# Some adaptive estimators for slope parameter

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*Abstract.* An adaptive estimator (of a slope parameter) based on rank statistics is constructed and its asymptotic optimality is studied. A complete orthonormal system is incorporated in the adaptive determination of the score generating function. The proposed sequential procedure is based on a suitable stopping rule. Various properties of the sequential adaptive procedure and the stopping rule are studied. Asymptotic linearity results of linear rank statistics are also studied and some rates of the convergence are established.

*Keywords:* asymptotically optimal score generating function, Fisher information, orthonormal system, rank $(R-)$-estimator, stopping rule, asymptotically optimal estimators

*Classification:* 62G05, 62G20

## 1. Introduction.

For $n = 1, 2, \ldots$, let $X_1, \ldots, X_n$ be independent observations such that

$$(1.1) \qquad X_i = \theta_0 + \theta_1 c_i + e_i, \quad i = 1, 2, \ldots, n,$$

where $\theta_0, \theta_1$ are unknown parameters, $c_1, \ldots, c_n$ are known regression constants, $e_1, \ldots, e_n$ are independent random errors fulfilling certain regularity assumptions.

The problem is to estimate $\theta_1$ (slope parameter). The estimator is based on ranks and a score-generating function $\varphi$ defined on $(0, 1)$. If the distribution function $F$ of $e_i$'s possesses an absolutely continuous density function (p.f.d) $f$ with a finite Fisher information $I(f) = \int (\frac{f'}{f})^2 \, dF \ (< \infty)$, where $f'$ stands for the first derivative of $f$, then for the estimation problem, the score-generating function $\varphi_f = -\frac{f'(F^{-1})}{f(F^{-1})}$ ($F^{-1}$ stands for the quantile function corresponding to $F$) is asymptotically optimal in the sense that the asymptotic variance of the estimator of $\theta_1$ attains the Cramer-Rao lower bound.

In practice, however, $F$ and hence $\varphi_f$ are rarely known, so the estimation of $\varphi_f$ is of a considerable interest. Several types of estimations of $\varphi_f$ have been developed. Here we shall concentrate on the Fourier expansion type via the estimation of the Fourier coefficients of $\varphi_f$. In this approach, Beran [1] used the trigonometric system to study this type, he described the construction of uniformly asymptotically efficient rank estimates in the two-sample location model. For this two-sample location model, Hušková [4] used a general type of Fourier series to estimate the Fourier coefficients and these estimators were based on the linearity of rank statistics. And in [6], Hušková and Sen considered the Legendre polynomials to estimate $\varphi_f$. Using

similar ideas, Rödel [7] developed an adaptive rank statistics for testing independence. Towards this problem, he used the system of Legendre polynomials and the Fourier expansion to estimate the bivariate density.

In the present paper, asymptotically efficient rank estimators are constructed, and differ from those mentioned above. In fact, we use the general type of Fourier series to estimate the score function and construct an asymptotically optimal estimator for the slope parameter in the simple regression model (1.1). Our proposed procedure is a sequential one based on a well defined stopping rule. This is presented in Section 2. The main results on the asymptotic optimality of the proposed procedure are considered in Section 3. The proofs of the main results are mentioned in Section 4.

## 2. Assumptions and notation.

We shall adopt the following assumptions in the sequel:

**Assumption A.** The regression constants $c_1, \ldots, c_n$ fulfil:

  (i) $n^{-1} C_n \to C > 0$,
  (ii) $n^{-1} \sum_{i=1}^{n} c_i^4 = O(1)$ as $n \to \infty$, with

$$C_n = \begin{bmatrix} n & \sum_{i=1}^{n} c_i \\ \sum_{i=1}^{n} c_i & \sum_{i=1}^{n} c_i^2 \end{bmatrix}$$

and $C > 0$ is a positively definite matrix.

The condition (ii) is slightly stronger than one usually considered for rank estimates, however, still reasonable when $\varphi_f$ is known.

**Assumption B.** $e_1, \ldots, e_n$ are iid random variables with distribution function $F$ satisfying:

  (i) $f(x) = \frac{dF(x)}{dx}$ exists and is absolutely continuous on $(-\infty, \infty)$,
  (ii) the Fisher information in nonzero and finite, i.e.

$$0 < I(f) = \int_{-\infty}^{\infty} \left( \frac{f'(x)}{f(x)} \right)^2 dx < \infty \,;$$

(i) and (ii) are the usual regularity assumptions.

Further, throughout the paper we shall work with a complete orthonormal system $\{P_k(u), \ 0 \le u \le 1, \ k \ge 0\}$ in $L^2([0,1])$. Suppose that the system satisfies the following properties:

**Assumption C.** $\{P_k(u), \ 0 \le u \le 1, \ k \ge 0\}$ is a complete orthonormal system in $L^2([0,1])$ fulfilling: The first three derivatives $P_k^{(i)}(u)$, $i = 0, 1, 2, 3$ exist and

$$D_{ki} = \sup_{0 \le u \le 1} |P_k^{(i)}(u)| < \infty, \quad i = 0, 1, 2, 3,$$

where

$$P_k^0 = P_k \,.$$

The Legendre polynomial system and trigonometric system satisfy this assumption.

If Assumption B is satisfied, one can easily realize that $I(f)=\|\varphi_f\|^2 \; (= \int (\frac{f'}{f})^2 \, dF)$ and $\varphi_f \in L^2([0,1])$. Hence $\varphi_f$ can be written:

$$(2.1) \qquad \varphi_f(u) \sim \sum_{k \geq 1} \gamma_k P_k(u), \quad 0 \leq u \leq 1,$$

where

$$(2.2) \qquad \gamma_k = \langle \varphi_f, P_k \rangle = \int_0^1 \varphi_f(u) P_k(u) \, du = \int_{-\infty}^{\infty} P_k'(u)(F(x)) f^2(x) \, du.$$

