

The index ${}_2F_1$ -transform of generalized functions

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Abstract. In this paper the index transformation

$$F(\tau) = \int_0^\infty f(t) {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right) t^\alpha dt$$

${}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right)$ being the Gauss hypergeometric function, is defined on certain space of generalized functions and its inversion formula established for distributions of compact support on $\mathbf{I} = (0, \infty)$.

Keywords: hypergeometric function, index integral transform, generalized functions

Classification: 44A15, 46F12

1. Introduction.

The index ${}_2F_1$ -transform (see [6]) of a real valued function f is defined by:

$$(1.1) \quad F(\tau) = \int_0^\infty \mathbf{F}(\mu, \alpha, \tau, t) f(t) dt$$

where

$$(1.2) \quad \mathbf{F}(\mu, \alpha, \tau, t) = {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right) t^\alpha$$

and ${}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right)$ is the Gauss hypergeometric function, α and μ are complex parameters and τ real.

In this paper, according to Zemanian [14], we introduce the testing function space $U_{a,\mu,\alpha}$ containing the kernel of the transform. As usual, $U'_{a,\mu,\alpha}$ denotes the dual space of $U_{a,\mu,\alpha}$. The generalized index ${}_2F_1$ -transformation of $f \in U'_{a,\mu,\alpha}$ is defined by:

$${}_2\mathcal{F}_1(f) = F(\tau) = \langle f(t), \mathbf{F}(\mu, \alpha, \tau, t) \rangle, \quad \tau \in \mathbb{R}_+.$$

An inversion formula on the space $\mathcal{E}'(\mathbf{I})$ is proved.

The notation and terminology used here is that of Zemanian [14]. In the following \mathbf{I} denotes the open interval $(0, \infty)$ and \mathbb{R}_+ the set of the positive real numbers. The spaces $\mathcal{D}(\mathbf{I})$, $\mathcal{D}'(\mathbf{I})$, $\mathcal{E}(\mathbf{I})$ and $\mathcal{E}'(\mathbf{I})$ have their usual meaning [11]. The parameter a is always in $[0, \frac{1}{2})$.

2. The testing function space and its dual.

Let $U_{a,\mu,\alpha}$ be the linear space of C^∞ -functions on \mathbf{I} according to:

$$U_{a,\mu,\alpha} = \{ \phi \in C^\infty : \gamma_{k,a,\mu,\alpha}(\phi) < \infty, \quad \text{for } k \in \mathbb{N} \cup \{0\} \}$$

where

$$(2.1) \quad \gamma_{k,a,\mu,\alpha}(\phi) = \sup_{0 < t < \infty} \left| (2t + 1)^a t^{\frac{\mu}{2} - \alpha} (t + 1)^{\frac{\mu}{2}} A_t^k \phi(t) \right|$$

A_t being the differential operator:

$$(2.2) \quad t^{\alpha - \mu} (t + 1)^{-\mu} D_t t^{\mu + 1} (t + 1)^{\mu + 1} D_t t^{-\alpha}$$

$U_{a,\mu,\alpha}$ equipped with the topology arising from the family $\{ \gamma_{k,a,\mu,\alpha} \}$ of seminorms of which $\gamma_{0,a,\mu,\alpha}$ is a norm, is a countably multinormed, locally convex, Hausdorff space. By using a technique of Zemanian [14] it follows immediately that $U_{a,\mu,\alpha}$ is sequentially complete, i.e. a Fréchet space.

From the relation:

$$(2.3) \quad A_t \mathbf{F}(\mu, \alpha, \tau, t) = - \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right] \mathbf{F}(\mu, \alpha, \tau, t)$$

and by the asymptotic behavior of the hypergeometric function it follows that $\mathbf{F}(\mu, \alpha, \tau, t) \in U_{a,\mu,\alpha}$.

The dual space $U'_{a,\mu,\alpha}$ of $U_{a,\mu,\alpha}$ is a space of generalized functions. Equipped with the usual weak topology it is a separated multinormed space which is sequentially complete.

The assertions of the following proposition can be proved by using standard techniques (cf. [14]):

Proposition 2.1.

(i) $\mathcal{D}(\mathbf{I})$ is a subspace of $U_{a,\mu,\alpha}$ and the topology of $\mathcal{D}(\mathbf{I})$ is stronger than that induced on it by $U_{a,\mu,\alpha}$. Consequently, the restriction of any $f \in U'_{a,\mu,\alpha}$ to $\mathcal{D}(\mathbf{I})$ is in $\mathcal{D}'(\mathbf{I})$. $\mathcal{D}(\mathbf{I})$ is not dense in $U_{a,\mu,\alpha}$.

(ii) $U_{a,\mu,\alpha}$ is a dense subspace of $\mathcal{E}(\mathbf{I})$. Hence $\mathcal{E}'(\mathbf{I})$ is a subspace of $U'_{a,\mu,\alpha}$.

(iii) For $f \in U'_{a,\mu,\alpha}$ there exists $C > 0$ and $r \in \mathbb{N} \cup \{0\}$ such that

$$| \langle f, \phi \rangle | \leq C \max_{0 \leq k \leq r} \gamma_{k,a,\mu,\alpha}(\phi)$$

for all $\phi \in U_{a,\mu,\alpha}$.

