Selfduality of the system of convex subsets of a partially ordered set

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Abstract. For a partially ordered set P let us denote by CoP the system of all convex subsets of P. It is found the necessary and sufficient condition (concerning P) under which CoP (as a partially ordered set) is selfdual.

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1. Introduction.

For a partially ordered set P we denote by CoP the system of all convex subsets of P. The system CoP is partially ordered by the set-theoretical inclusion. It is not difficult to see that CoP is a lattice. The aim of this paper is to find a necessary and sufficient condition (concerning P) under which CoP is selfdual.

An analogous question for $IntP$ (the system of all intervals of a partially ordered set P) was investigated by J. Jakubík.

In [1], the following theorem was proved:

(T) Let P be a partially ordered set. Then the following conditions are equivalent:

(i) The partially ordered set $IntP$ is selfdual.

(ii) P is a lattice such that either card $P \leq 2$, or card $P = 4$ and P has two atoms.

2. Results.

Let Q be a partially ordered set. A subset C of Q is called convex if the following holds:

If $x, y \in C$, $z \in Q$ and $x \leq z \leq y$, then $z \in C$.

Thus the empty set is convex.

The partially ordered set Q is said to be selfdual if there exists a dual automorphism of Q, i.e. such a bijection $f : Q \to Q$, that for all $x, y \in Q$ we have

$$x \leq y \Leftrightarrow f(x) \geq f(y).$$

An element $q \in Q$ is said to be extremal if it is minimal or maximal element of Q. Finally, for $r, s \in Q$ we put $[r, s] = \{\, u \in Q : r \leq u \leq s \,\}$.

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From now on let P denote a partially ordered set. The system CoP is a lattice (intersection of convex subsets is clearly convex) with the least element ($\emptyset$) and the greatest element (P).

Let $X \in CoP$. The set X is an atom of CoP if and only if there is $a \in P$ with $X = \{a\}$.

Lemma. *Let $X \subseteq P$. Then X is a dual atom of CoP if and only if there exists $x \in P$ such that $X = P \setminus \{x\}$ and x is extremal.*

Proof: Let x be an extremal element of P such that $X = P \setminus \{x\}$. In order to prove the convexity of $P \setminus \{x\}$, suppose that $u, v \in P \setminus \{x\}$, $z \in P$, $u \le z \le v$. If $z = x$, then either $u = x$ or $v = x$, since x is extremal. But that is a contradiction, because $u, v \in P \setminus \{x\}$. We have proved that $X = P \setminus \{x\}$ is convex and hence X is a dual atom of CoP.

Now let $X \subseteq P$ be a dual atom of CoP. We distinguish two cases:

(1) There exists an extremal element $y \in P$ such that $y \notin X$.
Then $X \subseteq P \setminus \{y\}$ and $P \setminus \{y\}$ is a dual atom, according to the first part of the proof. Since X is a dual atom, necessarily $X = P \setminus \{y\}$. Thus in the case (1) the assertion is proved.

(2) X contains all extremal elements of P.
Since X is a dual atom, there exists $x \in P \setminus X$. Consider $C(X \cup \{x\})$—the convex closure of $X \cup \{x\}$. It is not difficult to see that $C(X \cup \{x\}) = \{\, z \in P : \text{there exist } t, u \in X \cup \{x\} \text{ such that } t \le z \le u \,\}$. Next, $x \in P \setminus X$, X contains all extremal elements of P, therefore x is not extremal. Then there are $x_1, x_2 \in P$ such that $x_1 < x < x_2$.

Let $x_1, x_2 \in C(X \cup \{x\})$. Then there exist $t_1, u_1, t_2, u_2 \in X \cup \{x\}$ such that $t_1 \le x_1 \le u_1$ and $t_2 \le x_2 \le u_2$. We have $t_1 \le x_1 < x < x_2 \le u_2$, so $t_1, u_2 \in X$. Since X is convex, we obtain $x \in X$, which is a contradiction.

Hence either x_1 or x_2 does not belong to $C(X \cup \{x\})$, which means $C(X \cup \{x\}) \ne P$. Then $X \subsetneq C(X \cup \{x\}) \subsetneq P$, which is a contradiction with the fact that X is a dual atom. Thus the case (2) cannot occur. $\square$

Theorem. *Let P be a partially ordered set. Then the following conditions are equivalent:*

(i) *CoP is selfdual.*
(ii) *P does not contain a three-element chain.*
(iii) *Each subset of P is convex.*

Proof: (i) $\Rightarrow$ (ii) Let CoP be selfdual and let P contain a three-element chain. Then we have $a, b, c \in P$ with $a < b < c$. As we know, $\{a\}, \{b\}, \{c\}$ are atoms of CoP. Let f be a dual automorphism of CoP. Then $f(\{a\}), f(\{b\}), f(\{c\})$ are dual atoms and by Lemma there exist distinct extremal elements $x, y, z \in P$ such that $f(\{a\}) = P \setminus \{x\}$, $f(\{b\}) = P \setminus \{y\}$ and $f(\{c\}) = P \setminus \{z\}$. Consider intervals $[a, b]$ and $[a, c]$. It is easily seen that $\{a\} \vee \{c\} = [a, c]$ in CoP. Using the dual automorphism f we get $f([a, c]) = f(\{a\} \vee \{c\}) = f(\{a\}) \wedge f(\{c\}) = (P \setminus \{x\}) \wedge (P \setminus \{z\}) = (P \setminus \{x\}) \cap (P \setminus \{z\}) = P \setminus \{x, z\}$. Analogously, $f([a, b]) = P \setminus \{x, y\}$. Since $a < b < c$,

$[a, b] \subseteq [a, c]$, which implies $f([a, c]) \subseteq f([a, b])$. Hence $P \setminus \{x, z\} \subseteq P \setminus \{x, y\}$, which is a contradiction.

(ii) $\Rightarrow$ (iii) Suppose that P does not contain a three-element chain. Let $X \subseteq P$ and let $x, y \in X$, $z \in P$, $x \leq z \leq y$. Now P does not contain a three-element chain, so either $z = x$ or $z = y$. Hence $z \in X$ and X is convex.

(iii) $\Rightarrow$ (i) We define the dual automorphism f as follows: $f(X) = P \setminus X$. $\square$

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References

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