha < \mu,\ \alpha(x) = \alpha\}$$

is an open refinement of $\mathcal{A}$.

Since uniform coverings are normal coverings (in the sense of Tukey), by the condition (g) in [9, Theorem 1.2] it follows that the covering $\mathcal{B}$ has an open star-refinement.

□

A family $\mathcal{F}$ of subsets of X is said to be uniformly locally finite if there exists a uniform covering $\mathcal{B}$ such that every element of $\mathcal{B}$ meets $\mathcal{F}$ in only a finite number of elements.

Step 2. *Every locally finite family $\mathcal{F}$ is uniformly locally finite.*

PROOF: We proceed by contradiction. Suppose that for every α there exist an element $U_\alpha \in \mathcal{U}_\alpha$ and a countable subfamily $\mathcal{F}_\alpha$ of $\mathcal{F}$ such that $U_\alpha \cap F \neq \emptyset$ for every $F \in \mathcal{F}_\alpha$.

Let $\mathcal{G}$ be the union of all subfamilies $\mathcal{F}_\alpha$. $\mathcal{G}$ is a locally finite family of subsets and the power of $\mathcal{G}$ is at most μ. By Lemma 1, there exists an open covering $\mathcal{B}$ of power at most μ such that every element of $\mathcal{B}$ meets $\mathcal{G}$ only in a finite number of elements. $\mathcal{B}$ is an open covering of power at most μ, which cannot be uniform because the family $\mathcal{G}$ is not uniformly locally finite. This is a contradiction with the hypothesis of Theorem 2. □

Step 3. *Every locally finite open covering $\mathcal{A}$ is uniform.*

Proof: By Step 2, $\mathcal{A}$ is uniformly locally finite. For every α, by possibly refining coverings $\mathcal{U}_\alpha$, we can assume that every element of $\mathcal{U}_\alpha$ meets $\mathcal{A}$ only for a finite number of elements.

If $\mathcal{A}$ is not a uniform covering, then for every α there exists $U_\alpha \in \mathcal{U}_\alpha$ such that $U_\alpha \setminus A \neq \emptyset$ for every $A \in \mathcal{A}$.

Let $\mathcal{A}_\alpha = \{A \in \mathcal{A} : A \cap U_\alpha \neq \emptyset\}$ and let $\mathcal{C} = \bigcup_\alpha \mathcal{A}_\alpha$.

Every $\mathcal{A}_\alpha$ is a finite family, thus the power of $\mathcal{C}$ is at most μ. Therefore we have a contradiction, because the open covering $\mathcal{C} \cup \{\bigcup(\mathcal{A} \setminus \mathcal{C})\}$, of power at most μ, cannot be uniform. In fact, for every α, $U_\alpha \cap (\bigcup(\mathcal{A} \setminus \mathcal{C})) = \emptyset$ and $U_\alpha \setminus A \neq \emptyset$ for every $A \in \mathcal{C}$. □

The conclusion of Theorem 2 follows from Step 1 and Step 3.

Remark. One might conjecture that every open covering is uniform, provided that the open coverings of power less than μ are uniform coverings. This, however, is not the case. For a counterexample, let X be the space of ordinals less than ω_1, equipped with the unique (precompact) uniformity (an open covering is uniform iff it has a finite subcovering).

By countable compactness of X, every countable open covering is uniform and it is easy to verify that the uniform weight of X is ω_1. Furthermore, X is not a paracompact topological space [4, p. 380].

2. Denote by $\mathbf{C}^*$ the weak uniformity of continuous bounded real functions on a completely regular Hausdorff space X.

X is a normal space iff every open covering consisting of two elements belongs to $\mathbf{C}^*$.

It is an interesting question to see when a uniformity finer than $\mathbf{C}^*$ is fine. For example, the implication $3 \Rightarrow 1$ of Theorem 1 says that metric uniformities finer than $\mathbf{C}^*$ are fine. Another example of uniform space with this property are sequentially uniform spaces [3].

In the next theorem, we extend the equivalences $1 \Leftrightarrow 3 \Leftrightarrow 4$ of Theorem 1 to ω_μ-metric spaces. Notice that the proof of this theorem follows from ordinal properties.

An ω_μ-metric space is a uniform space which admits a base of uniform coverings

$$\mathbf{B} = \{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$$

well ordered by refinement (hence by star-refinement) by a regular cardinal ω_μ.

An ω_μ-metric space is paracompact (ultra-paracompact if $\mu > 0$) [1].

Let λ be a cardinal number. A topological space is said to be λ-compact if every open covering has a subcovering of power less than λ. A weakly paracompact space X is λ-compact iff the power of every discrete closed subset of X is less than λ (as one can prove by mimicking the proof of [4, Theorem 5.3.2]).

In the proof of Theorem 3, the base $\mathbf{B}$ is assumed well ordered by star-refinement. The equivalence $1 \Leftrightarrow 2$ has been already proved in [8].