(iv) The differential operator A_t is a continuous linear mapping from $U_{a,\mu,\alpha}$ into $U_{a,\mu,\alpha}$. Its adjoint operator A'_t maps $U'_{a,\mu,\alpha}$ continuously into $U'_{a,\mu,\alpha}$.

(v) A locally integrable function f on \mathbf{I} such that

$$(2t + 1)^{-a} t^{\alpha - \frac{\mu}{2}} (t + 1)^{-\frac{\mu}{2}} f(t)$$

is absolutely integrable on \mathbf{I} , gives rise to a regular generalized function on $U'_{a,\mu,\alpha}$ with

$$\langle f, \phi \rangle = \int_0^\infty f(t) \phi(t) dt, \quad \phi \in U_{a,\mu,\alpha}$$

(vi) If $Re(2\alpha - \frac{\mu}{2}) > -1$ and $a + Re(\mu - \alpha) < -\frac{1}{2}$, $U_{a,\mu,\alpha}$ is contained in $U'_{a,\mu,\alpha}$.

Lemma 2.1. For each compact subset \mathbb{K} contained in \mathbf{I} and $k \in \mathbb{N} \cup \{0\}$ let the seminorm $\gamma_{k,K}$ be defined by

$$\gamma_{k,K}(\phi) = \sup_{t \in K} \left| A_t^k \phi(t) \right|, \quad \phi \in \mathcal{E}(\mathbf{I})$$

where A_t is defined by (2.2). Then, $\{\gamma_{k,K}\}$ gives rise to a topology in $\mathcal{E}(\mathbf{I})$ which coincides with its usual topology.

PROOF: From an inductive argument it can be proved that:

$$A_t^k \phi(t) = \sum_{j=0}^{2k} t^{j-k} p_{j,k}(t) D_t^j \phi(t)$$

with

$$p_{2k,k}(t) = (t + 1)^k \quad \text{and} \quad p_{2k-1,k}(t) = k(t + 1)^{k-1} [\mu - 2\alpha + k + 2t(\mu - \alpha + k)]$$

$p_{j,k}(t)$ being polynomials of degree k , $0 \leq j \leq 2k$. Therefore, if a sequence $\{\phi_n(t)\}_{n \in \mathbb{N}} \subset \mathcal{E}(\mathbf{I})$ converges to zero in the usual topology on $\mathcal{E}(\mathbf{I})$, then ϕ_n converges to zero in the topology generated from $\gamma_{k,K}$.

Conversely, let $\{\phi_n(t)\}_{n \in \mathbb{N}}$ be a sequence on $\mathcal{E}(\mathbf{I})$ converging to zero in the topology generated from $\gamma_{k,K}$. Obviously, $\phi_n(t)$ and $A_t \phi_n(t)$ tend to zero uniformly on every compact $\mathbb{K} \subset \mathbf{I}$.

Moreover,

$$(2.4) \quad \begin{aligned} A_t \phi_n(t) &= t(t + 1) D_t^2 \phi_n(t) + [\mu - 2\alpha + 1 + 2t(\mu - \alpha + 1)] D_t \phi_n(t) + \\ &+ \left[\alpha(\alpha - 2\mu - 1) + \frac{\alpha(\alpha - \mu)}{t} \right] \phi_n(t). \end{aligned}$$

Thus,

$$(2.5) \quad \begin{aligned} A_t \phi_n(t) - \left[\alpha(\alpha - 2\mu - 1) + \frac{\alpha(\alpha - \mu)}{t} \right] \phi_n(t) &= \\ = t(t + 1) D_t^2 \phi_n(t) + [\mu - 2\alpha + 1 + 2t(\mu - \alpha + 1)] D_t \phi_n(t) \end{aligned}$$

tends uniformly to zero on \mathbb{K} . Now, taking into account that (2.5) can be written as:

$$(2.6) \quad t^{2\alpha-\mu} (t + 1)^{-\mu} D_t \left[t^{\mu-2\alpha+1} (t + 1)^{\mu+1} D_t \phi_n(t) \right]$$

by an integration it follows that $D_t \phi_n(t)$ and also $D_t^2 \phi_n(t)$ tends to zero uniformly in \mathbb{K} . By a similar argument it is proved for every non negative integer k , that $D_t^k \phi_n(t)$ converges uniformly to zero in \mathbb{K} .

Finally, since $\mathcal{E}(\mathbf{I})$ is a metrizable space, the conclusion follows. □

3. The generalized transform.

For $f \in U'_{a,\mu,\alpha}$ the generalized index ${}_2F_1$ -transform is defined by

$$(3.1) \quad {}_2\mathcal{F}_1(f) = F(\tau) = \langle f(t), \mathbf{F}(\mu, \alpha, \tau, t) \rangle, \quad \tau \in \mathbb{R}^+.$$

For regular generalized functions this formula coincides with (1.1).

Proposition 3.1. *For all $f \in U'_{a,\mu,\alpha}$, and $k \in \mathbb{N} \cup \{0\}$, one has:*

$${}_2\mathcal{F}_1(A'_t{}^k f) = (-1)^k \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k {}_2\mathcal{F}_1(f)$$

A'_t being the adjoint operator of A_t .

PROOF: By making use of the relation (2.3) the conclusion follows. □

Now, the analyticity of the index ${}_2F_1$ -transform will be established. For it, the next two lemmas are required.

