On hit-and-miss hyperspace topologies

Gerald Beer, Robert K. Tamaki

Abstract. The Vietoris topology and Fell topologies on the closed subsets of a Hausdorff uniform space are prototypes for hit-and-miss hyperspace topologies, having as a subbase all closed sets that hit a variable open set, plus all closed sets that miss (= fail to intersect) a variable closed set belonging to a prescribed family Δ of closed sets. In the case of the Fell topology, where Δ consists of the compact sets, a closed set A misses a member B of Δ if and only if A is far from B in a uniform sense. With the Fell topology as a point of departure, one can consider proximal hit-and-miss hyperspace topologies, where "miss" is replaced by "far from" in the above formulation. Interest in these objects has been driven by their applicability to convex analysis, where the Mosco topology, the slice topology, and the linear topology have received close scrutiny in recent years.

In this article we look closely at the relationship between hit-and-miss and proximal hit-and-miss topologies determined by a class Δ. In the setting of metric spaces, necessary and sufficient conditions on Δ are given for one to contain the other. Particular attention is given to these topologies when Δ consists of the family of closed balls in a metric space, and their interplay with the Wijsman topology is considered in some detail.

Keywords: hyperspace, hit-and-miss topology, proximal topology, Wijsman topology, Kuratowski-Painlevé convergence, almost convex metric

Classification: 54B20

1. Introduction.

Let $\langle X,\mathcal{U}\rangle$ be a Hausdorff uniform space, and let $CL(X)$ and $K(X)$ be the nonempty closed and compact subsets of X, respectively. Two of the most familiar *hyperspace topologies*, i.e. topologies on $CL(X)$, are the Vietoris topology and the Fell topology. Generically, these are "hit-and-miss" topologies. To explain this terminology, we introduce some notation. If $E \subset X$, we write E^- and E^+ for the following collections of subsets of $CL(X)$:

$$E^- \equiv \{A \in CL(X) : A \cap E \neq \emptyset\},$$
$$E^+ \equiv \{A \in CL(X) : A \subset E\}.$$

Sets in E^- *hit* E, whereas sets in E^+ *miss* the complement E^c of E. The *Vietoris topology* τ_V [Mi], [KT], [FLL], [BLLN] has as subbase all sets of the form V^- where V is an open subset of X, plus all sets of the form W^+ where W is an open subset of X. The weaker *Fell topology* τ_F [Fe], [Po], [At], [KT] has as a subbase all sets of the form V^- where V is open, plus all sets of the form W^+ where W has compact complement.

There is a significant qualitative difference between these two topologies, which in our view, explains the greater applicability of the latter (see e.g. [At]): if a closed set A misses a compact set K, it is automatically *far* from A, in that there exists an entourage $U \in \mathcal{U}$ such that $A \cap U[K] = \emptyset$. On the other hand, if A misses a closed set F, then A and F can be asymptotic to one another. In the last six years, so-called *proximal topologies* [DCNS] have been under intense investigation, wherein the family $K(X)$ of far sets is replaced by a different family Δ. If $B \subset X$, we now write $(B^c)^{++}$ for the closed sets that are far from B, that is, $(B^c)^{++} \equiv \{A \in CL(X) : \exists U \in \mathcal{U} \text{ with } A \cap U[B] = \emptyset\}$.

Definition 1.1. Let $\langle X, \mathcal{U} \rangle$ be a uniform space, and let Δ be a subfamily of $CL(X)$. Then the Δ-*proximal topology* τ_Δ^{++} on $CL(X)$ has as subbase all sets of the form V^- where V is open, plus all sets of the form $(B^c)^{++}$, where $B \in \Delta$.

In contrast, we will write τ_Δ^{+} for the topology having as subbase all sets of the form V^- where V is open, plus all sets of the form $(B^c)^{+}$, where $B \in \Delta$, as studied in the abstract by Poppe [Po]. Although proximal topologies perhaps were first explicitly noted by Nachman [Na], they did not receive much attention until it was shown that one of them was compatible with the celebrated Mosco convergence [Mo1], [Mo2], [At], [So] of sequences of closed convex sets in an arbitrary Banach space: take $\Delta =$ the nonempty weakly compact subsets of the Banach space [Be2, Theorem 3.1]. Again for convex sets, with $\Delta =$ the nonempty closed convex subsets of a normed linear space, the induced proximal topology is the supremum of the weak topologies induced by support functionals and distance functionals, viewed as functions of a set argument with fixed point variable [Be4], [He]. With $\Delta =$ the nonempty closed and bounded convex subsets, the induced proximal topology, called the *slice topology*, agrees with the fundamental *Joly topology* [Jo] formulated in the context of locally convex spaces, and is stable with respect to duality in any normed linear space [Be5]. Proximal topologies defined on the closed subsets of a metric space $\langle X, d \rangle$ induced by the families $\Delta = CL(X)$ and $\Delta =$ the nonempty closed and bounded sets are considered in [BLLN], [BL1], [BL2], [SZ].

For the proximal topologies that have received much scrutiny — including all of those mentioned above, it has been the case that $\tau_\Delta^{+} \supset \tau_\Delta^{++}$. But this containment need not always hold; in particular, it need not hold when $\Delta =$ the closed balls of a metric space. It is the main purpose of this note to display in the context of an arbitrary metric space necessary and sufficient conditions for the inclusions $\tau_\Delta^{+} \supset \tau_\Delta^{++}$ and for $\tau_\Delta^{+} \subset \tau_\Delta^{++}$. Particular attention is given to the case when $\Delta =$ the family of closed balls, and we look closely at the relationship of the hyperspaces τ_Δ^{+} and τ_Δ^{++} to the Wijsman topology, i.e. the topology of pointwise convergence of distance functionals [Wi], [Co], [FLL], [LL], [Be1], [BaP], [BLLN], [BL1], ...

