Comment.Math.Univ.Carolin. 34,4 (1993)597–611

Some remarks on the regularity of minimizers of integrals with anisotropic growth

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Abstract. We prove higher integrability for minimizers of some integrals of the calculus of variations; such an improved integrability allows us to get existence of weak second derivatives.

Keywords: regularity, minimizers, integral functionals, anisotropic growth

Classification: 49N60, 35J60

0. Introduction.

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$; u be such that $u : \Omega \to \mathbb{R}^N$, $N \geq 1$. Consider the integral functional

$$I(u) = \int_\Omega F(Du(x))\, dx, \tag{0.1}$$

where F satisfies an anisotropic growth condition, namely

$$a \sum_{i=1}^{n} |\xi_i|^{q_i} - b \leq F(\xi) \leq c \sum_{i=1}^{n} |\xi_i|^{q_i} + d, \tag{0.2}$$

$\forall\, \xi \in \mathbb{R}^{nN}$. Here a, b, c and d are positive constants and $1 \leq q_i$, $i = 1, \dots, n$. It is well known that the standard results of the isotropic case, i.e. $q_i = q$, $i = 1, \dots, n$, fail to hold if the q_i's are too far apart [10], [14], [15]. The main aim of this paper is to show that under some restrictions on the q_i's, an improved integrability result holds for minimizers u of (0.1) verifying (0.2) and some additional restrictions. The prototype for our work is the integral

$$I(u) = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n-1} |D_i u(x)|^2 + \frac{1}{p}(1 + |D_n u(x)|^2)^{p/2} \right) dx, \tag{0.3}$$

where $Du = (D_1 u, \dots, D_n u)$ and $1 < p < 2$, for which (0.2) holds with $q_1 = \cdots = q_{n-1} = 2$ and $q_n = p$. We have arranged our work as follows. In Section 1 we state the main result, Section 2 contains some preliminaries while Sections 3 and 4 deal with the proofs of the results of the paper.

This work has been supported by MURST and GNAFA-CNR.

1. Notation and main results.

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, u be a vector-valued function, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$; we consider integrals

$$\int_\Omega F(Du(x))\, dx, \tag{1.1}$$

based on (0.3). More precisely, we assume that $F : \mathbb{R}^{nN} \to \mathbb{R}$ is in $C^2(\mathbb{R}^{nN})$ and satisfies, for some positive constants c, m, M, p,

$$|F(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p); \tag{1.2}$$

$$|\frac{\partial F}{\partial \xi_i^\alpha}(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p)^{1/2} \quad \text{if } \ i = 1, \dots, n-1; \tag{1.3}$$

$$|\frac{\partial F}{\partial \xi_n^\alpha}(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p)^{1-1/p}; \tag{1.4}$$

and

$$\begin{aligned} m\Big(\sum_{i=1}^{n-1} |\lambda_i|^2 + (1 + |\xi_n|^2)^{(p-2)/2} |\lambda_n|^2\Big) &\leq \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 F}{\partial \xi_j^\beta \partial \xi_i^\alpha}(\xi) \lambda_i^\alpha \lambda_j^\beta \\ &\leq M\Big(\sum_{i=1}^{n-1} |\lambda_i|^2 + (1 + |\xi_n|^2)^{(p-2)/2} |\lambda_n|^2\Big), \end{aligned} \tag{1.5}$$

for every $\lambda, \xi \in \mathbb{R}^{nN}$. Here, $\lambda = \{\lambda_i^\alpha\}$, $\xi = \{\xi_i^\alpha\}$, $|\lambda_i|^2 = \sum_{\alpha=1}^{N} |\lambda_i^\alpha|^2$, etc. About p, we assume that

$$1 < p < 2. \tag{1.6}$$

We remark that the integrand of (0.3) satisfies (1.2), … ,(1.5). We say that u minimizes the integral (1.1) if $u : \Omega \to \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ with $D_i u \in L^2(\Omega)$, $i = 1, \dots, n-1$, and for every $\phi : \Omega \to \mathbb{R}^N$ with $\phi \in W_0^{1,p}(\Omega)$ and $D_i \phi \in L^2(\Omega)$, $i = 1, \dots, n-1$, we have

$$I(u) \leq I(u + \phi). \tag{1.7}$$

We have the following regularity results.

Theorem 1. *Let $u : \Omega \to \mathbb{R}^N$ satisfy $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$ with $D_i u \in L^2(\Omega)$, $i = 1, \dots, n-1$, where*

$$1 < p < 2 \qquad \text{if } \ n = 2, 3; \tag{1.8}$$

$$98/97 < p < 2 \qquad \text{if } \ n = 4; \tag{1.9}$$

and

$$2 - 4/n < p < 2 \qquad \text{if } n \geq 5. \tag{1.10}$$

If F satisfies (1.2), ... , (1.5) *and u minimizes the integral* (1.1) *in the sense of* (1.7), *then*

$$D_n u \in L^2_{\mathrm{loc}}(\Omega). \tag{1.11}$$

This result of higher integrability implies the following improved differentiability.