Lemma 3.1. *For each non negative integer m and $Re\mu > -\frac{1}{2}$, one has:*

$$(3.2) \quad |D_\tau^m \mathbf{F}(\mu, \alpha, \tau, t)| \leq \\ \leq Mt^{Re\alpha} \left[\log \left(2t + 1 + 2\sqrt{t(t+1)} \right) \right]^m [t(t+1)]^{-Re\frac{\mu}{2}} P_{-\frac{1}{2}}^{-Re\mu}(2t+1)$$

$P_{-\frac{1}{2}}^{-Re\mu}$ being the well-known associated Legendre function.

PROOF: The integral representation ([1, p. 155]),

$$(3.3) \quad \mathbf{F}(\mu, \alpha, \tau, t) = \\ = \frac{\Gamma(\mu+1)t^\alpha}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})} \int_0^\pi \left(2t+1+2\sqrt{t(t+1)}\cos\xi \right)^{-\mu-\frac{1}{2}-i\tau} (\sin\xi)^{2\mu} d\xi$$

is valid for $Re\mu > -\frac{1}{2}$. Now, differentiating with respect to the parameter τ , (3.2) holds. □

Lemma 3.2. *Let μ be a complex parameter with $Re\mu > -\frac{1}{2}$ and k, m non negative integers. Then there exists $C > 0$ such that:*

$$(3.4) \quad \gamma_{k,a,\mu,\alpha}(D_\tau^m \mathbf{F}(\mu, \alpha, \tau, t)) \leq C \left| \left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right|^k.$$

PROOF: For $k = 0$, making use of the asymptotic behavior:

$$P_{-\frac{1}{2}}^{-Re\mu}(2t+1) \sim \frac{1}{\Gamma(\mu+\frac{1}{2})} \left(\frac{2}{\pi(2t+1)} \right)^{\frac{1}{2}} \log(2t+1), \quad t \rightarrow \infty$$

(cf. [9, p. 173 (12.20)]), it follows from Lemma 3.1:

$$\begin{aligned} \gamma_{0,a,\mu,\alpha}(D_\tau^m \mathbf{F}(\mu, \alpha, \tau, t)) &\leq \\ &\leq M_1 \sup_{0 < t < \infty} \left| (2t + 1)^a \left[\log \left(2t + 1 + 2\sqrt{t(t+1)} \right) \right]^m P_{-\frac{1}{2}}^{-Re\mu}(2t + 1) \right| \leq M_2 \end{aligned}$$

with $M_1, M_2 > 0$.

For $k > 0$, by using the commutativity of A_t^k and D_τ^m , (2.3) and Lemma 3.1, one has:

$$\begin{aligned} \gamma_{k,a,\mu,\alpha}(D_\tau^m \mathbf{F}(\mu, \alpha, \tau, t)) &\leq \\ &\leq \sum_{j=0}^m \binom{m}{j} H_j \left| D_\tau^j \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k \right| \leq C \left| \left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right|^k \end{aligned}$$

$H_j, j = 1, 2, \dots, m$ and C being suitable constants. □

Theorem 3.1. For $f \in U'_{a,\mu,\alpha}$, $Re\mu > -\frac{1}{2}$, the generalized transform $F(\tau)$ defined by (3.1) is an analytic function and

$$(3.5) \quad D_\tau^m F(\tau) = \langle f(t), D_\tau^m \mathbf{F}(\mu, \alpha, \tau, t) \rangle.$$

PROOF: By Lemmas 3.1 and 3.2 it follows that (3.5) has a sense. Moreover, set

$$\frac{F(\tau + \Delta\tau) - F(\tau)}{\Delta\tau} - \langle f(t), D_\tau \mathbf{F}(\mu, \alpha, \tau, t) \rangle = \langle f(t), \Upsilon_{\Delta\tau}(t) \rangle$$

where

$$\begin{aligned} (3.6) \quad \Upsilon_{\Delta\tau}(t) &= \frac{1}{\Delta\tau} [\mathbf{F}(\mu, \alpha, \tau + \Delta\tau, t) - \mathbf{F}(\mu, \alpha, \tau, t)] - D_\tau \mathbf{F}(\mu, \alpha, \tau, t) = \\ &= \frac{1}{\Delta\tau} \int_\tau^{\tau+\Delta\tau} dx \int_\tau^x D_y^2 \mathbf{F}(\mu, \alpha, y, t) dy. \end{aligned}$$

Thus, from (2.3), for any k non negative integer,

$$\begin{aligned} &\left| (2t + 1)^a t^{\frac{\mu}{2} - \alpha} (t + 1)^{\frac{\mu}{2}} A_t^k \Upsilon_{\Delta\tau}(t) \right| \leq \\ &\leq \frac{|\Delta\tau|}{2} \left| (2t + 1)^a t^{\frac{\mu}{2} - \alpha} (t + 1)^{\frac{\mu}{2}} \right| \sup_{y \in \Lambda} \left| D_y^2 \left[\left(\mu + \frac{1}{2} \right)^2 + y^2 \right]^k \mathbf{F}(\mu, \alpha, y, t) \right| \end{aligned}$$

Λ being the interval $\tau - |\Delta\tau| < y < \tau + |\Delta\tau|$.