Following the idea of Hušková and Sen [6], we introduce the stopping rules as follows:

$$(2.3) \qquad L_n = \min\{k \geq K : \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \leq \varepsilon_n\},$$

where $K$ is a predetermined positive integer and $\hat{\gamma}_{n,j}$ is the estimator of $\gamma_j$ defined by: (for $t \neq 0$)

$$(2.4) \quad \hat{\gamma}_{n,j} =$$
$$= -\frac{1}{t} \sum_{i=1}^{n} c_{in} \left[ P_k \left( (n+1)^{-1} \overset{\circ}{R}_i \left( \overline{\theta}_{1n} - \frac{t}{[\sum_{i=1}^{n}(c_i - \overline{c}_n)^2]^{1/2}} \right) \right) - P_k \left( \frac{\overset{\circ}{R}_i(\overline{\theta}_{1n})}{n+1} \right) \right]$$

with $\overset{\circ}{R}_i(u)$ being the rank of $X_i - c_i u$ among $X_1 - c_1 u, \ldots, X_n - c_n u$, and

$$(2.5) \qquad c_{in} = \frac{c_i - \overline{c}_n}{[\sum_{i=1}^{n}(c_i - \overline{c}_n)^2]^{1/2}} \quad (\overline{c}_n = n^{-1} \sum_{i=1}^{n} c_i),$$

$\overline{\theta}_{1n}$ is a preliminary estimator of $\theta_1$ satisfying:

$$(2.6) \qquad \left[ \sum_{i=1}^{n}(c_i - \overline{c}_n)^2 \right]^{1/2} (\overline{\theta}_{1n} - \theta_1) = 0_p(1) \text{ as } n \to \infty$$

and $\{r_n\}$ and $\{\varepsilon_n\}$ are sequences of positive integers and positive real numbers such that

**Assumption D.**
    (i) $\{r_n\}$ is increasing but $r_n n^{-s} \to 0$ as $n \to \infty$, for some $s > 0$,
    (ii) $\{\varepsilon_n\}$ is nonincreasing with $\lim_n \varepsilon_n = 0$;

if $r_n = O(\log n)$ and $\varepsilon_n = O(n^{-\alpha}(\log n)^\beta)$ with $\alpha, \beta > 0$, then (i) and (ii) are satisfied.

Along with $L_n$ we need

$$(2.7) \qquad L_n^*(\lambda) = \min\{k \geq K : \sum_{j=k+1}^{k+r_n} \gamma_j^2 \leq \lambda\varepsilon_n\}, \quad \lambda > 0.$$

Together with the stopping rules we consider the following adaptive estimators $\hat{\varphi}_n(u)$ of $\varphi_f(u)$ (following the idea of [1] and [6])

$$(2.8) \qquad \hat{\varphi}_n(u) = \sum_{k \leq L_n + r_n} \hat{\gamma}_{n,k} P_k(u), \quad u \in [0, 1],$$

where $L_n$, $\hat{\gamma}_{n,k}$ are given by (2.3), (2.4), respectively.

As an estimator of the Fisher information, we use

$$(2.9) \qquad \hat{I}_n = \sum_{k \leq L_n + r_n} \hat{\gamma}_{n,k}^2.$$

Finally, we are ready to introduce the adaptive estimator of $\theta_1$ as follows:

$$(2.10) \qquad \hat{\theta}_{1n} = \overline{\theta}_{1n} + \frac{1}{\hat{I}_n[\sum_{i=1}^n (c_i - \overline{c}_n)^2]^{1/2}} \sum_i^n c_{in}\hat{\varphi}_n(\overset{\circ}{R}_i(\overline{\theta}_{1n})/(n+1)),$$

where $\hat{\varphi}_n$, $\hat{I}_n$ are given by (2.8) and (2.9).

We shall investigate the asymptotic properties of the stopping rules, of $\hat{\varphi}_n$, $\hat{I}_n$ and, of course, of the resulting adaptive estimators.

The following assumptions will be needed:

**Assumption E.** For some $\delta > 0$, some $0 < \lambda_2 < 1 < \lambda_1$ and some $s > 0$, the first three derivatives of $\{P_k\}_{k=1}^\infty$ satisfy:

  (i) $\sum_{k=L_n^*(\lambda_2)+1}^{L^*(\lambda_2)+r_n} [D_{k1}^2 n^{-1+\delta} + (D_{k2}^2 + D_{k3}^2)n^{-2+\delta}] \varepsilon_n^{-1} \to 0$ as $n \to \infty$,

  (ii) $\max_{K \leq k < L_n^*(\lambda_1)} \sum_{j=k+1}^{k+r_n} [D_{k1}^2 n^{-1+\delta} + (D_{k2}^2 + D_{k3}^2)n^{-2+\delta}] \varepsilon_n^{-1} \to 0$ as $n \to \infty$,

  (iii) $(L_n^*(\lambda_1) + r_n)n^{-s} \to 0$ as $n \to \infty$,

where $K$ is a predetermined positive integer.

**Assumption F.** For some $\delta > 0$, some $0 < \lambda_2 < 1 < \lambda_1$ and some $s > 0$

  (i) $\sum_{k=1}^{L_n^*(\lambda_2)+r_n} [D_{k1}^2 n^{-1+\delta} + (D_{k2}^2 + D_{k3}^2)n^{-2+\delta}] \to 0$ as $n \to \infty$,

  (ii) $\sum_{k=L_n^*(\lambda_1)+r_n+1}^\infty \gamma_k^2 \to 0$ as $n \to \infty$,

  (iii) $(L_n^*(\lambda_2) + r_n)n^{-s} \to 0$ as $n \to \infty$,

where $K$ is a predetermined positive integer.

**Assumption G.** For some $\delta > 0$ and some $0 < \lambda_2 < 1$

$$\sum_{k=1}^{L_n^*(\lambda_2)+r_n} [D_{k1}n^{-1/2+\delta} + (D_{k2} + D_{k3})n^{-1+\delta}] \to 0 \text{ as } n \to \infty.$$

In practice one can consider either the trigonometric system or Legendre polynomial system (see [6], [5]) and Assumptions E, F, G can be replaced by:

**Assumption H** (trigonometric system). For some $0 < \lambda_2 < 1 < \lambda_1$ and some $0 < \delta < \frac{1}{2}$

   (i) $L_n^*(\lambda_1) + r_n \to \infty$ as $n \to \infty$,
   (ii) $(L_n^*(\lambda_2) + r_n)^2 n^{-1/2+\delta} \to 0$ as $n \to \infty$,
   (iii) $\limsup\limits_{n\to\infty} \frac{r_n n^{-1/2}}{\varepsilon_n} < \infty$.