Corollary 1. *Under the assumptions of Theorem* 1, *we obtain the existence of the weak second derivatives. Furthermore,*

$$D_i Du \in L^2_{\mathrm{loc}}(\Omega), \quad i = 1, \dots, n-1 \;\text{ and }\; D_n Du \in L^p_{\mathrm{loc}}(\Omega).$$

Remark 1. We prove Theorem 1 by employing a technique in [6]. The idea is to gain a fractional order derivative of Du thereby improving its integrability. Also see [4], [7], [13].

Remark 2. It is not clear to us whether the restriction $2-4/n < p$ is a consequence of the technique we have used. We are unable to prove or disprove Theorem 1 outside this range. It must be mentioned that the same restriction was arrived at in a slightly different context in the work [7].

Remark 3. It is to be noted that local boundedness of scalar valued minimizers has been proved without any restrictions on p from below [8], [9].

2. Preliminaries.

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h} f(x) = f(x + h e_s) - f(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R. We will often suppress x_0 whenever there is no danger of confusion. We now state several lemmas that are crucial to our work. In the following $f : \Omega \to \mathbb{R}^k$, $k \geq 1$; B_R, B_{2R} and B_{3R} are concentric balls.

Lemma 2.1. *If $f, D_s f \in L^t(B_{3R})$ with $1 \leq t < \infty$ then*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq |h|^t \int_{B_{2R}} |D_s f(x)|^t \, dx,$$

for every h with $|h| < R$. (*See* [11, p. 45], [5, p. 28].)

Lemma 2.2. *Let $f \in L^t(B_{2R})$, $1 < t < \infty$; if there exists a positive constant C such that*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq C|h|^t,$$

for every h with $|h| < R$, then there exists $D_s f \in L^t(B_R)$. (*See* [11, p. 45], [5, p. 26].)

Lemma 2.3. *If $f \in L^2(B_{3R})$ and for some $d \in (0,1)$ and $C > 0$*

$$\sum_{s=1}^{n} \int_{B_R} |\tau_{s,h} f(x)|^2 \, dx \leq C|h|^{2d},$$

for every h with $|h| < R$, then $f \in L^r(B_{R/4})$ for every $r < 2n/(n-2d)$.

Proof: The previous inequality tells us that $f \in W^{b,2}(B_{R/2})$ for every $b < d$, so we can apply the imbedding theorem for fractional Sobolev spaces [3, Chapter VII]. $\square$

Lemma 2.4. *For every t with $1 \leq t < \infty$ there exists a positive constant C such that*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq C \int_{B_{2R}} |f(x)|^t \, dx,$$

for every $f \in L^t(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$.

Lemma 2.5 (Anisotropic Sobolev imbedding theorem)**.** *If $q_i \geq 1$, $i = 1, \ldots, n$, we assume that $f \in W^{1,1}(Q)$ and $f, D_i f \in L^{q_i}(Q)$, $\forall\, i = 1, \ldots, n$, where $Q \subset \mathbb{R}^n$ is a cube with faces parallel to the coordinate planes. Define $\overline{q}$ by*

$$\frac{1}{\overline{q}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{q_i} \quad \text{and set} \quad \overline{q}^{\,*} = \begin{cases} n\overline{q}/(n - \overline{q}), & \text{if } \overline{q} < n, \\ \text{any number}, & \text{if } \overline{q} \geq n. \end{cases}$$

If $q_i < \overline{q}^{\,*}$, $\forall\, i = 1, \ldots, n$, then $f \in L^{\overline{q}^{\,*}}(Q)$. (*See* [16], [1].)

Now we state some basic inequalities.

Lemma 2.6. *For every $\gamma \in (-1/2, 0)$ we have*

$$1 \leq \frac{\int_0^1 (1 + |b + t(a-b)|^2)^\gamma \, dt}{(1 + |a|^2 + |b|^2)^\gamma} \leq \frac{8}{2\gamma + 1},$$

for all $a, b \in \mathbb{R}^k$. (*See* [2].)

Lemma 2.7. *For every $\gamma \in (-1/2, 0)$ we have*

$$(2\gamma + 1)|a - b| \leq \frac{|(1 + |a|^2)^\gamma a - (1 + |b|^2)^\gamma b|}{(1 + |a|^2 + |b|^2)^\gamma} \leq \frac{c(k)}{2\gamma + 1}|a - b|,$$

for all $a, b \in \mathbb{R}^k$. (*See* [2].)