Now, by the boundedness on $0 < t < \infty$ of

$$\left| (2t + 1)^a t^{\frac{\mu}{2} - \alpha} (t + 1)^{\frac{\mu}{2}} \right| \sup_{y \in \Lambda} \left| D_y^2 \left[\left(\mu + \frac{1}{2} \right)^2 + y^2 \right]^k \mathbf{F}(\mu, \alpha, y, t) \right|$$

for $|\Delta\tau| < 1$, it follows that $\Upsilon_{\Delta\tau}(t) \rightarrow 0$ in $U_{a,\mu,\alpha}$ as $\Delta\tau \rightarrow 0$. With this the proof is finished. □

Theorem 3.2. *Let $F(\tau)$ be the generalized ${}_2F_1$ -transform of f given by (3.1). Then:*

$$(3.7) \quad \begin{cases} \text{(i) For } \tau \rightarrow 0, \text{ one has } F^{(m)}(\tau) = O(1), \text{ for all } m \in \mathbb{N} \cup \{0\}. \\ \text{(ii) There exists a } p \in \mathbb{N} \cup \{0\} \text{ such that } F(\tau) = O\left(\tau^{2p - R_e\mu - \frac{1}{2}}\right), \tau \rightarrow \infty. \end{cases}$$

PROOF: It follows immediately from (2.3), Proposition 2.1 (iii), and taking into account that:

$$|\mathbf{F}(\mu, \alpha, \tau, t)| \leq Mt^{-\frac{1}{2} - R_e(\alpha + \mu)}(t + 1)^{-\frac{1}{2} - R_e\frac{\mu}{2}}\tau^{-\frac{1}{2} - R_e\mu}, \quad \tau \rightarrow \infty$$

(cf. [10, (24), p. 231]).

4. Generalized inversion formula.

In this paragraph we state the main result of this work. For it we recall the definition of the $\mathcal{M}_{c,\gamma}^{-1}(L)$ spaces introduced in [13].

Let c and γ be real numbers such that $2 \operatorname{sgn} c + \operatorname{sgn} \gamma \geq 0$. The space of functions $f(x)$ which can be represented in the form of:

$$f(x) = \frac{1}{2\pi i} \int_{\sigma} \rho(s)x^{-s} ds, \quad x \in (0, \infty), \quad \sigma = \{s \in \mathbb{C} : R_e s = \frac{1}{2}\}$$

where

$$\rho(s) = s^{-\gamma} e^{-c\pi|Im s|} F(s) \quad \text{with} \quad \int_{\sigma} |F(s)| ds < \infty,$$

is denoted by $\mathcal{M}_{c,\gamma}^{-1}(L)$. Before giving the inversion theorem we need to prove the following lemmas:

Lemma 4.1. *If $2 \operatorname{sgn} (c + 1) + \operatorname{sgn} (\gamma - R_e\mu) > 0$, there exists the integral*

$$F(\tau) = \frac{1}{2\pi i} \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \frac{1}{2} + i\tau)\Gamma(\mu + \frac{1}{2} - i\tau)} \cdot \int_{\sigma} \frac{\Gamma(\mu + \frac{1}{2} - \alpha + i\tau - s)\Gamma(\mu + \frac{1}{2} - \alpha - i\tau - s)\Gamma(\alpha + s)}{\Gamma(1 + \mu - \alpha - s)} f^*(1 - s) ds$$

f^* being the Mellin transform of $f \in \mathcal{M}_{c,\gamma}^{-1}(L)$, $\alpha, \mu \in \mathbb{C}$, $\tau \in \mathbb{R}_+$, $\sigma = \{s \in \mathbb{C} : R_e s = \frac{1}{2}\}$.

Moreover, if $R_e\alpha > -\frac{1}{2}$ and $R_e(\mu - \alpha) > 0$, then:

$$(4.2) \quad F(\tau) = \int_0^{\infty} f(t) \mathbf{F}(\mu, \alpha, \tau, t) dt.$$

PROOF: From the asymptotic behavior of the Gamma function (see [1, p. 47]) and since $f \in \mathcal{M}_{c,\gamma}^{-1}(L)$ it follows the existence of the first integral.

On the other hand, if $Re\alpha > -\frac{1}{2}$ and $Re(\mu - \alpha) > 0$,

$$\int_0^\infty \mathbf{F}(\mu, \alpha, \tau, t)t^{s-1} dt$$

converges absolutely $\forall s \in \sigma$.

Moreover, $f^* \in L(\sigma)$ and consequently:

$$\int_0^\infty f(t)\mathbf{F}(\mu, \alpha, \tau, t) dt = \frac{1}{2\pi i} \int_0^\infty \mathbf{F}(\mu, \alpha, \tau, t) dt \int_\sigma f^*(1-s)t^{s-1} ds.$$