2. Notation and terminology.

Let $\langle X, d \rangle$ be a metric space. If $x \in X$ and A is a nonempty subset of X, we write $d(x, A)$ for the distance from x to A, i.e. $d(x, A) = \inf_{a \in A} d(x, A)$. If A and B are nonempty subsets of X, we write $D_d(A, B)$ for the gap between A and B, i.e. $D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. If $x \in X$ and $\alpha > 0$, we write $S_\alpha[x]$ and

$\overline{S}_\alpha[x]$ for the open and closed balls about x of radius α, respectively. We denote the family of closed balls by $\mathcal{B}$. If A is a nonempy subset of X, then $S_\alpha[A] \equiv \{x : d(x, A) < \alpha\} = \bigcup_{a\in A} S_\alpha[a]$, and $\overline{S}_\alpha[A] \equiv \{x : d(x, A) \leq \alpha\}$, which may contain $\bigcup_{a\in A} \overline{S}_\alpha[a]$ properly. We call sets of the form $S_\alpha[A]$ and $\overline{S}_\alpha[A]$ *enlargements* of A. When working with the proximal topologies in the context of metric spaces, the metric uniformity will be understood. Thus, nonempty sets A and B are far if and only if for some $\alpha > 0$, we have $S_\alpha[A] \cap B = \emptyset$, that is, $D_d(A, B) > 0$. Clearly, A and B are far if and only if cl A and cl B are far, and so that when considering proximal topologies, there is no loss in generality in requiring members of Δ to belong to $CL(X)$.

3. Results.

It is clear that the relationship between the topologies τ_Δ^+ and τ_Δ^{++} is determined by the relationship between the "upper topologies" generated by the families $\{(B^c)^+ : B \in \Delta\}$ and $\{(B^c)^{++} : B \in \Delta\}$. Nevertheless, we choose to formulate our results for the full topologies, although their equal "lower halves" never come into play.

In a recent article, Di Maio and Naimpally [DMN] claimed to give a counterexample to the inclusion $\tau_\Delta^+ \supset \tau_\Delta^{++}$ when Δ is $\mathcal{B}$, the family of closed balls in a metric space. It is instructive to see exactly why their construction is valid, and we pause to provide the details.

Example 3.1. Let ℓ_∞ be the normed linear space of bounded real sequences with the usual sup norm. Write θ for the origin of the space, and let $\{e_n : n \in \mathbf{Z}^+\}$ be the standard set of unit vectors. Consider the metric subspace

$$X = \{\theta\} \cup \{e_{2n} : n \in \mathbf{Z}^+\} \cup \{\frac{n+1}{n} e_n : n \in \mathbf{Z}^+\},$$

and let $A = \{\frac{n+1}{n} e_n : n \text{ odd}\}$. Note that every point in A has distance more than 1 from every other point of X. The space X and the set A are shown in Figure 1.

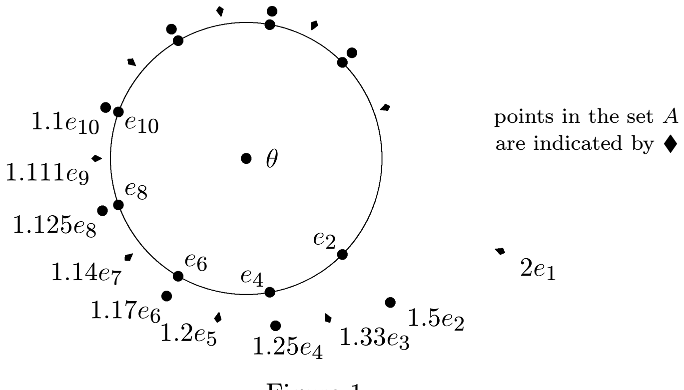

Figure 1

We claim that $(\overline{S}_1[\theta]^c)^{++} \in \tau_{\mathcal{B}}^{++} - \tau_{\mathcal{B}}^+$. To see this, observe that $A \in (\overline{S}_1[\theta]^c)^{++}$ and let

$$V = (\overline{S}_\delta[\theta]^c)^+ \cap (\overline{S}_{\delta_1}[\lambda_1 e_{s_1}]^c)^+ \cap \cdots \cap (\overline{S}_{\delta_k}[\lambda_k e_{s_k}]^c)^+ \cap U_1^- \cap \cdots \cap U_n^-$$

be an arbitrary basic $\tau_{\mathcal{B}}^{+}$-open neighborhood of A, where s_k is an even positive integer, and either $\lambda_k = 1$ or $\lambda_k = (s_k+1)/s_k$, and $U_1, U_2, \ldots, U_n$ are open subsets of X each of which meets A. Defining

$$F = A \cup \{\frac{n+1}{n} e_n : n \ \text{ even, and } n \neq s_1, \ldots, s_k\},$$

we claim that $F \in V$, whereas $F \notin (\overline{S}_1[\theta]^c)^{++}$ because F contains a terminal tail of the set

$$E = \{\frac{n+1}{n} e_n : n \ \text{ even}\}.$$

Since $A \in V$, $\delta \leq 1$, and so $\overline{S}_\delta[\theta]$ cannot contain any points of E. Each $\overline{S}_{\delta_i}[\lambda_i e_{s_i}]$ cannot contain more than one point of E (viz. the point $\frac{s_i+1}{s_i} e_{s_i}$), since otherwise it would contain points of A. Finally, since $F \supset A$, F hits all of the U_i's used in the definition of V. □

For many classes Δ, we have the following property which is clearly sufficient for $\tau_{\Delta}^{++} \subset \tau_{\Delta}^{+}$: whenever $B_0 \in \Delta$ and $A \in CL(X)$ are far, then there exists $B_1 \in \Delta$ and $\varepsilon > 0$ such that $S_\varepsilon[B_0] \subset B_1 \subset A^c$. This is true for $\Delta = CL(X)$, $\Delta =$ the closed and bounded sets, and in a normed linear space, $\Delta =$ the closed convex sets, $\Delta =$ the closed and bounded convex sets, $\Delta =$ the closed connected sets, $\Delta =$ the closed balls, … . A natural first guess for a necessary and sufficient condition is this somewhat weaker sufficient condition: whenever $B_0 \in \Delta$ and $A \in CL(X)$ are far, then there exists $\{B_1, B_2, \ldots, B_n\}$ in Δ and $\varepsilon > 0$ such that $S_\varepsilon[B_0] \subset \bigcup_{i=0}^n B_i \subset A^c$. For example, if X is the line with the usual topology, and Δ consists of the set of all closed balls in X with radius 1, then Δ satisfies the second condition but not the first. But this, too, is not necessary, for if $\Delta = K(X)$ (in which case $(B^c)^+ = (B^c)^{++}$ for each $B \in \Delta$ with no restriction on X), then the latter condition forces local compactness on X.