Theorem 3. *Let X be an ω_μ-metric space. The following conditions are equivalent:*

(1) *every open covering is uniform;*
(2) *the set K of limit points is ω_μ-compact and for every α the subspace $X \setminus \mathrm{St}(K,\mathcal{U}_\alpha)$ is uniformly discrete;*
(3) *every finite open covering is uniform.*

Proof: $1 \Rightarrow 3$ Obvious.

$3 \Rightarrow 2$ By way of contradiction, assume that there exists a closed discrete subset $D = \{x_\alpha : \alpha < \omega_\mu\}$ of pairwise distinct limit points.

We shall prove that, for every α, one can choose $\beta(\alpha) \geq \alpha$ such that the collection $\mathcal{F} = \{\mathrm{St}(x_\alpha,\mathcal{U}_{\beta(\alpha)}) : \alpha < \omega_\mu\}$ consists of pairwise disjoint subsets. We proceed by transfinite induction. Choose $\beta(0) \geq 0$ such that $\overline{\mathrm{St}(x_0,\mathcal{U}_{\beta(0)})}$ is disjoint from $D \setminus \{x_0\}$. Let $\alpha > 0$ and $C_\alpha = \bigcup_{\gamma<\alpha} \overline{\mathrm{St}(x_\gamma,\mathcal{U}_{\beta(\gamma)})}$. The set C_α is closed, because X is an ω_μ-additive topological space [1]. The set $C = C_\alpha \cup \{x_\gamma : \gamma > \alpha\}$ is closed and therefore there exists $\beta(\alpha) \geq \alpha$ such that the subset $\overline{\mathrm{St}(x_\alpha,\mathcal{U}_{\beta(\alpha)})}$ is disjoint from C.

Choose $y_\alpha \in \mathrm{St}(x_\alpha,\mathcal{U}_{\beta(\alpha)})$, $y_\alpha \neq x_\alpha$. The subset $F = \{y_\alpha : \alpha < \omega_\mu\}$ has no limit points and the open covering $\{X \setminus F, X \setminus D\}$ cannot be uniform, because the subsets F and D cannot be separated by a uniform covering.

What we still need to prove is that for every $\mathcal{U} \in \mathbf{B}$ the subspace $Y = X \setminus \mathrm{St}(K,\mathcal{U})$ is uniformly discrete (notice that every subset of Y is closed in X). We argue by way of contradiction. Again using transfinite induction, it is easy to choose elements x_α, y_α such that $y_\alpha \in \mathrm{St}(x_\alpha,\mathcal{U}_\alpha)$ and $x_\alpha \neq y_\beta$ for every α, β. Thus the open covering consisting of $X \setminus \{x_\alpha : \alpha < \omega_\mu\}$ and $X \setminus \{y_\alpha : \alpha < \omega_\mu\}$ cannot be uniform. This contradiction concludes the proof.

$2 \Rightarrow 1$

It is enough to prove that the trace on K of every open covering is a uniform covering of K. This trace can be refined by a covering of the form $\{\mathrm{St}(\mathrm{St}(x,\mathcal{U}_{\alpha(x)}),\mathcal{U}_{\alpha(x)}) : x \in K\}$. As K is ω_μ-compact, the covering $\{\mathrm{St}(x,\mathcal{U}_{\alpha(x)}) : x \in K\}$ has a subcovering of power less than ω_μ, say $\{\mathrm{St}(x_i,\mathcal{U}_{\alpha(x_i)}) : i < \delta\}$ for a suitable $\delta < \omega_\mu$. Thus $\{\mathrm{St}(x,\mathcal{U}_\gamma) : x \in K\}$ where $\gamma = \sup\{\alpha(x_i) : i < \delta\}$ is the required uniform refinement (see [4, Theorem 4.3.31]). □

Remark. It follows from the proof that the condition 2 of the above theorem can be strengthened as follows:

(2') *the set K of limit points is ω_μ-compact and every closed discrete subset of X is uniformly discrete.*

Remark. It is well-known that the fine uniformity on a metrizable topological space X is a metric uniformity iff the set of limit points is compact (see for example [10]).

We can see that an analogous result holds for ω_μ-metrizable spaces: precisely, if the subset K of limit points of an ω_μ-metrizable space X is ω_μ- compact, then the fine uniformity is an ω_μ-metric uniformity.

Let

$$\mathbf{B} = \{\mathcal{U}_\alpha :\ \alpha < \omega_\mu\}$$

be a well ordered base for a compatible uniformity. For every α, consider the open covering

$$\mathcal{V}_\alpha = \big\{\{x\}, U :\ U \in \mathcal{U}_\alpha, U \cap K \neq \emptyset, x \in X \setminus \mathrm{St}(K, \mathcal{U}_\alpha)\big\}\,.$$

It is easy to check that $\mathbf{C} = \{\mathcal{V}_\alpha :\ \alpha < \omega_\mu\}$ is a well ordered (by refinement) base which induces the fine uniformity (see Theorem 3).

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(*Received* September 23, 1992)