Now the absolute convergence of this integral allows us to interchange the order of integration to obtain:

$$\frac{1}{2\pi i} \int_\sigma f^*(1-s) ds \int_0^\infty \mathbf{F}(\mu, \alpha, \tau, t)t^{s-1} dt$$

and the conclusion follows. □

Lemma 4.2. *Let α, μ and s be complex parameters with $Re\alpha > 0, Re\mu > 0, Re s = \frac{1}{2}, \frac{1}{8} < Re(\mu - \alpha) < \frac{1}{4}, Re(\mu - 2\alpha) < -1$. Then one has the following integral representation:*

$$(4.3) \quad \frac{1}{2\Gamma(\mu + 1)} sh \pi\tau\Gamma(\mu + \frac{1}{2} - \alpha + i\tau - s)\Gamma(\mu + \frac{1}{2} - \alpha - i\tau - s)t^{\alpha-\mu}\mathbf{G}(\mu, \alpha, \tau, t) = \\ = \int_0^\infty z^{\mu-\alpha-s}C_\mu(tz) dz \int_{-\infty}^\infty e^{2\theta(\mu-\alpha+\frac{1}{2}-s)} d\theta \int_{|\theta|}^\infty C_0(ze^\theta\Psi) \sin 2\tau u du$$

where $\Psi = 2 ch u - 2 ch \theta$ and

$$\mathbf{G}(\mu, \alpha, \tau, t) = x^{\mu-\alpha} {}_2F_1\left(\frac{1}{2} + i\tau, \frac{1}{2} - i\tau; \mu + 1; -t\right).$$

Remark 4.1. C_μ denotes the Bessel-Clifford function of the first kind and order μ . This function is related with the Bessel function J_μ through $C_\mu(z) = z^{-\frac{\mu}{2}}J_\mu(2\sqrt{z})$ (see [4]).

PROOF: Let us consider the integral representation (cf. [7]):

$$(4.4) \quad \frac{2}{\pi}K_{2\tau i}(2\sqrt{z})K_{2\tau i}(2\sqrt{y}) sh 2\pi\tau = \\ = \int_{|\frac{1}{2} \log \frac{y}{z}|}^\infty C_0(2\sqrt{zy}chu - z - y) \sin 2\tau u du$$

and also that (cf. [12, p. 248] and [2, 10.2(2)] resp.)

$$\int_0^\infty K_{2\tau i}(2\sqrt{z})z^{-\frac{1}{2}}C_\mu(tz) dz = \frac{\Gamma(\frac{1}{2} + i\tau)\Gamma(\frac{1}{2} - i\tau)}{2\Gamma(\mu + 1)}t^{\alpha-\mu}\mathbf{G}(\mu, \alpha, \tau, t) \\ \int_0^\infty y^{\mu-\alpha-\frac{1}{2}-s}K_{2\tau i}(2\sqrt{y})dy = \frac{1}{2}\Gamma(\mu + \frac{1}{2} - \alpha + i\tau - s)\Gamma(\mu + \frac{1}{2} - \alpha - i\tau - s).$$

Now, by means of the change $\frac{1}{2} \log \frac{y}{z} = \theta$, one has (4.3). The existence of the integral (4.3) follows from the asymptotic behavior of the Bessel-Clifford functions $C_\nu(x)$ (cf. [4]) and the hypotheses. □

Lemma 4.3. Let $F(\tau) = {}_2\mathcal{F}_1(f)$, $\phi \in \mathcal{D}(\mathbf{I})$ be and set

$$(4.5) \quad \varphi(\tau) = S(\mu, \tau) \int_0^\infty \phi(t) \mathbf{G}(\mu, \alpha, \tau, t) dt$$

then

$$(4.6) \quad \int_0^N \varphi(\tau) \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle d\tau = \left\langle f(x), \int_0^N \varphi(\tau) \mathbf{F}(\mu, \alpha, \tau, x) d\tau \right\rangle$$

where

$$S(\mu, \tau) = \frac{2}{\pi \Gamma(\mu + 1)^2} \tau \operatorname{sh} \pi \tau \Gamma(\mu + \frac{1}{2} + i\tau) \Gamma(\mu + \frac{1}{2} - i\tau).$$

PROOF: By the asymptotic behavior of the hypergeometric function, one has that

$$(4.7) \quad \Theta_N(x) = \int_0^N \varphi(\tau) \mathbf{F}(\mu, \alpha, \tau, x) d\tau$$

belongs to $U_{a,\mu,\alpha}$.

Moreover, if we put

$$Q(x, n) = \frac{N}{n} \sum_{p=1}^n \varphi\left(\frac{pN}{n}\right) \mathbf{F}\left(\mu, \alpha, \frac{pN}{n}, x\right)$$

it follows

$$(4.8) \quad \langle f(x), Q(x, n) \rangle = \frac{N}{n} \sum_{p=1}^n \varphi\left(\frac{pN}{n}\right) \left\langle f(x), \mathbf{F}\left(\mu, \alpha, \frac{pN}{n}, x\right) \right\rangle$$

and it can be easily proved that (4.8) tends to

$$\int_0^N \varphi(\tau) \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle d\tau \quad \text{for } n \rightarrow \infty.$$

Now, by (2.3) and the asymptotic behavior of $\mathbf{F}(\mu, \alpha, \tau, x)$ it follows the existence of an $X > 0$ and $n_0 \in \mathbb{N}$ such that

$$\left| (2x + 1)^a x^{\frac{\mu}{2} - \alpha} (x + 1)^{\frac{\mu}{2}} A_x^k [\Theta_N(x) - Q(x, n)] \right| < \varepsilon$$

for $x > X$ and $n > n_0$.