We now come to the anticipated characterization theorem.

Theorem 3.2. *Let $\langle X, d\rangle$ be metric space, and Δ a family of nonempty closed subsets of X. The following are equivalent:*

(1) $\tau_{\Delta}^{+} \supset \tau_{\Delta}^{++}$ *on* $CL(X)$*;*
(2) *whenever $A \in CL(X)$ and $B_0 \in \Delta$ are far, then there exists a finite subset $\{B_1, B_2, \ldots, B_n\}$ of Δ such that $\bigcup_{i=0}^n B_i \subset A^c$ and such that each sequence $\langle x_k\rangle$ in $(B_0 \cup B_1 \cup \cdots \cup B_n)^c$ with $\lim_{k\to\infty} d(x_k, B_0) = 0$ has a cluster point.*

Proof: (2) $\Rightarrow$ (1). It suffices to show that for each $B \in \Delta$, $(B^c)^{++}$ contains a τ_{Δ}^{+}-neigborhood of each of its points. Fix $B_0 \in \Delta$ and $A \in CL(X)$ with $A \in (B_0^c)^{++}$; then A is far from B_0. Pick $B_1, B_2, \ldots, B_n$ as guaranteed by (2). Since $\bigcup_{i=0}^n B_i \subset A^c$, we have $A \in \bigcap_{i=0}^n (B_i^c)^+$. We claim that if $F \in CL(X)$ and $F \in \bigcap_{i=0}^n (B_i^c)^+$, then F must be far from B_0. Otherwise, there exists a sequence $\langle x_k\rangle$ in F with $\lim_{k\to\infty} d(x_k, B_0) = 0$. By (2), $\langle x_k\rangle$ has a cluster point, which must be simultaneously in F and B_0. However, this violates $F \in (B_0^c)^+$. We have shown

that
$$A \in \bigcap_{i=0}^{n} (B_i^c)^+ \subset (B_0^c)^{++},$$
as required.

(1) $\Rightarrow$ (2). We prove the contrapositive. Suppose (2) fails. Then there exist $B_0 \in \Delta$ and $A \in CL(X)$ far from B_0 such that for each finite subset $\{B_1, B_2, \dots, B_n\}$ of Δ with $\bigcup B_i \subset A^c$, there exists a sequence $s(\{B_1, B_2, \dots, B_n\})$ in $(\bigcup_{i=0}^n B_i)^c$ approaching B_0 having no cluster point. Since A is far from B_0, without loss of generality, we may assume that the range of $s(\{B_1, B_2, \dots, B_n\})$ is contained in A^c. Now let Ω be the family of finite subsets of Δ whose union misses A, partially ordered by inclusion, and for each $\mathcal{F} \in \Omega$, denote by $T(\mathcal{F})$ the range of $s(\mathcal{F})$, a closed nonempty subset of A^c satisfying $D_d(B_0, T(\mathcal{F})) = 0$. Define $\psi : \Omega \to CL(X)$ by $\psi(\mathcal{F}) = A \cup T(\mathcal{F})$. Clearly, the net ψ converges to A in τ_Δ^+ but not in τ_Δ^{++}, as the net $\psi(\mathcal{F})$ is never far from B_0, although A is. Thus, (1) fails. $\square$

Corollary 3.3. *Suppose $\langle X, d\rangle$ is a metric space and Δ is a family of nonempty closed subsets of X that is closed under finite unions. The following are equivalent*:

1. $\tau_\Delta^+ \supset \tau_\Delta^{++}$ *on* $CL(X)$;
2. *whenever $A \in CL(X)$ and $B_0 \in \Delta$ are far, then there exists $B_1 \in \Delta$ with $B_0 \subset B_1 \subset A^c$ such that each sequence $\langle x_k\rangle$ in B_1^c with $\lim_{k\to\infty} d(x_k, B_0) = 0$ has a cluster point.*

Corollary 3.4. *Suppose $\langle X, d\rangle$ is a metric space, and $\Delta \subset CL(X)$. Suppose that whenever $B_0 \in \Delta$ and $A \in CL(X)$ are far, then there exists $\{B_1, B_2, \dots, B_n\}$ in Δ and $\varepsilon > 0$ such that $S_\varepsilon[B_0] \subset \bigcup_{i=0}^n B_i \subset A^c$. Then $\tau_\Delta^+ \supset \tau_\Delta^{++}$ on $CL(X)$.*

We now present an application of Theorem 3.2.

Proposition 3.5. *Let $\langle X, d\rangle$ be a metric space. Let Δ be the family of nonempty closed nowhere dense subsets of X, i.e. $A \in \Delta$ provided* int $A = \emptyset$. *Then $\tau_\Delta^+ \supset \tau_\Delta^{++}$ on $CL(X)$ if and only if the set of limit points X' of X is compact.*

Proof: Suppose X' is noncompact. We show that the condition (2) of Theorem 3.2 fails. Choose a sequence $\langle w_k\rangle$ in X' with distinct terms with no cluster point. Clearly $B_0 \equiv \{w_k : k \in \mathbf{Z}^+\}$ is a nowhere dense closed proper subset of X. Take $a_0 \in B_0^c$; then $A \equiv \{a_0\}$ is far from B_0. We can find a sequence $\langle \varepsilon_k\rangle$ of positive scalars such that $\langle \varepsilon_k\rangle \to 0$, $\{S_{\varepsilon_k}[w_k] : k \in \mathbf{Z}^+\}$ is a disjoint family, and such that $d(w_k, a_0) > \varepsilon_k$. Now if $\{B_1, B_2, \dots, B_n\} \subset \Delta$, then $\forall\, k$ $\{B_0, B_1, B_2, \dots, B_n\}$ fails to cover $S_{\varepsilon_k}[w_k]$, because nowhere dense sets are closed under finite unions. Choosing $x_k \in S_{\varepsilon_k}[w_k] - \bigcup_{i=0}^n B_i$ does the job, because $\limsup d(x_k, B_0) \le \limsup d(x_k, w_k) = 0$, and if $\langle x_k\rangle$ has a cluster point, then so would $\langle w_k\rangle$.