3. Proof of Theorem 1.

Since u minimizes the integral (1.1) with growth conditions as in (1.2), ... ,(1.4), u solves the Euler equation,

$$\int_\Omega \sum_{i=1}^{n}\sum_{\alpha=1}^{N} \frac{\partial F}{\partial \xi_i^\alpha}(Du(x))D_i\phi^\alpha(x)\,dx = 0, \tag{3.1}$$

for all functions $\phi : \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$ and $D_1\phi, \dots, D_{n-1}\phi \in L^2(\Omega)$. Let $R > 0$ be such that $\overline{B_{3R}} \subset \Omega$ and let B_ϱ and B_R be concentric balls, $0 < \varrho < R \leq 1$. Fix s, take $0 < |h| < R$ and let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a "cut off" function in $C_0^1(B_R)$ with

$$\eta \equiv 1 \quad \text{on} \quad B_\varrho, \qquad 0 \leq \eta \leq 1 \quad \text{and} \quad |D_\eta| \leq c/(R-\varrho).$$

Using $\phi = \tau_{s,-h}(\eta^2\tau_{s,h}u)$ in (3.1), via a standard reduction, we get the following Caccioppoli estimate, i.e. for some positive constants $C_0 = C_0(n, N, p, m, M)$,

$$\begin{aligned}
&\int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h}D_iu(x)|^2\,dx \\
&\qquad + \int_{B_\varrho} (1{+}|D_nu(x)|^2{+}|D_nu(x{+}he_s)|^2)^{(p-2)/2}|\tau_{s,h}D_nu(x)|^2\,dx \\
&\leq \frac{C_0}{(R{-}\varrho)^2}\int_{B_R} \{1{+}(1{+}|D_nu(x)|^2{+}|D_nu(x{+}he_s)|^2)^{(p-2)/2}\}|\tau_{s,h}u(x)|^2\,dx \\
&\leq \frac{2C_0}{(R{-}\varrho)^2}\int_{B_R} |\tau_{s,h}u(x)|^2\,dx,
\end{aligned} \tag{3.2}$$

where we have used the fact that $p < 2$. Set

$$\hat{V}(\xi) = |V(\xi_n)| + \sum_{i=1}^{n-1}|\xi_i|, \quad V(\xi_n) = (1+|\xi_n|^2)^{(p-2)/4}\xi_n, \quad \forall\, \xi \in \mathbb{R}^{nN}. \tag{3.3}$$

Clearly,

$$|\tau_{s,h}\hat{V}(Du)| \leq |\tau_{s,h}V(D_nu)| + \sum_{i=1}^{n-1}|\tau_{s,h}D_iu| \tag{3.4}$$

and

$$\hat{V}(Du) \in L^r \quad \text{if and only if} \quad \begin{cases} D_iu \in L^r, & i = 1, \dots, n-1, \\ D_nu \in L^{rp/2}. \end{cases} \tag{3.5}$$

Using Lemma 2.7 we find

$$\begin{aligned}
C_1|\tau_{s,h}D_nu(x)| &\leq \frac{|\tau_{s,h}V(D_nu(x))|}{(1+|D_nu(x)|^2+|D_nu(x+he_s)|^2)^{(p-2)/4}} \\
&\leq C_2|\tau_{s,h}D_nu(x)|,
\end{aligned} \tag{3.6}$$

where C_1, C_2 depend only on N and p. From (3.4), (3.6) and (3.2) we get

$$\begin{aligned}\int_{B_\varrho} |\tau_{s,h}\hat{V}(Du)|^2\,dx &\le C_3 \int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h}D_i u|^2\,dx + C_3 \int_{B_\varrho} |\tau_{s,h}V(D_n u)|^2\,dx \\ &\le \frac{C_4}{(R-\varrho)^2} \int_{B_R} |\tau_{s,h}u|^2\,dx,\end{aligned} \tag{3.7}$$

for some positive constants $C_3 = C_3(n)$ and $C_4 = C_4(n, N, p, m, M)$. Recalling that $D_s u \in L^2$ for $s = 1, \dots, n-1$, we may use Lemma 2.1 in order to get

$$\int_{B_R} |\tau_{s,h}u|^2\,dx \le C_5 |h|^2 \qquad \forall\, s = 1, \dots, n-1, \qquad \forall\, h : |h| < R, \tag{3.8}$$

with C_5 independent of h. Since we do not know apriori that $D_n u \in L^2$, the integral corresponding to $s = n$ in (3.8) is dealt with as follows. We write

$$\int_{B_R} |\tau_{n,h}u|^2\,dx = \int_{B_R} |\tau_{n,h}u|^a |\tau_{n,h}u|^{2-a}\,dx, \tag{3.9}$$

where $0 < a < 2$ is to be chosen later. Let us first assume that

$$u, D_i u \in L^r_{\mathrm{loc}},\ 2 \le r,\ \forall\, i = 1, \dots, n-1, \quad \text{and} \quad D_n u \in L^t_{\mathrm{loc}},\ p \le t < 2. \tag{3.10}$$

In order to apply the anisotropic Sobolev imbedding theorem contained in Lemma 2.5, let $\overline{r}$ be the harmonic mean of the numbers $q_i = r$, $i = 1, \dots, n-1$ and $q_n = t$, i.e.