Furthermore, by the uniform continuity of $\mathbf{F}(\mu, \alpha, \tau, x)$ ($R_e \alpha > 0$) on the domain $E = \{(x, \tau) : 0 \leq x \leq X, 0 \leq \tau \leq N\}$, there exists $n_1 \in \mathbb{N}$ such that

$$\left| (2x + 1)^a x^{\frac{\mu}{2} - \alpha} (x + 1)^{\frac{\mu}{2}} A_x^k [\Theta_N(x) - Q(x, n)] \right| < \varepsilon$$

for $0 \leq x \leq X$ and $n > n_1$. This fact implies that $Q(x, n) \rightarrow \Theta_N(x)$ in $U_{a,\mu,\alpha}$ as $n \rightarrow \infty$ and therefore (4.6) holds. □

Lemma 4.4. Assume that $\phi \in \mathcal{D}(\mathbf{I})$ and let $\Theta_N(x)$ be given as in Lemma 4.5. If α and μ are complex parameters such that $R_e\alpha > 0$, $R_e\mu > 0$, $\frac{1}{8} < R_e(\mu - \alpha) < \frac{1}{4}$ and $R_e(\frac{\mu}{2} - \alpha) < -\frac{1}{2}$, then $\Theta_N(x)$ converges in $\mathcal{E}(\mathbf{I})$ to $\phi(x)$ as $N \rightarrow \infty$.

PROOF: Let ϕ be in $\mathcal{D}(\mathbf{I})$. If the support of ϕ is contained in the closed interval $[c, d]$, $0 < c < d < \infty$, one has:

$$\Theta_N(x) = \int_0^N S(\mu, \tau) \mathbf{F}(\mu, \alpha, \tau, x) d\tau \int_c^d \phi(t) \mathbf{G}(\mu, \alpha, \tau, t) dt.$$

By virtue of the smoothness of the functions and the finiteness of the limits of integration we may repeatedly differentiate under the integral sign. By using the identity (2.3) we get:

$$\begin{aligned} A_x^k \Theta_N(x) &= \\ &= \int_0^N S(\mu, \tau) (-1)^k \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k \mathbf{F}(\mu, \alpha, \tau, x) d\tau \cdot \\ &\quad \int_c^d \phi(t) \mathbf{G}(\mu, \alpha, \tau, t) dt = \\ &= \int_0^N S(\mu, \tau) \mathbf{F}(\mu, \alpha, \tau, x) d\tau \int_c^d \phi(t) (-1)^k \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k \mathbf{G}(\mu, \alpha, \tau, t) dt. \end{aligned}$$

Integrating by parts and using the identity

$$(4.9) \quad A_t' \mathbf{G}(\mu, \alpha, \tau, t) = - \left[\left(\mu + \frac{1}{2} \right)^2 + \tau^2 \right] \mathbf{G}(\mu, \alpha, \tau, t)$$

A_t' being the adjoint operator of A_t . It follows by applying some properties of the hypergeometric function that (see [1, p. 105]):

$$(4.10) \quad \begin{aligned} A_x^k \Theta_N(x) &= \\ &= \int_0^N S(\mu, \tau) x^\alpha (x+1)^{-\mu} \mathbf{G}(\mu, \alpha, \tau, x) d\tau \int_c^d t^{\mu-2\alpha} (t+1)^\mu A_t^k \phi(t) \mathbf{F}(\mu, \alpha, \tau, t) dt. \end{aligned}$$

By virtue of our assumptions, $\mathcal{D}(\mathbf{I}) \subset \mathcal{M}_{0,n}^{-1}(L)$ ($\forall n \in \mathbb{N}$) and by Lemma 4.1, (4.10) can be rewritten as follows:

$$(4.11) \quad \begin{aligned} &2x^\alpha (x+1)^{-\mu} \int_0^N S(\mu, \tau) \mathbf{G}(\mu, \alpha, \tau, x) d\tau \frac{1}{2\pi i} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{1}{2}+i\tau)\Gamma(\mu+\frac{1}{2}-i\tau)} \cdot \\ &\int_\sigma \frac{\Gamma(\mu+\frac{1}{2}-\alpha+i\tau-s)\Gamma(\mu+\frac{1}{2}-\alpha-i\tau-s)\Gamma(\alpha+s)}{\Gamma(1+\mu-\alpha-s)} \cdot \\ &\quad \left[t^{\mu-2\alpha} (t+1)^\mu A_t^k \phi(t) \right]^* (1-s) ds \end{aligned}$$

with $\sigma = \{s \in \mathbb{C} : R_e s = \frac{1}{2}\}$, and where

$$\left[t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t) \right]^* (1-s)$$

is the Mellin transform of the function within the square brackets calculated at the point $1-s$.

Taking into account that

$$\int_0^N \tau \sin 2\tau u \, d\tau = -\frac{\partial}{\partial u} \frac{\sin 2Nu}{u}$$

by reversing the order of integration and using Lemma 4.2 we obtain:

$$\begin{aligned} & \frac{x^\alpha(x+1)^{-\mu}}{2\pi i} \int_\sigma \frac{\Gamma(\alpha+s)}{\Gamma(1+\mu-\alpha-s)} \left[t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t) \right]^* (1-s) \, ds \\ & \frac{1}{\pi} \int_0^\infty z^{\mu-\alpha-s} C_\mu(xz) \, dz \int_{-\infty}^\infty e^{2\theta(\mu-\alpha+\frac{1}{2}-s)} \, d\theta. \\ & \int_{|\theta|}^\infty C_0(z e^\theta \Psi) \left(-\frac{\partial}{\partial u} \frac{\sin 2Nu}{u} \right) \, du. \end{aligned}$$