For the converse, suppose X' is compact and B_0 is a closed nowhere dense subset of X far from $A \in CL(X)$. Since B_0 can contain no isolated points of X, we have $B_0 \subset X'$. Condition (2) of Theorem 3.2 is fulfilled with $\{B_1, B_2, \dots, B_n\} = \{B_0\}$, because if $\lim_{k\to\infty} d(x_k, B_0) = 0$, then by the compactness of B_0, $\langle x_k\rangle$ must have a cluster point. $\square$

We note that the metrizable spaces X described in Proposition 3.5 are those having a compatible metric for which disjoint closed sets are far [Ng], [Be3], equivalently, a metric with respect to which each continuous function on X is uniformly continuous [At]. Metric spaces of this kind are called *Atsuji spaces* or *UC spaces* in the literature. For additional characterizations of metrizable spaces for which X' is compact, the reader may consult [Ra].

We now look at the reverse inclusion $\tau_\Delta^+ \subset \tau_\Delta^{++}$.

Lemma 3.6. *Let $\langle X, d\rangle$ be metric space, and let Δ be a family of nonempty closed subsets of X. The following are equivalent*:

(i) $\tau_\Delta^+ \subset \tau_\Delta^{++}$ *on $CL(X)$;*
(ii) *for $A \in CL(X)$ and $B \in \Delta$, $A \cap B = \emptyset \Rightarrow A$ and B are far.*

Proof: (ii) $\Rightarrow$ (i). This is obvious, for (ii) yields $(B^c)^{++} = (B^c)^+$ for each $B \in \Delta$.

(i) $\Rightarrow$ (ii). Suppose $A \in (B^c)^+$; by (i), there exists $\{B_1, B_2, \ldots, B_n\} \subset \Delta$ such that $A \in \bigcap_{i=1}^n (B_i^c)^{++} \subset (B^c)^+$. Clearly, $B \subset \bigcup_{i=1}^n B_i$, else choosing $b \in B - \bigcup_{i=1}^n B_i$, we would have $\{b\}$ far from each B_i, a contradiction, as $\{b\}$ meets B. But then $B \cap B_i$ far from A for each i implies that $B = \bigcup_{i=1}^n (B \cap B_i)$ is far from A, as required. $\square$

Theorem 3.2 and Lemma 3.6 together yield

Theorem 3.7. *Let $\langle X, d\rangle$ be metric space, and let Δ be a family of nonempty closed subsets of X. The following are equivalent*:

(i) $\tau_\Delta^+ = \tau_\Delta^{++}$ *on $CL(X)$;*
(ii) $\tau_\Delta^+ \subset \tau_\Delta^{++}$ *on $CL(X)$;*
(iii) *for $A \in CL(X)$ and $B \in \Delta$, $A \cap B = \emptyset \Rightarrow A$ and B are far.*

Proof: (i) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (iii). This is a consequence of the last result.

(iii) $\Rightarrow$ (i). Condition (iii) says that for each $B \in \Delta$ we have $(B^c)^+ = (B^c)^{++}$, and so subbasic open sets for the two topologies agree. This implication also follows from Lemma 3.6 upon verifying the condition (2) of Theorem 3.2. This is simple: if $A \in CL(X)$ and $B_0 \in \Delta$ are far, then we claim that the choice $\{B_1, B_2, \ldots, B_n\} = \{B_0\}$ works. To see this, suppose to the contrary that $\langle x_k\rangle$ is a sequence in B_0^c with $\lim_{k\to\infty} d(x_k, B_0) = 0$ that has no cluster point. Then $\{x_k : k \in \mathbf{Z}^+\}$ and B_0 are disjoint closed sets that are not far, in violation of (ii). $\square$

When $\Delta = CL(X)$ in the condition (iii) of Theorem 3.7, we have the Atsuji spaces. The case $\Delta =$ the nonempty closed and bounded sets has been recently considered in [BDC]. Metrizable spaces that admit a metric of this kind are those for which X' is locally compact and separable.

4. On the Wijsman topology and ball hyperspace topologies.

The ball topology $\tau_{\mathcal{B}}^{+}$ and the proximal ball topology $\tau_{\mathcal{B}}^{++}$ arose in the first place in an essentially unsuccessful attempt to find alternative presentations of the Wijsman topology in a general metric space. In this section we intend to survey the terrain. Following [FLL], we introduce the Wijsman topology as a weak topology. We may regard $d(x, A)$ as a function of a set variable by holding x fixed and letting A vary.

Definition 4.1. Let $\langle X, d\rangle$ be a metric space. The *Wijsman topology* τ_{W_d} is the weakest topology τ on $CL(X)$ such that for each $x \in X$, $A \to d(x, A)$ is a τ-continuous functional.

Basic facts about this topology are established in [Co], [FLL], [LL]. It may be argued that the Wijsman topology is the most important construction in the theory of hyperspaces of a metric space, given that so many important hit-and-miss topologies can be expressed as suprema of Wijsman topologies, including the Vietoris and slice topologies [BLLN], [Be5], [BL1]. The inclusion $\tau_{W_d} \subset \tau_{\mathcal{B}}^{+}$ is valid in any metric space (see e.g. [FLL, Proposition 2.3] and [Be1, Lemma 2.0]). The inclusion $\tau_{W_d} \supset \tau_{\mathcal{B}}^{+}$ requires extremely strong conditions; for example, it is necessary (but not sufficient) that whenever B is a closed ball and A is a closed set with $B \cap A = \emptyset$, then A and B are far [Be1, Lemma 2.8]. If closed and bounded subsets of X are compact, then the inclusion is satisfied [FFL, Proposition 2.5], although it need not be valid in an Atsuji space [Be1, Example 2.9].