**Assumption I** (Legendre polynomials). For some $0 < \lambda_2 < 1 < \lambda_1$ and some $0 < \delta < \frac{1}{2}$

   (i) $L_n^*(\lambda_1) + r_n \to \infty$ as $n \to \infty$,
   (ii) $(L_n^*(\lambda_2) + r_n)^{7/2} n^{-1/2+\delta} \to 0$ as $n \to \infty$,
   (iii) $\limsup\limits_{n\to\infty} \frac{r_n (L_n^*(\lambda_2)+r_n)^{3/2} n^{-1/2}}{\varepsilon_n} < \infty$,
   (iv) $(L_n^*(\lambda_2) + r_n)^8 n^{-1} \to 0$ as $n \to \infty$.

In these examples some assumptions are stronger, some are weaker than the above ones.

F (ii) is fulfilled e.g. if $L_n^*(\lambda_1) + r_n \to \infty$ as $n \to \infty$ or $\gamma_k = 0$, for all $k \geq M$ and $L_n^*(\lambda_1) + r_n > M$.

If $D_{ki} \geq D_{k-1,i}$, $D_{ki} \leq D_{k,i+1}$, $k = 1, 2, 3, \dots$, $i = 0, 1, 2, 3$, we can formulate the above assumptions in a simple way.

## 3. Main theorems.

In this section the results concerning properties of the stopping rules, $\hat{\varphi}_n$ as well as the asymptotic distribution of the adaptive estimator $\hat{\theta}_{1n}$ of $\theta_1$ will be formulated.

**Theorem 3.1.** *Let Assumptions* A–E *and* (2.6) *be satisfied, then*

$$L_n^*(\lambda_1) \leq L_n \leq L_n^*(\lambda_2) \text{ in probability as } n \to \infty,$$

*for some* $0 < \lambda_2 < 1 < \lambda_1$ *in the sense that for every* $\varepsilon > 0$ *there exists a positive integer* $n_0$ *such that:*

$$P(L_n^*(\lambda_1) \leq L_n \leq L_n^*(\lambda_2)) \geq 1 - \varepsilon, \text{ for } n \geq n_0,$$

*where* $L_n$ *and* $L_n^*(\lambda)$ *are defined by* (2.3) *and* (2.7), *respectively, i.e. the stopping rule* $L_n$ *is bounded in probability by a nonrandom lower and upper bound in the above sense.*

Next, we state a result concerning $\hat{\varphi}_n$ and $\hat{I}_n$.

**Theorem 3.2.** *Let Assumptions* A–F *and* (2.6) *be satisfied, then*

(3.2)        $\|\hat{\varphi}_n - \varphi_f\| \to 0$,  *in probability as*  $n \to \infty$,

(3.3)        $\displaystyle\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \leq \hat{I}_n \leq \sum_{k=1}^{L_n^*(\lambda_2)+r_n} \gamma_k^2$  *in probability as*  $n \to \infty$

*in the sense as in Theorem 3.1 and hence*

$$\lim_{n\to\infty} \cdot \hat{I}_n = I(f)  \text{ in probability as }  n \to \infty,$$

*where* $\hat{\varphi}_n$ *and* $\hat{I}_n$ *are given by* (2.8) *and* (2.9).

Thus $\hat{\varphi}_n$ and $\hat{I}_n$ are consistent estimators of $\varphi_f$ and $I(f)$, respectively.

Now, we shall present a result on the asymptotic distribution of $[\sum_1^n (c_i - \overline{c}_n)^2]^{1/2}$ $(\hat{\theta}_{1n} - \theta_1)$ as $n$ tending to infinity with $\hat{\theta}_{1n}$ given by (2.10).

**Theorem 3.3.** *Let Assumptions* A–G *and* (2.6) *be satisfied, then* $[\sum_1^n (c_i - \overline{c}_n)^2]^{1/2}$ $(\hat{\theta}_{1n} - \theta_1)$ *has asymptotically normal distribution of* $(0, I(f)^{-1})$, *i.e. is an asymptotically optimal estimator of* $\theta_1$.

Theorems 3.2 and 3.3 imply that $[\sum_1^n (c_i - \overline{c}_n)^2]^{1/2}$ $(\hat{\theta}_{1n} - \theta_1)$ has asymptotically normal distribution $N(0,1)$ and hence, for some given $\alpha \in (0,1)$, we can find a $(1-\alpha)$ confidence interval for $\theta_1$.

## 4. Proof of Theorems 3.1, 3.2, 3.3.

At first, we derive a certain extension of the asymptotic linearity result of [6] and then we use it as a main tool in the proof of Theorems 3.1, 3.2 and 3.3.

Let $Z_1, Z_2, \ldots, Z_n$ be iid random variables. Let $R_i(t)$ denote the rank of $Z_i - c_{in}t$ among $Z_1 - c_{1n}t, \ldots, Z_n - c_{nn}t$ with $c_{in}$ defined by (2.5). Let $P_k$ be defined on $[0,1]$. Define

(4.1)        $$S_n(t, P_k) = \sum_{i=1}^{n} c_{in} \cdot P_k\Big(\frac{R_i(t)}{n+1}\Big).$$

**Theorem 4.1.** *Let* $Z_1, Z_2, \ldots, Z_n$ *be iid random variables with absolute continuous density* $f$ *satisfying Assumption* B. *Let Assumptions* A, C *be satisfied. Then for every* $s > 0$, $\delta > 0$ *and* $A > 0$ *there exist* $d > 0$ *and* $n_0$ *such that for all* $n \geq n_0$:

(4.2)        $$P\big(\sup_{|t|\leq a} \{|S_n(t, P_k) - S_n(0, P_k) + t\gamma_k|\} \geq du_{nk}\big) < n^{-s},$$

*where*

(4.3)        $$u_{nk} = n^{-1/2+\delta} \cdot D_k + n^{-1+\delta}(D_{k2} + D_{k3}),$$

$\delta > 0$ arbitrary, $S_n$ is defined by (4.1), $\gamma_k$ is defined by (2.2).