The absolute convergence allows the interchanging of the order of integration and it follows

$$\begin{aligned} (4.12) \quad A_x^k \Theta_N(x) &= \frac{x^\alpha(x+1)^{-\mu}}{\pi} \int_0^\infty \left(-\frac{\partial}{\partial u} \frac{\sin 2Nu}{u} \right) \, du \int_{-u}^u e^{2\theta(\mu-\alpha+\frac{1}{2}-s)} \, d\theta. \\ & \int_0^\infty z^{\mu-\alpha} C_\mu(xz) C_0(z e^\theta \Psi) \, dz. \\ & \frac{1}{2\pi i} \int_\sigma \left(t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t) \right)^* (1-s) \left(z e^{2\theta} \right)^{-s} \, ds. \end{aligned}$$

Observe that

$$(4.13) \quad \frac{1}{2\pi i} \int_\sigma \left(t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t) \right)^* (1-s) \left(z e^{2\theta} \right)^{-s} \, ds$$

represents the G_{02}^{10} -transform of

$$t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t)$$

evaluated at the point $z e^{2\theta}$ (see [13]). This transform exists since it can be proved that $\mathcal{D}(\mathbf{I}) \subset \mathcal{M}_{0,n}^{-1}(L)$, $\forall n \in \mathbb{N}$. We denote (4.13) by $G(\phi_k)(z e^{2\theta})$.

Now, by making the change of variable $z e^{2\theta} = y$, (4.12) can be written as

$$\begin{aligned} (4.14) \quad & \frac{x^\alpha(x+1)^{-\mu}}{\pi} \int_0^\infty \left(-\frac{\partial}{\partial u} \frac{\sin 2Nu}{u} \right) \, du \int_{-u}^u e^{-\theta} \, d\theta. \\ & \int_0^\infty G(\phi_k)(y) y^{\mu-\alpha} C_\mu(x y e^{-2\theta}) C_0(y e^{-2\theta} \Psi) \, dy. \end{aligned}$$

A partial integration leads to:

$$\begin{aligned}
 A_x^k \Theta_N(x) &= \\
 &= -\frac{\sin 2Nu}{u} \frac{x^\alpha(x+1)^{-\mu}}{\pi} \int_{-u}^u e^{-\theta} d\theta \cdot \\
 (4.15) \quad &\int_0^\infty G(\phi_k)(y)y^{\mu-\alpha} C_\mu(xy e^{-2\theta}) C_0(ye^{-2\theta}\Psi) dy \Big|_0^\infty \\
 &+ \frac{x^\alpha(x+1)^{-\mu}}{\pi} \int_0^\infty \Phi(x, u) \frac{\sin 2Nu}{u} du
 \end{aligned}$$

where

$$\begin{aligned}
 \Phi(x, u) &= e^{-u} \int_0^\infty G(\phi_k)(y)y^{\mu-\alpha} C_\mu(xy e^{2u}) dy + \\
 (4.16) \quad &+ e^u \int_0^\infty G(\phi_k)(y)y^{\mu-\alpha} C_\mu(xy e^{-2u}) dy + \\
 &+ \int_{-u}^u e^{-\theta} d\theta \frac{\partial}{\partial u} \int_0^\infty G(\phi_k)(y)y^{\mu-\alpha} C_\mu(xy e^{-2\theta}) C_0(ye^\theta\Psi) dy.
 \end{aligned}$$

It can be shown that the first term of (4.15) tends uniformly to zero for $u \rightarrow 0$ and $u \rightarrow \infty$ if $\frac{1}{8} < R_e(\mu - \alpha) < \frac{1}{4}$ when x belongs to any compact $\mathbb{K} \subset \mathbf{I}$.

Next, by the absolute convergence, one can differentiate under the integral sign in the last term of (4.16). By using the identity

$$\frac{\partial}{\partial u} C_0(ye^{-\theta}\Psi) = 2ye^{-2\theta}(e^{\theta-u} - 1)C_1(ye^{-\theta}\Psi) - \frac{\partial}{\partial \theta} C_0(ye^{-\theta}\Psi)$$

we obtain

$$\begin{aligned}
 (4.17) \quad \Phi(x, u) &= \\
 &= 2e^u \int_0^\infty G(\phi_k)(y)y^{\mu-\alpha} C_\mu(xy e^{-2u}) dy + F_1(x, u) - F_2(x, u) - F_3(x, u)
 \end{aligned}$$

where

$$\begin{aligned}
 F_1(x, u) &= 2e^{-u} \int_{-u}^u e^{-2\theta} d\theta \int_0^\infty G(\phi_k)(y)y^{\mu-\alpha+1} C_\mu(xy e^{-2\theta}) C_1(ye^{-\theta}\Psi) dy, \\
 F_2(x, u) &= 2 \int_{-u}^u e^{-3\theta} d\theta \int_0^\infty G(\phi_k)(y)y^{\mu-\alpha+1} C_\mu(xy e^{-2\theta}) C_1(ye^{-\theta}\Psi) dy, \\
 F_3(x, u) &= \int_{-u}^u e^{-\theta} d\theta \cdot \\
 &\int_0^\infty G(\phi_k)(y)y^{\mu-\alpha} \left[e^{-\theta} C_\mu(xy e^{-2\theta}) + 2xy e^{-3\theta} C_{\mu+1}(xy e^{-2\theta}) \right] dy.
 \end{aligned}$$