$$\overline{r} = \frac{nrt}{(n-1)t + r}\,. \tag{3.11}$$

Note that $\overline{r} < n$ if and only if $r < t(n-1)/(t-1)$; define $\overline{r}^*$ as

$$\overline{r}^* = \begin{cases} n\overline{r}/(n-\overline{r}), & \text{if } \ \overline{r} < n, \\ \text{any number } \ > r, & \text{if } \ \overline{r} \ge n. \end{cases}$$

In either case, $\overline{r}^* > r$ and Lemma 2.5 yields

$$u \in L^{\overline{r}^*}_{\mathrm{loc}}\,. \tag{3.12}$$

Thus applying Hölder's inequality on (3.9), with exponents t/a, $t/(t-a)$, provided $0 < a < t$, it follows that

$$\int_{B_R} |\tau_{n,h}u|^2\,dx \le \Big(\int_{B_R} |\tau_{n,h}u|^t\,dx\Big)^{a/t} \Big(\int_{B_R} |\tau_{n,h}u|^{(2-a)t/(t-a)}\,dx\Big)^{(t-a)/t}. \tag{3.13}$$

Because of (3.10) we may use Lemma 2.2 in order to get

$$\Big(\int_{B_R} |\tau_{n,h}u|^t \, dx\Big)^{a/t} \le C_6|h|^a \quad \forall\, h : |h| < R, \tag{3.14}$$

with C_6 independent of h. If

$$(2-a)t/(t-a) \le \overline{r}^*, \tag{3.15}$$

then we may use Lemma 2.4 in order to get

$$\Big(\int_{B_R} |\tau_{n,h}u|^{(2-a)t/(t-a)} \, dx\Big)^{(t-a)/t} \le C_7 \qquad \forall\, h : |h| < R, \tag{3.16}$$

with C_7 independent of h. The inequalities (3.14), (3.16) and (3.13) yield

$$\int_{B_R} |\tau_{n,h}u|^2 \, dx \le C_8|h|^a \qquad \forall\, h : |h| < R, \tag{3.17}$$

with C_8 independent of h. Thus, noting that $a < 2$ and $R \le 1$, (3.8), (3.17) and (3.7) yield

$$\sum_{s=1}^{n} \int_{B_\varrho} |\tau_{s,h}\hat{V}(Du)|^2 \, dx \le C_9|h|^a \qquad \forall\, h : |h| < R, \tag{3.18}$$

with C_9 independent of h. Straightforward computations in (3.15) yield that

$$\begin{cases} 0 < a \le \frac{r(tn+2(t-1))-2(n-1)t}{r(n+t-1)-(n-1)t}, & \text{if } \overline{r} < n, \\ a \ \text{ any number in } \ (0,t), & \text{if } \overline{r} \ge n. \end{cases} \tag{3.19}$$

Let us remark that, when $\overline{r} < n$,

$$0 < \frac{r(tn+2(t-1))-2(n-1)t}{r(n+t-1)-(n-1)t} < t. \tag{3.20}$$

Now via Lemma 2.3 we improve on integrability:

$$\hat{V}(Du) \in L^{\hat{r}}_{\rm loc} \quad \forall\, \hat{r} < 2n/(n-a).$$

This implies via (3.5) that $D_i u \in L^{\hat{r}}_{\rm loc}$, $i = 1, \dots, n-1$ and $D_n u \in L^{\hat{r}p/2}_{\rm loc}$. Elementary computations from (3.12) yield

$$2n/(n-a) \le \overline{r}^*, \tag{3.21}$$

implying that $u \in L^{\hat{r}}_{\rm loc}$. Let us summarize as follows. If

$$u, D_i u \in L^r_{\rm loc}, \quad 2 \le r, \quad \forall\, i = 1, \dots, n-1 \ \text{ and } \ D_n u \in L^t_{\rm loc}, \quad p \le t < 2,$$

then

$$u, D_i u \in L^{\hat{r}}_{\rm loc},\ \forall\, i = 1, \ldots, n-1 \ \text{ and } \ D_n u \in L^{\hat{r}p/2}_{\rm loc},\ \forall\, \hat{r} < 2n/(n-a). \tag{3.22}$$

It is useful to remark that (3.22) continues to hold if (3.10) is replaced by a weaker condition, namely

$$u, D_i u \in L^{\tilde{r}}_{\rm loc},\ \forall\, \tilde{r} < r,\ \forall\, i = 1, \ldots, n-1 \ \text{ and } \ D_n u \in L^{\tilde{t}}_{\rm loc},\ \forall\, \tilde{t} < t, \tag{3.23}$$

provided $2 < r$ and $p < t < 2$. Assuming that $\hat{r} > r$ and $\hat{r}p/2 > t$, we may improve upon $\overline{r}^*$ by using Lemma 2.5 and hence in turn improve on a. Thus the whole analysis behind higher integrability depends upon whether the above process leads to an augmented value of a at each stage of iteration. In what follows we show that this can actually be realized. Although some improvement in t is always possible we can show that t can be boosted to 2, i.e. $D_n u \in L^2_{\rm loc}$, for only a limited range of p. We now describe the iterative process that will be used to boost integrability. At each stage we will compare r to the initial values of $r = 2$ and $t = p$. In the following we have broken down the analysis into two steps. Also, we will firstly assume $n \geq 5$ and although the most of the analysis is valid for $n = 2, 3$ and 4, we treat these separately for better presentation.