PROOF: The proof is similar to that of Theorem 4 in [6]. Therefore, we shall provide only the necessary modifications.

In order to prove (4.2) we shall use exponential inequalities and replace the "sup" in (4.2) by a "max" over a set of gridpoints, noticing

$$
\begin{aligned}
(4.4) \quad & \sup_{|t| \leq a} \{|S_n(t, P_k) - S_n(0, P_k) - t\gamma_k|\} \leq \\
& \leq \max_{q=1,\ldots,N} \{|S_n(t_q, P_k) - S_n(0, P_k) - t_q\gamma_k| + \\
& \quad + \sup_{t_q \leq t \leq t_{q+1}} \{|S_n(t, P_k) - S_n(t_q, P_k) - (t - t_q)\gamma_k|\}\},
\end{aligned}
$$

where

$$
t_0 = -a, \quad t_1 = -a + 1a/n, \ldots, t_n = a, \quad N = 2n.
$$

Now, we have

$$
\begin{aligned}
(4.5) \quad & \sup_{t_q \leq t \leq t_{q+1}} \{|S_n(t, P_k) - S_n(t_q, P_k)|\} \leq \\
& \leq \sup_{t_q \leq t \leq t_{q+1}} \left\{ \sum_{i=1}^{n} |c_{in}| \cdot \left| \frac{R_i(t) - R_i(t_q)}{n+1} \right| \cdot D_{k1} \right\} = \\
& = \sum_{i=1}^{n} |c_{in}|(n+1)^{-1} D_{k1} \cdot \sum_{j \neq i} W_{ij}(Z_i),
\end{aligned}
$$

where

$$
\begin{aligned}
W_{ij} = I\{&\min((c_{jn} - c_{in})t_q, (c_{jn} - c_{in})t_{q+1}) \leq Z_j - z \leq \\
& \leq \max((c_{jn} - c_{in})t_q, (c_{jn} - c_{in})t_{q+1})\}, \quad z \in R_1, \quad 1 \leq i, \quad j \leq n.
\end{aligned}
$$

Then one observes that by the exponential inequality and the independence of $Z_1, \ldots, Z_n$ for $z \in R_1$, $\lambda > 0$,

$$
\begin{aligned}
(4.6) \quad & P\left\{ \sum_{j \neq i} (W_{ij}(z) \geq \lambda) \right\} \leq e^{-\lambda} \cdot E\left[ \exp\left\{ \sum_{j \neq i} W_{ij}(z) \right\} \right] \leq \\
& \leq e^{-\lambda} \prod_{j=1}^{n} \left( 1 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \cdot EW_{ij}(z) \right) \leq e^{-\lambda} \exp\left\{ e \cdot \sum_{j \neq i} EW_{ij}(z) \right\} \leq \\
& \leq \exp\left\{ -\lambda + e \cdot K_1(|c_{in}| + \frac{1}{\sqrt{n}}) \right\} \quad \text{for some} \quad K_1 > 0,
\end{aligned}
$$

where we used the fact that for $z \in R_1$

$$
E\{W_{ij}^p(z)\} \leq EW_{ij}(z) \leq K_1|c_{in} - c_{jn}|/n, \quad p = 1, 2, \ldots \quad \text{for some} \quad K_1 > 0.
$$

Then putting $\lambda = d_1 \log n$ one can realize for $z \in R_1$, $d_1 > 0$ and $n$ large

$$(4.7) \qquad P\Big\{\sum_{j\neq i}(W_{ij}(z) \geq d_1 \log n)\Big\} \leq K_2^{-d_1} \quad \text{for some} \quad K_2 > 0.$$

From (4.5)–(4.7) one can conclude that for every $d_1 > 0$ there exist $d_2 > 0$ and $n_0$ such that for $n \geq n_0$

$$P(\sup_{t_q \leq t \leq t_{q+1}} \{|S_n(t, P_k) - S_n(t_q, P_k)|\} \geq d_2 \cdot D_{k1}n^{-1/2}\log n) \leq K_2 \cdot n^{-d_1},$$

which further yields

$$P(\max_{q=1,\dots,N}\sup_{t_q \leq t \leq t^{q+1}} \{|S_n(t, P_k) - S_n(t_q, P_k) - (t - t_q)\cdot\gamma_k|\} >$$
$$> d_2 D_{k1}n^{-1/2}\log n) < n^{-d_1+2} < n^{-s}$$

if $d_1$ is chosen such that $d_1 - 2 > s$ and $n \geq n_0$.

Consequently, it remains to show that for every $d_1 > 0$ there exist $d_2 > 0$ and $n_0 > 0$ such that for $n \geq n_0$

$$P[|S_n(t, P_k) - S_n(0, P_k) - t\gamma_k| \geq d_2(D_{k1}n^{-1/2+\delta} + (D_{k2} + D_{k3})n^{-1+\delta})] <$$
$$< n^{-d_1} \quad \text{for} \quad |t| \leq a$$

the proof of which is similar to that of Theorem 4 of [6] with $t$ fixed, and hence it can be omitted. $\qquad\qquad\square$

PROOF OF THEOREM 3.1: At first, we prove the first inequality.

For every $\lambda > 0$ and positive integer $p \leq L_n^*(\lambda) + r_n$, put

$$A_n(\lambda) = \bigcap_{k=L_n^*(\lambda)+1}^{L_n^*(\lambda)+r_n} A_{nk}$$

and

$$\overset{\circ}{A}_n(\lambda) = \bigcap_{k=p}^{L_n^*(\lambda)+r_n} A_{nk},$$

where

$$(4.8) \qquad\qquad A_{nk} = \{|\hat{\gamma}_{n,k} - \gamma_k| \leq du_{nk}\}$$

with $u_{nk}$ and $\hat{\gamma}_{n,k}$ being defined by (4.3) and (2.4), respectively.

Then for every $\varepsilon > 0$ from Theorem 4.1 and (2.6) one can easily prove that there exists a positive integer $n_0$ such that for $n \geq n_0$

$$(4.9) \qquad\qquad P(\overset{\circ}{A}_n^c) \leq (L_n^*(\lambda) + r_n - p)n^{-s} + \varepsilon,$$

where $\overset{\circ}{A}{}^c_n$ is the complement of the event $\overset{\circ}{A}_n$.