Relative to the relationship between the proximal ball topology $\tau_{\mathcal{B}}^{++}$ and the Wijsman topology, we introduce the following condition $(*)$:

$$(*) \quad \forall\, x \in X, \quad \forall\, \mu > 0, \quad \forall\, \alpha > 0, \quad \exists\, \delta > \mu \text{ such that } \overline{S}_\delta[x] \subset \overline{S}_\alpha[\overline{S}_\mu[x]].$$

Proposition 4.2. *Let $\langle X, d\rangle$ be a metric space. Then the proximal ball topology $\tau_{\mathcal{B}}^{++}$ on $CL(X)$ contains the Wijsman topology τ_{W_d}. Conversely, if the metric d satisfies $(*)$, then τ_{W_d} contains $\tau_{\mathcal{B}}^{++}$.*

PROOF: The inclusion $\tau_{W_d} \subset \tau_{\mathcal{B}}^{++}$, valid in any metric space, is established in [DMN]. For the reverse inclusion, we show that the Wijsman topology contains each subbasic open set for the proximal ball topology, subject to the condition $(*)$. That $V^- \in \tau_{W_d}$ for each open V requires no assumptions whatsoever on the metric (see [FLL, Proposition 2.1]). Now suppose $A_0 \in (B^c)^{++}$ where B is a closed ball, say $B = \overline{S}_\mu[x_0]$. Choosing $\alpha > 0$ with $D_d(A_0, \overline{S}_\mu[x_0]) > \alpha$, we see that $\overline{S}_\alpha[\overline{S}_\mu[x_0]] \cap A_0 = \emptyset$, and the condition $(*)$ now gives $\overline{S}_\delta[x_0] \cap A_0 = \emptyset$ for some $\delta > \mu$. Thus, $d(x_0, A_0) \geq \delta$, and as a result,

$$A_0 \in \{A \in CL(X) : d(x_0, A) > \frac{1}{2}(\delta + \mu)\} \subset (B^c)^{++},$$

completing the proof that $\tau_{W_d} \supset \tau_{\mathcal{B}}^{++}$. $\square$

Example 4.3. The condition $(*)$ does not imply the following relative of a condition of Francaviglia, Levi, and Lechicki [FLL]: $\forall\, x \in X$, $\forall\, \mu > 0$, $\forall\, \alpha > 0$, $\exists\, \delta > \mu$ such that $S_\delta[x] \subset S_\alpha[S_\mu[x]]$! Consider $L = \{(x,0) : x \geq 0\} \cup \{(0,y) : y \geq 0\}$ as a metric subspace of the plane equipped with the box metric, $d[(x_1,y_1),(x_2,y_2)] = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. This space satisfies much more than the condition $(*)$; in fact, for each $x \in L$ and $\mu > 0$, there exists $\alpha > 0$ such that $\overline{S}_{\alpha+\mu}[x] = \overline{S}_\alpha[\overline{S}_\mu[x]]$. However, $S_{1/2}[S_1[(0,1)]]$ contains no open ball about $(0,1)$ of radius greater than one. □

The condition $(*)$ holds in particular in any normed linear space, where the equality of the Wijsman and proximal ball topologies was observed by Sonntag and Zalinescu [SZ]. In our condition $(*)$, the scalar δ depends not only on μ and α but also on x. We now look at a stronger condition $(**)$, where δ may be chosen independent of x.

$$(**) \qquad \forall\, \mu > 0, \quad \forall\, \alpha > 0, \quad \exists\, \delta > \mu \ \text{ such that } \ \forall\, x \in X, \quad \overline{S}_\delta[x] \subset \overline{S}_\alpha[\overline{S}_\mu[x]].$$

With $(**)$, Wijsman convergence is equivalent to the classical Kuratowski-Painlevé convergence [KT], [Mr], [FLL], [Ah] [Do] of all closed enlargements of fixed radius.

Definition 4.4. Let $\langle X, \tau\rangle$ be a topological space and let $\langle A_\lambda\rangle$ be a net in $CL(X)$. The *limit inferior* $Li_\lambda A_\lambda$ and *limit superior* $Ls_\lambda A_\lambda$ of the net $\langle A_\lambda\rangle$ are defined by the formulas

$$Li_\lambda A_\lambda \equiv \{x \in X\colon \text{ each neighborhood of } x \text{ meets } A_\lambda \text{ eventually}\};$$
$$Ls_\lambda A_\lambda \equiv \{x \in X\colon \text{ each neighborhood of } x \text{ meets } A_\lambda \text{ frequently}\}.$$

The net $\langle A_\lambda\rangle$ is declared *Kuratowski-Painlevé convergent* to a (closed) set A provided $A = Li_\lambda A_\lambda = Ls_\lambda A_\lambda$. In this case, we write $A = K\text{-}\lim_\lambda A_\lambda$.

As is well-known [FLL], [Be1], in an arbitrary metric space $\langle X, d\rangle$, Wijsman convergence ensures Kuratowski-Painlevé convergence in $CL(X)$; the converse holds if and only if each proper closed ball in the metric space is compact [Be1, Theorem 2.3].

Theorem 4.5. *Let $\langle X, d\rangle$ be a metric space satisfying $(**)$. Let $\langle A_\lambda\rangle$ be a net in $CL(X)$ and let $A \in CL(X)$. The following are equivalent*:

(i) $A = \tau_{W_d}\text{-}\lim_\lambda A_\lambda$;
(ii) $A = \tau_{\mathcal{B}}^{++}\text{-}\lim A_\lambda$;
(iii) *for each* $\mu > 0$, $\overline{S}_\mu[A] = K\text{-}\lim_\lambda \overline{S}_\mu[A_\lambda]$.

Proof: Since the condition $(**)$ gives the condition $(*)$, the conditions (i) and (ii) are equivalent by Proposition 4.2. We establish the equivalence of (i) and (iii).