Step 1. Since $u, D_i u \in L^2$, $i = 1, \ldots, n-1$, and $D_n u \in L^p$, (3.10) holds with $r = 2$ and $t = p$; we insert the values $r = 2$ and $t = p$ into (3.11). Call $\overline{r}(0)$ the resulting expression, i.e.

$$\overline{r}(0) = \frac{2pn}{(n-1)p + 2}\,. \tag{3.24}$$

We remark that $\overline{r}(0) < n$ so that, by the first line of (3.19) with $r = 2$ and $t = p$, we choose $a(0)$ to be the maximum value allowed for a, that is

$$a(0) = \frac{2(3p-2)}{n(2-p) + (3p-2)}\,. \tag{3.25}$$

We set

$$\varepsilon(0) = \frac{2n}{n - a(0)} - 2 = \frac{4(3p-2)}{n^2(2-p) + (n-2)(3p-2)}\,. \tag{3.26}$$

From (3.22) we find

$$\begin{aligned} u, D_i u \in L^{\hat{r}}_{\rm loc}, \quad &\forall\, \hat{r} < 2 + \varepsilon(0) \quad \forall\, i = 1, \ldots, n-1 \\ &\text{and} \quad D_n u \in L^{\hat{t}}_{\rm loc}, \quad \forall\, \hat{t} < p(1 + \varepsilon(0)/2). \end{aligned} \tag{3.27}$$

We now describe an intermediate stage in the iterative process.

Step 2. Let $\varepsilon > 0$, take $r(\varepsilon) = 2 + \varepsilon$, $t(\varepsilon) = p(1 + \varepsilon/2)$; assume that

$$u, D_i u \in L^{\tilde{r}}_{\mathrm{loc}},\ \forall \tilde{r} < r(\varepsilon),\ \forall i = 1, \dots, n-1 \ \text{ and } \ D_n u \in L^{\tilde{t}}_{\mathrm{loc}},\ \forall \tilde{t} < t(\varepsilon). \tag{3.28}$$

We now split the discussion into three cases, namely (i) $0 < \varepsilon < 2(2-p)/p$, (ii) $\varepsilon = 2(2-p)/p$ and (iii) $\varepsilon > 2(2-p)/p$.

Case (i). We assume that

$$0 < \varepsilon < 2(2-p)/p. \tag{3.29}$$

Then $2 < r(\varepsilon)$ and $p < t(\varepsilon) < 2$. Clearly, (3.23) holds with $r = r(\varepsilon)$ and $t = t(\varepsilon)$. The improvements as in (3.22) are as follows. We insert $r = r(\varepsilon) = 2 + \varepsilon$ and $t = t(\varepsilon) = p(1 + \varepsilon/2)$ into (3.11); setting $\overline{r}(\varepsilon)$ as the resulting expression, we have

$$\overline{r}(\varepsilon) = \frac{2pn}{(n-1)p + 2}(1 + \varepsilon/2). \tag{3.30}$$

Note that, for $n \geq 3$, the condition (3.29) implies $\overline{r}(\varepsilon) < n$, so that we use the first line in (3.19) with $r = r(\varepsilon)$ and $t = t(\varepsilon)$. We choose $a(\varepsilon)$ to be the maximum value allowed for a, that is

$$a(\varepsilon) = \frac{2(3p-2) + (n+2)\varepsilon p}{n(2-p) + (3p-2) + \varepsilon p}\,. \tag{3.31}$$

Set

$$I(\varepsilon) = \frac{2n}{n - a(\varepsilon)} = \frac{4(3p-2) + 2(n+2)\varepsilon p}{n^2(2-p) + (n-2)(3p-2) - 2\varepsilon p} \tag{3.32}$$

and thus in (3.22) we get

$$\begin{aligned} u, D_i u \in L^{\hat{r}}_{\mathrm{loc}}, \quad &\forall \hat{r} < 2 + I(\varepsilon) \quad \forall i = 1, \dots, n-1 \\ &\text{and } \ D_n u \in L^{\hat{t}}_{\mathrm{loc}}, \quad \forall \hat{t} < p(1 + I(\varepsilon)/2). \end{aligned} \tag{3.33}$$