It follows that

(4.10)                              $P(A^c_n(\lambda)) \to 0 \;\; \text{as} \;\; n \to \infty \,,$

where $A^c_n(\lambda)$ is the complement of the event $A_n(\lambda)$.

Further, from the inequality

(4.11)              $\left| \Big( \sum_1^m a_i^2 \Big)^{1/2} - \Big( \sum_1^m b_i^2 \Big)^{1/2} \right| \le \Big( \sum_1^m (a_i - b_i)^2 \Big)^{1/2}$

one receives

$$\Big[ \sum_{k=L^*_n(\lambda)+1}^{L^*_n(\lambda)+r_n} \hat\gamma_{n,k}^2 \Big]^{1/2} \le \Big[ \sum_{k=L^*_n(\lambda)+1}^{L^*_n(\lambda)+r_n} (\hat\gamma_{n,k} - \gamma_k)^2 \Big]^{1/2} + (\lambda \varepsilon_n)^{1/2},$$

which together with the definition of $L_n$ implies that (for $\lambda \in (0,1)$)

$$P(L^*_n(\lambda) < L_n) = P\Big( \Big[ \sum_{k=L^*_n(\lambda)+1}^{L^*_n(\lambda)+r_n} \hat\gamma_{n,k}^2 \Big]^{1/2} > \varepsilon_n^{1/2} \Big) \le$$

$$\le P\Big( \Big[ \sum_{k=L^*_n(\lambda)+1}^{L^*_n(\lambda)+r_n} (\hat\gamma_{n,k} - \gamma_k)^2 \Big]^{1/2} > \varepsilon_n^{1/2}(1 - \lambda^{1/2}) \Big).$$

Hence

(4.12)   $P(L^*_n(\lambda) < L_n) \le$

$$\le P\Big( \sum_{k=L^*_n(\lambda)+1}^{L^*_n(\lambda)+r_n} (\hat\gamma_{n,k} - \gamma_k)^2 > \varepsilon_n (1 - \lambda^{1/2})^2, A_n(\lambda) \Big) + P(A^c_n(\lambda)).$$

On the other hand, on the set $A_n(\lambda)$ and for $0 < \varepsilon < (1 - \lambda^{1/2})^2$ with $0 < \lambda < 1$ and under Assumptions E (i) and D one can easily see that there exists a positive integer $n_0$ such that the first summand term on the r.h.s. of (4.12) is equal to 0 for $n \ge n_0$ and by (4.10) one obtains

$$P(L^*_n(\lambda_2) < L_n) \to 0 \;\; \text{as} \;\; n \to \infty, \;\; \text{for any} \;\; 0 < \lambda_2 < 1.$$

Next, we prove the second inequality. We first note that:

$$P(L^*_n(\lambda) > L_n) = P\Big( \bigcup_{k=K}^{L^*_n(\lambda)-1} \{ L_n = k \} \Big) \le$$

$$\le P\Big( \bigcup_{k=K}^{L^*_n(\lambda)-1} \Big\{ \sum_{j=k+1}^{k+r_n} \hat\gamma_{n,j}^2 \le \varepsilon_n \Big\} \Big) = P\Big( \min_{K \le k < L^*_n(\lambda)} \sum_{j=k+1}^{k+r_n} \hat\gamma_{n,j}^2 \le \varepsilon_n \Big).$$

Hence

$$(4.13) \quad P(L_n^*(\lambda) > L_n) \leq P\Big(\min_{K \leq k < L_n^*(\lambda)} \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \leq \varepsilon_n, B_n(\lambda)\Big) + P(B_n^c(\lambda)),$$

where

$$(4.14) \qquad\qquad B_n(\lambda) = \bigcap_{K \leq k < L_n^*(\lambda)} A_{nk} \ \text{ for every } \ \lambda > 1,$$

with $A_{nk}$ being defined by (4.8).

Using the inequality (4.11) with $a = \gamma_j$, $b = \gamma_j - \hat{\gamma}_{n,j}$, and for every $K \leq k < L_n^*(\lambda)$, then one has

$$(4.15) \quad (\lambda \varepsilon_n)^{1/2} \leq \Big[ \sum_{j=k+1}^{k+r_n} \gamma_j^2 \Big]^{1/2} \leq \Big[ \sum_{j=k+1}^{k+r_n} (\hat{\gamma}_{n,j} - \gamma_j)^2 \Big]^{1/2} + \Big[ \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \Big]^{1/2}.$$

which implies that on the set $B_n$

$$(4.16) \quad \min_{K \leq k < L_n^*(\lambda)} \Big[ \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \Big]^{1/2} > (\lambda \varepsilon_n)^{1/2} - \max_{K \leq k < L_n^*(\lambda)} d \Big[ \sum_{j=k+1}^{k+r_n} u_{n,j}^2 \Big]^{1/2}.$$

By Assumption E (ii), there exists a positive integer $n_0$ such that for all $n \geq n_0$ the second member on the r.h.s. of (4.16) is larger than $-\varepsilon \varepsilon_n$ ($0 < \varepsilon < \lambda^{1/2} - 1$). It follows that

$$\min_{K \leq k < L_n^*(\lambda)} \sum_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 > (\lambda^{1/2} - \varepsilon)\varepsilon_n > \varepsilon_n, \ \text{ for } \ n \geq n_0,$$

which implies that for $n \geq n_0$, the first summand term on the r.h.s. of (4.12) is equal to 0. Hence there exists $\lambda_1 > 1$ for every $\varepsilon > 0$ and $n$ large such that

$$P(L_n^*(\lambda) > L_n) \leq P(B_n^c) \leq (L_n^*(\lambda_1) + r_n)n^{-s} + \varepsilon \ \text{ (by (4.10) and (4.14))},$$

which tends to 0 by Assumption E (iii). This completes the proof of Theorem 3.1.
□