(iii) ⇒ (i). This holds with no assumptions on the metric. First, suppose that $x_0 \in X$ is fixed and $d(x_0, A) < \mu$. Write $\beta = \frac{1}{2}(\mu + d(x_0, A))$; then $\beta > 0$ and we have $x_0 \in \overline{S}_\beta[A]$. Since $\overline{S}_\beta[A] \subset Li_\lambda \overline{S}_\beta[A_\lambda]$, there exists an index λ_0 such that for each

$\lambda \geq \lambda_0$, we have $S_{\mu-\beta}[x_0] \cap \overline{S}_\beta[A_\lambda] \neq \emptyset$, and for each such λ, we have $d(x_0, A_\lambda) < \mu$. On the other hand, if $d(x_0, A_\lambda) \leq \mu$ frequently, then $x_0 \in Ls_\lambda \overline{S}_\mu[A_\lambda] \subset \overline{S}_\mu[A]$, in which case $d(x_0, A) \leq \mu$. Thus, $d(x_0, A) > \mu$ ensures $d(x_0, A_\lambda) > \mu$ eventually.

(i) $\Rightarrow$ (iii). Fix $\mu > 0$. We show $\overline{S}_\mu[A] \subset Li_\lambda \overline{S}_\mu[A_\lambda]$ and $Ls_\lambda \overline{S}_\mu[A_\lambda] \subset \overline{S}_\mu[A]$. For the first inclusion, fix $x_0 \in \overline{S}_\mu[A]$ and let $\varepsilon > 0$. Choose by $(**)$ $\delta > \mu$ such that for each $x \in X$, we have $\overline{S}_\delta[x] \subset \overline{S}_{\varepsilon/2}[\overline{S}_\mu[x]]$. Clearly, $d(x_0, A) < \delta$, and by Wijsman convergence, there exists an index λ_0 such that for each $\lambda \geq \lambda_0$ we have $d(x_0, A_\lambda) < \delta$. For each such λ, there exists $a_\lambda \in A_\lambda$ with $d(x_0, a_\lambda) < \delta$. By the choice of δ there exists $w_\lambda \in X$ with both $d(x_0, w_\lambda) < \varepsilon$ and $d(w_\lambda, a_\lambda) \leq \mu$. This shows that $S_\varepsilon[x_0] \cap \overline{S}_\mu[A_\lambda] \neq \emptyset$ for $\lambda \geq \lambda_0$, and the inclusion $\overline{S}_\mu[A] \subset Li_\lambda \overline{S}_\mu[A_\lambda]$ follows.

We now turn to the second inclusion. By the equivalence of (i) and (ii), the net $\langle A_\lambda \rangle$ converges to A in the topology σ with subbase $\{(B^c)^{++} : B \text{ a closed ball}\}$. Suppose to the contrary that $Ls_\lambda \overline{S}_\mu[A_\lambda] \not\subset \overline{S}_\mu[A]$. Choose $x_0 \in Ls_\lambda \overline{S}_\mu[A_\lambda]$ with $d(x_0, A) > \mu$, and then β strictly between μ and $d(x_0, A)$. Although $A \in [(\overline{S}_\beta[x_0])^c]^{++}$, it is clear that A_λ hits $\overline{S}_\beta[x_0]$ frequently, so that convergence in σ fails, a contradiction. Thus, $Ls_\lambda \overline{S}_\mu[A_\lambda] \subset \overline{S}_\mu[A]$ is a consequence of Wijsman convergence, and the proof is complete. $\square$

Example 4.6. The condition $(**)$ cannot be replaced by the condition $(*)$ in the statement of Theorem 4.5. We revisit the space L of Example 4.3. In this space, closed and bounded sets are compact, and the Wijsman topology, the ball topology, and the proximal ball topologies all coincide. But Wijsman convergence — in fact even Hausdorff metric convergence — cannot guarantee Kuratowski-Painlevé convergence of closed enlargements. To see this, let $A = \{(0,1)\}$ and let $A_n = \{(0, (n+1)/n)\}$. We have

$$(1,0) \in \overline{S}_1[A] = \{(x,0) : x \in [0,1]\} \cup \{(0,y) : y \in [0,2]\},$$

whereas $(1,0) \notin Ls_{n\to\infty} \overline{S}_1[A_n]$. $\square$

Let σ be the topology on $CL(X)$ generated by $\{(B^c)^{++} : B \text{ a closed ball}\}$. The proof of Theorem 4.5 shows that with no assumptions on $\langle X, d \rangle$, the condition $A = \sigma\text{-}\lim A_\lambda$ implies $Ls_\lambda \overline{S}_\mu[A_\lambda] \subset \overline{S}_\mu[A]$ for each $\mu > 0$. The converse holds assuming the condition $(*)$; we leave this as an easy exercise for the reader. But the converse is not true in general. Returning to Example 3.1, let $A_n = A \cup \{\frac{i+1}{i} e_i : i \text{ even, and } i \geq 2n\}$. Clearly, $\langle A_n \rangle$ fails to converge to A in the topology generated by $\{(B^c)^{++} : B \text{ a closed ball}\}$, as $A \in (\overline{S}_1[\theta]^c)^{++}$. Still, whenever $\mu > 0$, we have $Ls_{n\to\infty} \overline{S}_\mu[A_n] \subset \overline{S}_\mu[A]$. There are two cases to consider: if $\mu < 1$, then $\overline{S}_\mu[A_n] \subset A \cup \{\frac{i+1}{i} e_i : i \text{ even, and } i \geq 2n\} \cup \{e_i : i \text{ even, and } i \geq 2n\}$, and $Ls_{n\to\infty} \overline{S}_\mu[A_n] = A = \overline{S}_\mu[A]$. If $\mu \geq 1$, then

$$Ls_{n\to\infty} \overline{S}_\mu[A_n] = X - \{\frac{i+1}{i} e_i : i \text{ even, and } \frac{i+1}{i} > \mu\} = \overline{S}_\mu[A].$$

With some effort, one can show that the condition $(**)$ is equivalent to the following cumbersome condition $(***)$:

$$(***)\qquad \begin{aligned} &\forall\,\mu>0,\ \forall\,\alpha>0,\ \exists\,\varepsilon_1>0 \text{ and } \varepsilon_2>0\\ &\text{such that } \forall\,y\in X,\ \forall\,A\in CL(X),\\ &\inf\{d(x,A): x\in \overline{S}_{\varepsilon_1}[y]\}<\mu+\varepsilon_2 \Rightarrow \exists\,v\in \overline{S}_\alpha[y] \text{ with } d(v,A)\le\mu. \end{aligned}$$

The reason that we bring this formulation to the attention of the reader is that when $(**)$ is viewed in this way, one can apply the quasi-equi-semicontinuity machinery of Dolecki [Do, p. 234] to produce an "epi-convergence proof" of Theorem 4.5. We leave this to the interested leader.