Case (ii). We now assume

$$\varepsilon = 2(2-p)/p\,; \tag{3.34}$$

then the assumption (3.28) implies that, for every $\varepsilon' < \varepsilon = 2(2-p)/p$ we have

$$\begin{aligned} u, D_i u \in L^{\tilde{r}}_{\mathrm{loc}}, \quad &\forall \tilde{r} < r(\varepsilon') \quad \forall i = 1, \dots, n-1 \\ &\text{and } \ D_n u \in L^{\tilde{t}}_{\mathrm{loc}}, \quad \forall \tilde{t} < t(\varepsilon'). \end{aligned} \tag{3.35}$$

Now $\varepsilon' < 2(2-p)/p$ so that we can apply the method in Case (i) with ε' instead of ε and we get (3.33), in particular,

$$D_n u \in L^{\hat{t}}_{\mathrm{loc}}, \quad \forall \hat{t} < p(1 + I(\varepsilon')/2), \quad \forall \varepsilon' < 2(2-p)/p. \tag{3.36}$$

As ε' approaches $2(2-p)/p$, $p(1 + I(\varepsilon')/2)$ goes to $p(1 + 2/(n-2))$ which is bigger than 2, provided $n \geq 3$ and $2 - 4/n < p$; then (3.36) implies

$$D_n u \in L^2_{\mathrm{loc}} \tag{3.37}$$

and Theorem 1 follows.

Case (iii). We assume that

$$\varepsilon > 2(2-p)/p. \tag{3.38}$$

Now $t(\varepsilon) = p(1+\varepsilon/2) > 2$, so that (3.28) implies (3.37) and the statement of Theorem 1 follows.

The preceding discussion indicates that (3.28) implies the result in Theorem 1, whenever $\varepsilon \geq 2(2-p)/p$. However, for $0 < \varepsilon < 2(2-p)/p$, we get only (3.33). This necessitates an iterative process where the new ε is given by $I(\varepsilon)$ as in (3.32). We now describe more precisely this process of bootstrapping ε. In (3.26), set

$$\varepsilon_0 = \varepsilon(0) = \frac{4(3p-2)}{n^2(2-p)+(n-2)(3p-2)}, \tag{3.26}$$

$$\varepsilon_{m+1} = I(\varepsilon_m) \ \text{ if } \ m \geq 0 \ \text{ and } \ 0 < \varepsilon_m < 2(2-p)/p. \tag{3.39}$$

We recall that the proof is achieved whenever, for some m, $\varepsilon_m \geq 2(2-p)/p$. We now prove that $m \to \varepsilon_m$ is strictly increasing. Set $a = 4(3p-2)$, $b = 2(n+2)p$, $c = n^2(2-p)+(n-2)(3p-2)$ and $d = 2p$; then

$$0 < \varepsilon_m < 2(2-p)/p \quad \Longrightarrow \quad c - d\varepsilon_m > 0 \tag{3.40}$$

and

$$\varepsilon_{m+1} = \frac{a+b\varepsilon_m}{c-d\varepsilon_m}. \tag{3.41}$$

Direct computations show that $\varepsilon_0 > 0$; moreover, if $\varepsilon_0 < 2(2-p)/p$, then

$$\varepsilon_1 - \varepsilon_0 = \frac{(b+d\varepsilon_0)\varepsilon_0}{c-d\varepsilon_0} > 0. \tag{3.42}$$

We are going to prove that

$$0 < \varepsilon_i < 2(2-p)/p, \ \forall\, i = 0,\ldots,m \quad \Longrightarrow \quad \varepsilon_j < \varepsilon_{j+1}, \ \forall\, j = 0,\ldots,m. \tag{3.43}$$

Let us set

$$\varepsilon_j < \varepsilon_{j+1}\,; \tag{3.44(j)}$$

we prove (3.44(j)) recursively on j: if $j = 0$ then (3.44(j)) reduces to (3.42); let us assume that (3.44(j)) holds true and $0 \leq j \leq j+1 \leq m$, then

$$\varepsilon_{j+2} - \varepsilon_{j+1} = \frac{(ad+bc)(\varepsilon_{j+1}-\varepsilon_j)}{(c-d\varepsilon_{j+1})(c-d\varepsilon_j)}.$$

Since ε_j and ε_{j+1} are between 0 and $2(2-p)/p$, by (3.40) we have $(c-d\varepsilon_{j+1})(c-d\varepsilon_j)>0$, so, using the recursive assumption (3.44(j)) we get $(\varepsilon_{j+1}-\varepsilon_j)>0$ and (3.44(j+1)) holds true. (3.43) is completely proved.