PROOF OF THEOREM 3.2: At first, we prove (3.2). Putting

$$(4.17) \qquad\qquad C_n(\lambda_2) = \bigcap_{k=1}^{L_n^*(\lambda_2)+r_n} A_{nk},$$

where $A_{nk}$ is defined by (4.8), then by (4.9) and F (iii) one has

$$P(C_n^c(\lambda_2)) \to 0 \text{ as } n \to \infty,$$

where $C_n^c(\lambda_2)$ is the complement of $C_n(\lambda_2)$ for $\lambda_2 < 1$. Note that

$$\|\hat{\varphi}_n - \varphi\|^2 = \int_0^1 (\hat{\varphi}_n - \varphi)^2(u)\,du = \sum_{k=1}^{L_n+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + \sum_{k=L_n(\lambda)+r_n+1}^{\infty} \gamma_k^2$$

and

(4.18) $P(\|\hat{\varphi}_n - \varphi\| \geq \varepsilon) \leq P(\|\hat{\varphi}_n - \varphi\| \geq \varepsilon, L_n \leq L_n^*(\lambda_2), C_n(\lambda_2)) +$
$$+ P(L_n > L_n^*(\lambda_2)) + P(C_n^c(\lambda_2)).$$

By Assumption F (i) and (ii), for every $\varepsilon > 0$ there exists $n_1$ such that for $n \geq n_1$ the first probability on the r.h.s. of (4.18) is equal to 0 and one gets

$$P(\|\hat{\varphi}_n - \varphi\| \geq \varepsilon) \leq P(L_n > L_n^*(\lambda_2)) + P(C_n^c(\lambda_2)), \text{ for } n \geq n_1,$$

which together with (4.17) and Theorem 3.1 implies (3.2).

Now, we shall prove (3.3) in two steps:

(i) $\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \leq \hat{I}_n$ in probability as $n \to \infty$.

For every $\varepsilon > 0$, we have

$$P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon\Big) \leq$$

$$\leq P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon, L_n \geq L_n^*(\lambda_1)\Big) + P(L_n < L_n^*(\lambda_1)).$$

On the other hand,

$$-\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k}^2 - \gamma_k^2) = -2\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)\gamma_k - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 \leq$$

$$\leq 2\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} + \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2.$$

Hence

(4.19) $P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon\Big) \leq P(C_n^c(\lambda_1)) +$

$$+ P\Big(\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon, C_n(\lambda_1)\Big) + P(L_n < L_n^*(\lambda_1)),$$

where $C_n(\lambda_1)$ is defined similarly as $C_n(\lambda_2)$ with $0 < \lambda_2 < 1 < \lambda_1$. From (4.9), F (iii) and Theorem 3.1 one obtains $P(C_n^c(\lambda_1)) \to 0$ as $n \to \infty$. By the same way as in the proof of (3.2), it follows from F (i) that the second probability on the r.h.s. (4.19) is equal to 0 as $n \geq n_2$ (for some positive integer $n_2$). And hence

$$P\Big(\hat{I}_n - \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 < -\varepsilon\Big) \to 0 \quad \text{as } n \to \infty, \quad \text{for every } \varepsilon > 0,$$

which implies the first inequality of (3.3).

(ii) $\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \geq \hat{I}_n$ in probability as $n \to \infty$.

Choosing $a_k = \hat{\gamma}_{n,k}$, $b_k = \gamma_k$ in the inequality (4.11) then, for every $\varepsilon > 0$, we have

$$P\Big(\hat{I}_n > \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 + \varepsilon\Big) \leq$$

$$\leq P\Big(\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + 2\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon\Big) +$$

$$+ P(L_n > L_n^*(\lambda_2)),$$

which ensures

$$(4.20)\quad P\Big(\hat{I}_n > \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 + \varepsilon\Big) \leq$$

$$\leq P\Big(\sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + 2\Big[I(f) \sum_{k=1}^{L_n^*(\lambda_1)+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2\Big]^{1/2} > \varepsilon, C_n(\lambda_2)\Big) +$$

$$+ P(L_n > L_n^*(\lambda_2)) + P(C_n^c(\lambda_2)),$$

where $C_n(\lambda_2)$ is defined by (4.17) and $C_n^c(\lambda_2)$ is its complement.

It follows from Assumption F (i) that the first probability on the r.h.s. of (4.20) is equal to 0 for $n \geq n_3$ ($n_3$ is some positive integer); and hence, the second inequality of (3.3) is proved. $\square$

PROOF OF THEOREM 3.3: Recall that

$$\hat{\theta}_{1n} = \overline{\theta}_{1n} + \Big[\hat{I}_n\Big(\sum_1^n (c_i - \overline{c}_n)^2\Big)^{1/2}\Big]^{-1} \sum_1^n c_{in} \hat{\varphi}_n(\overset{\circ}{R}_i(\overline{\theta}_{1n})/(n+1)),$$

where $\hat{\varphi}_n$, $\hat{I}_n$ are defined by (2.8) and by (2.9), respectively. $\overset{\circ}{R}_i(\overline{\theta}_{1n})$ is the rank of $X_i - c_i\overline{\theta}_{1n}$ among $X_1 - c_1\overline{\theta}_{1n}, \ldots, X_n - c_n\overline{\theta}_{1n}$.

In order to be able to apply Theorem 4.1 we rewrite $\hat{\theta}_{1n}$ as follows:

$$(4.21) \quad \hat{\theta}_{1n} = \overline{\theta}_{1n} + \left[\hat{I}_n\left(\sum_1^n (c_i - \overline{c}_n)^2\right)^{1/2}\right]^{-1} \cdot$$

$$\sum_1^n c_{in}\hat{\varphi}_n\left(R_i\left(\left[\sum_1^n (c_i - \overline{c}_n)^2\right]^{1/2}(\overline{\theta}_{1n} - \theta_1)\right)\right)/(n+1)\right),$$

where $R_i(t)$ is the rank of $e_i - c_{in}t$ among $e_1 - c_{1n}t, \ldots, e_n - c_{nn}t$.