Evidently, the condition $(**)$ holds in each metric space $\langle X,d\rangle$ in which the closed ball operator is "additive": $\forall\,x\in X, \forall\,\mu>0, \forall\,\alpha>0$, $\overline{S}_{\alpha+\mu}[x]=\overline{S}_\alpha[\overline{S}_\mu[x]]$. Such metric spaces have a number of simple characterizations. A metric is often called *convex* provided for each $\alpha\in(0,d(x_1,x_2))$, there exists $x_3\in X$ such that $d(x_1,x_3)=\alpha$ and $d(x_2,x_3)=d(x_1,x_2)-\alpha$ [Bl]. Metrics for which the ball operator is additive are almost convex, as we now define in the most convenient form for our purposes.

Definition 4.7. We call a metric d on a set X *almost convex* provided whenever $\{x_1,x_2\}\subset X$, $\alpha>d(x_1,x_2)$, and $0<\beta<\alpha$, there exists $w\in X$ such that both $d(x_1,w)<\beta$ and $d(w,x_2)<\alpha-\beta$.

It is easy to check that almost convexity amounts to the following condition, which better justifies its name: whenever $0<\alpha<d(x_1,x_2)$ and $\varepsilon>0$, there exists $x_3\in X$ such that $|d(x_1,x_3)-\alpha|<\varepsilon$ and $|d(x_2,x_3)-(d(x_1,x_2)-\alpha)|<\varepsilon$. Clearly, each convex metric is almost convex; in particular, each metric determined by a norm is almost convex. The rationals as a subspace of the line is an almost convex metric space, but the usual metric so restricted is not convex.

In closing, we verify that additivity of the ball operation, interpreted in a variety of ways, is equivalent to almost convexity of the metric.

Proposition 4.8. *Let $\langle X,d\rangle$ be a metric space. The following are equivalent*:

(i) *the metric d is almost convex;*
(ii) *for each $x_0\in X$, $\mu>0$ and $\alpha>0$, we have $S_\alpha[S_\mu[x_0]]=S_{\alpha+\mu}[x_0]$;*
(iii) *for each $A\subset X$, $\mu>0$ and $\alpha>0$, we have $S_\alpha[S_\mu[A]]=S_{\alpha+\mu}[A]$;*
(iv) *for each $A\subset X$, $\mu>0$ and $\alpha>0$, we have $\overline{S}_\alpha[\overline{S}_\mu[A]]=\overline{S}_{\alpha+\mu}[A]$;*
(v) *for each $x_0\in X$, $\mu>0$ and $\alpha>0$, we have $\overline{S}_\alpha[\overline{S}_\mu[x_0]]=\overline{S}_{\alpha+\mu}[x_0]$.*

Proof: (i) $\Rightarrow$ (ii). One always has $S_\alpha[S_\mu[x_0]]\subset S_{\alpha+\mu}[x_0]$. Let $x\in S_{\alpha+\mu}[x_0]$; since $d(x,x_0)<\alpha+\mu$, by almost convexity, there exists $w\in X$ with $d(w,x_0)<\mu$ and $d(x,w)<\alpha$. This shows that $x\in S_\alpha[S_\mu[x_0]]$.

(ii) $\Rightarrow$ (iii). According to (ii),

$$S_{\alpha+\mu}[A]=\bigcup_{a\in A}S_{\alpha+\mu}[a]=\bigcup_{a\in A}S_\alpha[S_\mu[a]]=S_\alpha[S_\mu[A]].$$

(iii) $\Rightarrow$ (iv). We will actually show that

$$\text{cl } S_\alpha[S_\mu[A]] = \overline{S}_\alpha[\overline{S}_\mu[A]] = \overline{S}_{\alpha+\mu}[A].$$

By the triangle inequality, we always have cl $S_\alpha[S_\mu[A]] \subset \overline{S}_\alpha[\overline{S}_\mu[A]] \subset \overline{S}_{\alpha+\mu}[A]$. By (iii),

$$(\#) \qquad S_{\alpha+\mu}[A] = S_\alpha[S_\mu[A]] \subset \text{cl } S_\alpha[S_\mu[A]].$$

Now fix $x_1 \in \overline{S}_{\alpha+\mu}[A]$, i.e. with $d(x_1, A) \leq \alpha + \mu$, and let $\varepsilon > 0$. Since $x_1 \in S_{\alpha+\mu+\varepsilon}[A] = S_\varepsilon[S_{\alpha+\mu}[A]]$, there exists $w \in X$ and $a \in A$ with $d(a, w) < \alpha + \mu$ and $d(w, x_1) < \varepsilon$. By condition (#), $x_1 \in$ cl $S_{\alpha+\mu}[A] \subset$ cl $S_\alpha[S_\mu[A]]$.

(iv) $\Rightarrow$ (v). This is trivial.