Let us summarize as follows; if $n\geq 3$ and $\max\{1,2-4/n\}<p<2$ we have shown that either (a) for some m, $\varepsilon_m\geq 2(2-p)/p$ and Theorem 1 follows, or (b) for every m, $0<\varepsilon_m<2(2-p)/p$, also implying that ε_m is increasing. We now confine ourselves to the latter case. Set

$$L=\lim_{m\to\infty}\varepsilon_m.$$

Recall

$$0<\varepsilon_m<2(2-p)/p,\quad \forall\, m=0,1,\dots \tag{3.45}$$

From (3.32)

$$I(n,p,\varepsilon)=\frac{4(3p-2)+2(n+2)\varepsilon p}{n^2(2-p)+(n-2)(3p-2)-2\varepsilon p}.$$

Moreover, for $1\leq p<2$

$$\frac{\partial I}{\partial p}(n,p,\varepsilon)>0,\quad \frac{\partial I}{\partial\varepsilon}(n,p,\varepsilon)>0\quad \text{for } 0<\varepsilon\leq 2(2-p)/p \tag{3.46}$$

and

$$\varepsilon\longrightarrow I(n,p,\varepsilon)\ \text{ is continuous in }\ (0,2(2-p)/p]. \tag{3.47}$$

By (3.39) we can see that ε_m depends on n,p; it is easy to prove that

$$p\longrightarrow\varepsilon_0(n,p)\ \text{ is increasing in }\ [1,2).$$

By (3.46) and (3.47) we get

$$p\longrightarrow\varepsilon_m(n,p)\ \text{ increasing }\quad\Longrightarrow\quad p\longrightarrow\varepsilon_{m+1}(n,p)\ \text{ increasing},$$

so that

$$p\longrightarrow\varepsilon_m(n,p)\ \text{ is increasing in }\ [1,2)\quad\forall\, m\geq 0. \tag{3.48}$$

Let us point out that L depends on n,p too:

$$L(n,p)=\lim_{m\to\infty}\varepsilon_m(n,p). \tag{3.49}$$

Because of (3.45) and (3.46) we have

$$0<L(n,p)\leq 2(2-p)/p; \tag{3.50}$$

since I is continuous with respect to ε, passing to the limit in (3.39) we get

$$L(n,p)=I(n,p,L(n,p)). \tag{3.51}$$

Now (3.48) implies

$$p\longrightarrow L(n,p)\ \text{ is increasing in }\ [1,2). \tag{3.52}$$

We now treat the cases $n=2$, $n=3$, $n=4$ and $n\geq 5$ separately.

Case A. Take $n \geq 5$.

From (1.10) and (3.52), we have

$$L(n, 2-4/n) \leq L(n,p). \tag{3.53}$$

Set $\hat{p} = 2-4/n$; then we have $2(2-\hat{p})/\hat{p} = 4/(n-2)$; because of (1.10), (3.50) and (3.53) we get

$$0 < L(n, 2-4/n) \leq 4/(n-2). \tag{3.54}$$

Moreover

$$L(n, 2-4/n) = I(n, 2-4/n, L(n, 2-4/n)). \tag{3.55}$$

Solving the equation $y = I(n, 2-4/n, y)$, we find $y_1 = 4/(n-2) < n-3 = y_2$, so that $L(n, 2-4/n) = 4/(n-2)$. Going back to (3.53),

$$4/(n-2) = L(n, 2-4/n) \leq L(n,p) \leq 2(2-p)/p < 4/(n-2), \tag{3.56}$$

where the last inequality holds as $y \to 2(2-y)/y$ is strictly decreasing and $2-4/n < p$. The inequalities in (3.56) imply that (3.45) does not hold and the Theorem follows when $n \geq 5$ (also see the discussion following (3.38)).

Case B. Let $n = 4$.

Solving the equation in (3.51),

$$pL^2 - (14-11p)L + (6p-4) = 0, \tag{3.57}$$

it turns out that

$$L = \frac{(14-11p) \pm \sqrt{\Delta}}{2p}, \quad \Delta = (14-11p)^2 - 4p(6p-4). \tag{3.58}$$

We have

$$\Delta < 0 \quad \text{if and only if} \quad 98/97 < p < 2. \tag{3.59}$$

We claim that, for $p \in (98/97, 2)$, $\varepsilon_m \geq 2(2-p)/p$ for some m. We argue by contradiction. If not, then $\varepsilon_m < 2(2-p)/p$ for every m, then $L = \lim_{m\to\infty} \varepsilon_m \in (0, 2(2-p)/p]$. Clearly, L solves (3.57), but by (3.58) L cannot be real. Hence Theorem 1 follows.

Case C. Now consider $n = 3$.

Again by (3.51),

$$pL^2 - 8(1-p)L + (6p-4) = 0; \tag{3.60}$$

it turns out that

$$L = \frac{4(1-p) \pm \sqrt{\Delta_1}}{p}, \quad \Delta_1 = 16 - 28p + 10p^2. \tag{3.61}$$

We have

$$\Delta_1 < 0 \ \text{ if and only if } \ 4/5 < p < 2, \tag{3.62}$$

so that, if $1 < p < 2$, then, as in the case $n = 4$, for some $m \geq 0$ we must have $\varepsilon_m \geq 2(2-p)/p$.

Case D. Lastly, we treat n=2.