Next,

$$(4.22)$$

$$\sup_{|t|\le a}\left|\sum_1^n c_{in}[\hat{\varphi}_n(R_i(t)/(n+1)) - \hat{\varphi}_n(R_i(0)/(n+1))]+\right.$$

$$\left. + t\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du\right| =$$

$$= \sup_{|t|\le a}\left|\sum_{k=1}^{L_n+r_n}\hat{\gamma}_{n,k}\left(\sum_1^n c_{in}[P_k(R_i(t)/(n+1)) - P_k(R_i(0)/(n+1))]+\right.\right.$$

$$\left.\left. + t\int_0^1 P_k(u)\varphi_f(u)\,du\right)\right| \le$$

$$\le \sup_{|t|\le a}\left|\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\hat{\gamma}_{n,k}^2 \sum_{k=1}^{L_n^*(\lambda_2)+r_n}\left[\sum_1^n c_{in}\left[P_k\left(\frac{R_i(t)}{n+1}\right) - P_k\left(\frac{R_i(0)}{n+1}\right)\right]+\right.\right.$$

$$\left.\left. + t\int_0^1 P_k(u)\varphi_f(u)\,du\right]^2\right|^{1/2} \quad \text{in probability as } n \to \infty.$$

Since $\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\hat{\gamma}_{n,k}^2 \le 2\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\gamma_k^2 + 2\sum_{k=1}^{L_n^*(\lambda_2)+r_n}(\hat{\gamma}_{n,k} - \gamma_k)^2$ and $\sum_{k=1}^{L_n^*(\lambda_2)+r_n}\gamma_k^2 \le I(f)$, one gets from Assumption F, Theorem 4.1, (2.6) and (4.21)

$$(4.23) \quad \sup_{|t|\le a}\left|\sum_1^n c_{in}[\hat{\varphi}_n(R_i(t)/(n+1)) - \hat{\varphi}_n(R_i(0)/(n+1))]+\right.$$

$$\left. + t\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du\right| \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Further, we have

$$\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du = \sum_{k=1}^{L_n+r_n}\hat{\gamma}_{n,k}\gamma_k = \sum_{k=1}^{L_n+r_n}\hat{\gamma}_{n,k}(\gamma_k - \hat{\gamma}_{n,k}) + \hat{I}_n.$$

Hence, by the Cauchy-Schwarz inequality one has

$$(4.24) \quad \left|1 - (\hat{I}_n)^{-1}\int_0^1 \hat{\varphi}_n(u)\varphi_f(u)\,du\right| \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

From (4.21), (4.23) and (4.24) we can write

$$(4.25) \quad \Big[\sum_1^n (c_i - \bar{c}_n)^2\Big]^{1/2} (\hat{\theta}_{1n} - \theta_1) = (\hat{I}_n)^{-1} \sum_1^n c_{in} \hat{\varphi}_n(R_i(0)/(n+1)) +$$

$$+ o_P(1) \quad \text{as} \quad n \to \infty.$$

Now, in order to prove the assertion of our theorem, it is sufficient to show that

$$(4.26) \quad (\hat{I}_n)^{-1} \sum_1^n c_{in} \hat{\varphi}_n(R_i(0)/(n+1)) \xrightarrow{D} N(0, (I(f))^{-1})$$

where $I(f)$ is the Fisher information.

Note that (4.26) will be implied by the following two assertions:

$$(4.27) \quad H_n = \sum_{k=1}^{L_n + r_n} (\hat{\gamma}_{n,k} - \gamma_k) \sum_1^n c_{in} P_k(R_i(0)/(n+1)) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty$$

and

$$(4.28) \quad \sum_1^n c_{in} \sum_{k=1}^{L_n + r_n} \gamma_k P_k(R_i(0)/(n+1)) \xrightarrow{D} N(0, I(f)) \quad \text{as} \quad n \to \infty.$$

Putting

$$H_{1n} = \sum_{k=1}^{L_n + r_n} (\hat{\gamma}_{n,k} - \gamma_k) I_{A_{nk}} \sum_1^n c_{in} P_k(R_i(0)/(n+1)),$$

where $A_{nk}$ is defined by (4.8), $I_A$ is the indicator function of the set $A$ and

$$H_{2n} = \sum_{k=1}^{L_n + r_n} (\hat{\gamma}_{n,k} - \gamma_k) I_{A_{nk}^c} \sum_1^n c_{in} P_k(R_i(0)/(n+1)).$$

We shall prove that $H_{in} \xrightarrow{P} 0$ as $n \to \infty$, $i = 1, 2$. Clearly

$$(4.29) \quad E|H_{1n}| \le d \sum_{k=1}^{L_n^*(\lambda_2) + r_n} u_{nk} \Big[ E\Big(\sum_1^n c_{in} P_k(R_i(0)/(n+1))^2\Big) \Big]^{1/2},$$

where

$$E\Big(\sum_1^n c_{in} P_k(R_i(0)/(n+1))^2\Big) = \text{var}\Big(\sum_1^n c_{in} P_k(R_i(0)/(n+1))^2\Big) =$$

$$= (n-1)^{-1} \sum_{i=1}^n \Big[ P_k(i/(n+1)) - n^{-1} \sum_{j=1}^n P_k(j/(n+1)) \Big]^2 \le$$

$$\le (n-1)^{-1} \sum_{i=1}^n P_k^2(i/(n+1)) \le$$

$$\le 1 + \frac{n}{n-1} \Big| \sum_{i=1}^n \int_{(i-1)/n}^{i/n} (P_k^2(i/(n+1)) - P_k^2(u)) \, du \Big| \le 1 + 4(D_{k1} + D_{k1}^2) n^{-1}$$

(by the mean value theorem for the function $P_k^2(u)$).

Hence, by Assumptions F, G and (4.29), one receives

$$EH_{1n} \to 0 \quad \text{and then} \quad H_{1n} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

As for $H_{2n}$, by (4.8) and (4.9) we have for every $\varepsilon > 0$ and $n$ large

$$P(|H_{2n}| > \eta) \le P(H_{2n} \ne 0) \le P(H_{2n} \ne 0, L_n \le L_n^*(\lambda_2)) + P(L_n > L_n^*(\lambda_2)) \le$$

$$\le P(L_n > L_n^*(\lambda_2)) + P\left( \bigcup_{k=1}^{L_n^*(\lambda_2)+r_n} \{|\hat{\gamma}_{n,k} - \gamma_k| > du_{nk}\} \right) \le$$

$$\le P(L_n > L_n^*(\lambda_2)) + (L_n^*(\lambda_2) + r_n)n^{-s} + \varepsilon,$$

which tends to 0 as $n$ goes to infinity. Hence (4.27) is proved.