(v) $\Rightarrow$ (i). Suppose $d(x_1, x_2) < \alpha$ and $\beta \in (0, \alpha)$. If $\beta > d(x_1, x_2)$, then with $w = x_2$, we have $d(x_1, w) < \beta$ and $d(w, x_2) = 0 < \alpha - \beta$. Otherwise we may assume that $\beta \leq d(x_1, x_2)$. Choose a positive ε with $0 < 2\varepsilon < \min\{\beta, \alpha - d(x_1, x_2)\}$, and let $\mu = \beta - \varepsilon/2$ and let $\gamma = d(x_1, x_2) - (\beta - \varepsilon)$; by (v), we have

$$x_2 \in \overline{S}_{\gamma+\mu}[x_1] = \overline{S}_\gamma[\overline{S}_\mu[x_1]]$$

and so there exists $w \in \overline{S}_\mu[x_1]$ with

$$d(x_2, w) < \gamma + \varepsilon = d(x_1, x_2) - (\beta - \varepsilon) + \varepsilon < \alpha - \beta.$$

Since $d(w, x_1) \leq \mu < \beta$, the metric d is almost convex. $\square$

References

[At] Atsuji M., *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. **8** (1958), 11–16.

[Ah] Attouch H., *Variational Convergence for Functions and Operators*, Pitman, New York, 1984.

[BaP] Baronti M., Papini P.-L., *Convergence of sequences of sets*, in Methods of Functional Analysis in Approximation Theory, ISNM #76, Birkhäuser-Verlag, 1986.

[Be1] Beer G., *Metric spaces with nice closed balls and distance functions for closed sets*, Bull. Australian Math. Soc. **35** (1987), 81–96.

[Be2] ———, *On Mosco convergence of convex sets*, Bull. Australian Math. Soc. **38** (1988), 239–253.

[Be3] ———, *UC spaces revisited*, Amer. Math. Monthly **95** (1988), 737–739.

[Be4] ———, *Support and distance functionals for convex sets*, Numer. Func. Anal. Optim. **10** (1989), 15–36.

[Be5] ———, *The slice topology: A viable alternative to Mosco convergence in nonreflexive spaces*, Sém. d'Anal. Convexe Montpellier (1991), exposé N° 3; Nonlinear Anal. **19** (1992), 271–290.

[BDC] Beer G., Di Concilio A., *A generalization of boundedly compact metric spaces*, Comment. Math. Univ. Carolinae **32** (1991), 361–367.

[BLLN] Beer G., Lechicki A., Levi S., Naimpally S., *Distance functionals and suprema of hyperspace topologies*, Annali Mat. Pura Appl. **162** (1992), 367–381.

[BL1] Beer G., Lucchetti R., *Weak topologies for the closed subsets of a metrizable space*, Trans. Amer. Math. Soc. **335** (1993), 805–822.

[BL2] ———, *Well-posed optimization problems and a new topology for the closed subsets of a metric space*, Rocky Mountain J. Math., to appear.

[Bl] Blumenthal L., *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1953.

[Ct] Cornet B., *Topologies sur les fermés d'un espace métrique*, Cahiers de mathématiques de la décision # 7309, Université de Paris Dauphine, 1973.

[DCNS] Di Concilio A., Naimpally S., Sharma P., *Proximal hypertopologies*, Proc. Top. Conf. Campinas, Brazil, August 1988.

[DMN] Di Maio G., Naimpally S., *Comparison of hypertopologies*, Rend. Instit. Mat. Univ. Trieste **22** (1990), 140–161.

[Do] Dolecki S., *Tangency and differentiation: some applications of convergence theory*, Ann. Mat. Pura Appl. **130** (1982), 223–255.

[Fe] Fell J., *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. **13** (1962), 472–476.

[FLL] Francaviglia S., Lechicki A., Levi S., *Quasi-uniformization of hyperspaces and convergence of nets of semicontinuous multifunctions*, J. Math. Anal. Appl. **112** (1985), 347–370.

[He] Hess C., *Contributions à l'étude de la mesurabilité, de la loi de probabilité, et de la convergence des multifunctions*, Thèse d'état, U.S.T.L. Montpellier, 1986.

[Jo] Joly J., *Une famille de topologies sur l'ensemble des fonctions convexes pour lesquelles la polarité est bicontinue*, J. Math. Pures Appl. **52** (1973), 421–441.

[KT] Klein E., Thompson A., *Theory of Correspondences*, Wiley, New York, 1984.

[LL] Lechicki A., Levi S., *Wijsman convergence in the hyperspace of a metric space*, Bull. Un. Mat. Ital. **1-B** (1987), 439–452.

[Mi] Michael E., *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[Mo1] Mosco U., *Convergence of convex sets and solutions of variational inequalities*, Advances in Math. **3** (1969), 510–585.

[Mo2] ———, *On the continuity of the Young-Fenchel transform*, J. Math. Anal. Appl. **35** (1971), 518–535.

[Mr] Mrowka S., *On the convergence of nets of sets*, Fund. Math. **45** (1958), 237–246.

[Na] Nachman L., *Hyperspaces of proximity spaces*, Math. Scand. **23** (1968), 201–213.

[Ng] Nagata J., *On the uniform topology of bicompactifications*, J. Inst. Polytech. Osaka **I** (1950), 28–38.

[Np] Naimpally S., *Wijsman convergence for function spaces*, Rend. Circ. Palermo II **18** (1988), 343–358.

[Po] Poppe H., *Eine Bemerkungen über den Raum der abgeschlossenen Mengen*, Fund. Math. **59** (1966), 159–169.

[Ra] Rainwater J., *Spaces whose finest uniformity is metric*, Pacific J. Math. **9** (1959), 567–570.

[So] Sonntag Y., *Convergence au sens de Mosco; théorie et applications à l'approximation des solutions d'inéquations*, Thèse, Université de Provence, Marseille, 1982.

[SZ] Sonntag Y., Zalinescu C., *Set convergences: An attempt of classification*, in Proceedings of Int. Conf. on Diff. Equations and Control Theory, Iasi, Romania, August 1990. Revised version to appear in Trans. Amer. Math. Soc.

[Wi] Wijsman R., *Convergence of sequences of convex sets, cones, and functions II*, Trans. Amer. Math. Soc. **123** (1966), 32–45.

Department of Mathematics, California State University, Los Angeles, CA 90032, USA

(*Received* December 2, 1992, *revised* March 30, 1993)