Computing $\varepsilon(0)$ from (3.26)

$$\varepsilon(0) = (3p-2)/(2-p). \tag{3.63}$$

We have

$$\begin{aligned} -3+\sqrt{17} < p < 2 \quad &\Longrightarrow \quad \varepsilon(0) > 2(2-p)/p, \\ 1 < p \le -3+\sqrt{17} \quad &\Longrightarrow \quad 0 < \varepsilon(0) \le 2(2-p)/p. \end{aligned}$$

In the case $-3+\sqrt{17} < p < 2$ the proof is finished. Let us consider the case $1 < p \le -3+\sqrt{17}$. The inequality (3.27) allows us to start from (3.28) (see Step 2) with any ε satisfying $0 < \varepsilon < \varepsilon(0)$. Since $(2-p)/p < \varepsilon(0) \le 2(2-p)/p$, we may select ε such that $(2-p)/p < \varepsilon < 2(2-p)/p$. Clearly, (3.29) holds and we have $\overline{r}(\varepsilon) \ge 2 = n$. By (3.19), $a(\varepsilon)$ can be chosen to be in $(0, p(\varepsilon))$ and we get as in (3.33),

$$D_n u \in L^{\hat{t}}_{\mathrm{loc}} \quad \forall \hat{t} < 2p/(2-a(\varepsilon)). \tag{3.33}$$

Since

$$\lim_{a(\varepsilon)\to p(\varepsilon)} \frac{2p}{2-a(\varepsilon)} = \frac{2p}{2-p(\varepsilon)} > 2,$$

we can select $a(\varepsilon)$ so that $2 < 2p/(2-a(\varepsilon))$, then (3.33) implies that $D_n u \in L^2_{\mathrm{loc}}$ and the proof is finished in the case $1 < p \le -3+\sqrt{17}$, too.

The theorem is completely proved. □

4. Proof of Corollary 1.

As in the proof of Theorem 1, we start from the Euler equation and we arrive at (3.7): for some positive constant $C_{10} = C_{10}(n, N, p, m, M)$ we have

$$\int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \, dx + \int_{B_\varrho} |\tau_{s,h} V(D_n u)|^2 \, dx \le \frac{C_{10}}{(R-\varrho)^2} \int_{B_R} |\tau_{s,h} u|^2 \, dx. \tag{3.7}$$

In Theorem 1 we have proved higher integrability of $D_n u$ so that

$$D_1 u, \ldots, D_{n-1} u, D_n u \in L^2_{\mathrm{loc}} \tag{4.1}$$

and we can apply Lemma 2.1 with $t = 2$ for $s = n$ too, compare with (3.8),

$$\int_{B_R} |\tau_{s,h} u|^2 \, dx \le |h|^2 \int_{B_{2R}} |D_s u|^2 \, dx \qquad \forall\, s = 1, \ldots, n-1, n, \quad \forall\, h : |h| < R. \tag{4.2}$$

We put together (3.7) and (4.2): for some positive constant C_{11} independent of h we have

$$\int_{B_\varrho} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \, dx + \int_{B_\varrho} |\tau_{s,h} V(D_n u)|^2 \, dx \le C_{11} |h|^2 \qquad \forall\, s = 1, \ldots, n-1, n, \quad \forall\, h : |h| < R. \tag{4.3}$$

We apply Lemma 2.2 in order to get

$$\exists\, D_sD_iu \in L^2_{\mathrm{loc}} \quad \exists\, D_s(V(D_nu)) \in L^2_{\mathrm{loc}} \qquad \forall\, s = 1,\dots,n-1,n, \quad \forall\, i = 1,\dots,n-1. \tag{4.4}$$

In order to prove existence of D_nD_nu, we use (3.6), Hölder's inequality, Lemma 2.4 and (4.3); thus, for some constants C_{12} and C_{13}, independent of h, we have

$$\begin{aligned} &\int_{B_\varrho} |\tau_{s,h}D_nu|^p\,dx \\ &\le C_{12}\int_{B_\varrho}\Big(1+|D_nu(x)|^2+|D_nu(x+he_s)|^2\Big)^{(2-p)p/4}|\tau_{s,h}V(D_nu(x))|^p\,dx \\ &\le C_{12}\left(\int_{B_\varrho}\Big(1+|D_nu(x)|^2+|D_nu(x+he_s)|^2\Big)^{p/2}dx\right)^{(2-p)/2} \\ &\qquad \left(\int_{B_\varrho}|\tau_{s,h}V(D_nu(x))|^2\,dx\right)^{p/2} \\ &\le C_{13}|h|^p \qquad \forall\, s=1,\dots,n, \quad \forall\, h : |h|<R. \end{aligned} \tag{4.5}$$

Inequality (4.5) with $s = n$ allows us to apply Lemma 2.2:

$$\exists D_nD_nu \in L^p_{\mathrm{loc}}(\Omega). \tag{4.6}$$

This ends the proof. □

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(*Received* October 20, 1992)