Next, we turn to (4.28). Obviously, it is sufficient to show that

$$(4.30) \qquad \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k \sum_{1}^{n} c_{in} P_k(R_i(0).(n+1)) \xrightarrow{D} N(0, I(f)),$$

and

$$(4.31) \qquad \sum_{k=L_n^*(\lambda_1)+r_n}^{L_n+r_n} \gamma_k \sum_{1}^{n} c_{in} P_k(R_i(0)/(n+1)) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

We start with (4.30). Note that Assumption A implies that

$$(4.32) \qquad \max_{1 \le i \le n} c_{in}^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Putting

$$a_n(i) = \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k P_k(i/(n+1)),$$

one can write

$$\sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k \sum_{1}^{n} c_{in} P_k(R_i(0)/(n+1)) = \sum_{i=1}^{n} c_{in} a_n(R_i(0))$$

and

$$\int_0^1 (a_n(1+[un]) - \varphi_f(u))^2 \, du = \int_0^1 \Big\{ \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k \Big( P_k\Big(\frac{1+[un]}{n+1}\Big) - P_k(u)\Big) +$$

$$+ \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k P_k(u) \Big\}^2 \, du \le$$

$$\le 2 \int_0^1 \Big\{ \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k ((P_k(1+[un])/(n+1)) - P_k(u)) \Big\}^2 \, du +$$

$$+ 2 \int_0^1 \Big\{ \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k P_k(u) \Big\}^2 \, du \le$$

$$\le 2 \Big\{ \sum_{k=1}^{L_n^*(\lambda_1)+r_n} |\gamma_k| D_{k1} n^{-1} \Big\}^2 + 2 \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k^2,$$

which implies that

$$\int_0^1 (a_n(1+[un]) - \varphi_f(u))^2 \, du \le 2 \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k^2 \sum_{k=1}^{L_n^*(\lambda_1)+r_n} D_{k1}^2 n^{-2} +$$

$$+ 2 \sum_{k=L_n^*(\lambda_1)+r_n+1}^{\infty} \gamma_k^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, by Theorem 1.6 (a) in [3, p. 163] and (4.32) we can conclude that

$$\sum_1^n c_{in} \sum_{k=1}^{L_n^*(\lambda_1)+r_n} \gamma_k P_k(R_i(0)/(n+1)) = \sum_1^n c_{in} a_n(R_i(0)) \xrightarrow{D} N(\mu, \sigma_f^2) \quad \text{as} \quad n \to \infty,$$

where

$$\mu = E \sum_1^n c_{in} a_n(R_i(0)) = 0$$

and

$$\sigma_f^2 = \lim_{n \to \infty} \text{var} \Big( \sum_1^n c_{in} a_n(R_i(0)) \Big) =$$

$$= \lim_{n \to \infty} \sum_1^n c_{in}^2 \int_0^1 (\varphi(u) - \overline{\varphi})^2 \, du = \int_0^1 (\varphi(u) - \overline{\varphi})^2 \, du = I(f)$$

with $\overline{\varphi} = \int_0^1 \varphi(u) \, du$.

Now, we prove (4.31). Putting

$$S_j = \sum_{k=L_n^*(\lambda_1)+r_n}^{L_n^*(\lambda_1)+r_n+j} \gamma_k \sum_1^n c_{in} P_k(R_i(0)/(n+1)), \quad 1 \leq j \leq N,$$

where $N = L_n^*(\lambda_2) - L_n^*(\lambda_1)$,

$S_0 = 0$,

$M_N = \max_{0 \leq j \leq N} |S_j|$.

$M_N' = \max_{0 \leq j \leq N} \min\{|S_j|, |S_n - S_j|\}$,

then

$$M_N' \leq M_N$$

and by [2, 12.4] we have

$$M_N \leq M_N' + S_N.$$

In order to prove (4.31), it is sufficient to check that $M_N'$ and $S_N$ converge in probability to 0. We start with the first assertion. To do this we will compute

$$E\Big[\sum_1^n c_{in} \sum_{k=p+1}^q \gamma_k P_k(R_i(0)/(n+1))\Big]^4 \quad \text{for every} \ \ p, q:$$

$$L_n^*(\lambda_1) + r_n \leq p \leq q \leq L_n^*(\lambda_2) + r_n.$$

After straightforward but long computations (cf. [8]) one receives

$$(4.33) \qquad E\Big[\sum_1^n c_{in} \sum_{k=p+1}^q \gamma_k P_k(R_i(0)/(n+1))\Big]^4 \leq K_3\Big(\sum_{p+1}^q \gamma_k^2\Big)^2,$$

for some $K_3 > 0$ and $n$ large. It follows that for every $i, j, k : 0 \leq i \leq j \leq k \leq N$ and for every $\lambda > 0$

$$P(|S_j - S_i| > \lambda, |S_k - S_j| > \lambda) \leq P(|S_j - S_i| > \lambda) \leq \lambda^{-4} E(S_j - S_i)^4 \leq$$

$$\leq K_4\Big(\sum_{m=i+1}^k \gamma_m^2\Big)^2, \quad \text{for some} \ \ K_4 > 0.$$

So, the assumption (12.11) of Theorem 12.1, [2], is satisfied, and now this theorem can be applied, and one gets

$$P(M_N' > \lambda) \leq \lambda^{-4} K_5 \Big[\sum_{k=L_n^*(\lambda_1)+r_n+1}^{L_n^*(\lambda_2)+r_n} \gamma_k^2\Big]^2 \quad (\gamma = \alpha = 2),$$

for some $K_5 > 0$, i.e. $M_N' \xrightarrow{P} 0$, and if we take $p = L_n^*(\lambda_1) + r_n$ and $q = L_n^*(\lambda_2) + r_n$ in (4.32) then $S_N \xrightarrow{P} 0$. These ensure that

$M_N = \max_{0 \leq j \leq N} |\sum_{k=L_n^*(\lambda_1)+r_n}^{L_n^*(\lambda_1)+r_n+j} \gamma_k \sum_1^n c_{in} P_k(R_i(0)/(n+1))| \xrightarrow{P} 0$ as $n \to \infty$, hence (4.31) is proved. The proof of the theorem is complete. $\quad\square$

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