Now, observe that (see [13])

$$G(\phi_k)(y) = \int_0^\infty t^{\mu-2\alpha}(t+1)^\mu A_t^k \phi(t) t^\mu C_\mu(ty) dt.$$

According to the inversion formula of the Hankel-Clifford transform (see [5] and [8]) we get:

$$(4.18) \quad \Phi(x, u) = 2x^{\alpha-\mu}(xe^{2u} + 1)^\mu e^{-2u(\mu-\alpha-\frac{1}{2})} A_x^k \phi(xe^{2u}) + F_1(x, u) - F_2(x, u) - F_3(x, u).$$

Thus

$$A_x^k \Theta_N(x) = \frac{1}{\pi} \int_0^\infty 2e^{-2u(\mu-\alpha-\frac{1}{2})} \left(\frac{xe^{2u} + 1}{x + 1}\right)^\mu A_x^k \phi(xe^{2u}) \frac{\sin 2Nu}{u} du + x^\alpha(x + 1)^{-\mu} \int_0^\infty (F_1(x, u) - F_2(x, u) - F_3(x, u)) \frac{\sin 2Nu}{u} du.$$

Let us consider now

$$(4.19) \quad \begin{aligned} & A_x^k(\Theta_N(x) - \phi(x)) = \\ & \frac{2}{\pi} \int_0^\infty \left[e^{-2u(\mu-\alpha-\frac{1}{2})} \left(\frac{xe^{2u} + 1}{x + 1}\right)^\mu A_x^k \phi(xe^{2u}) - A_x^k \phi(x) \right] \frac{\sin 2Nu}{u} du + \\ & + \frac{1}{\pi} \int_0^\infty (F_1(x, u) - F_2(x, u) - F_3(x, u)) \frac{\sin 2Nu}{u} du. \end{aligned}$$

For x in a compact $\mathbb{K} \subset \mathbf{I}$,

$$(4.20) \quad \begin{aligned} A_x^k(\Theta_N(x) - \phi(x)) &= \left(\int_0^\delta + \int_\delta^\infty \right) v(x, u) \sin 2Nu du + \\ &+ \frac{1}{\pi} \int_0^\infty (F_1(x, u) - F_2(x, u) - F_3(x, u)) \frac{\sin 2Nu}{u} du \end{aligned}$$

where

$$v(x, u) = \frac{2}{\pi u} \left[e^{-2u(\mu-\alpha-\frac{1}{2})} \left(\frac{xe^{2u} + 1}{x + 1}\right)^\mu A_x^k \phi(xe^{2u}) - A_x^k \phi(x) \right]$$

with $\delta > 0$.

From the boundedness of $v(x, u)$ on $E = \{(x, u) : x \in \mathbb{K}, 0 \leq u \leq 1\}$, for a given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that for each δ in the interval $(0, \delta_1]$, $x \in \mathbb{K}$ and $N > 0$, we have

$$\left| \int_0^\delta v(x, u) \sin 2Nu du \right| < \frac{\varepsilon}{2}.$$

In order to study \int_δ^∞ , set

$$\lambda(x, u) = \frac{1}{u} e^{-2u(\mu-\alpha-\frac{1}{2})} \left(\frac{x e^{2u} + 1}{x + 1} \right)^\mu A_x^k \phi(x e^{2u}).$$

Since $\phi \in \mathcal{D}(\mathbf{I})$ there exists a constant $m > 0$ such that the support of $\lambda(x, u)$ with respect to u is upperly bounded by m whatever $x \in \mathbb{K}$ may be. An integration by parts yields

$$\begin{aligned} & \int_\delta^\infty \sin 2Nu \lambda(x, u) du = \\ &= \frac{1}{2N} [(\cos 2N\delta) \lambda(x, \delta)] + \int_\delta^h (\cos 2Nu) \frac{\partial}{\partial u} \lambda(x, u) du. \end{aligned}$$

But $\lambda(x, u)$ is a bounded function of x and $\frac{\partial}{\partial u} \lambda(x, u)$ is a bounded function of (x, u) for all $x \in \mathbb{K}$ and $u \in [\delta, m]$. Moreover,

$$\int_{2N\delta}^\infty \frac{\sin u}{u} du \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

These facts imply that there exists an N_1 such that, for every $N > N_1$, and every $x \in \mathbb{K}$

$$\left| \int_\delta^\infty v(x, u) \sin 2Nu du \right| < \frac{\varepsilon}{2}.$$

Finally, by the boundedness of the functions $x^{\frac{1}{4}} C_0(x)$ and $x^{\frac{1}{4}} C_1(x)$ [4], the imposed conditions and the estimation

$$|G(\psi)(w)| < C w^{-\frac{1}{2}}$$

C being a suitable constant, it follows after some manipulations that

$$x^\alpha (1+x)^{-\mu} \frac{F_i(x, u)}{u} \in L(0, \infty), \quad i = 1, 2, 3.$$

Furthermore, from the Riemann lemma, we can conclude that

$$x^\alpha (x+1)^{-\mu} \int_0^\infty F_i(x, u) \frac{\sin 2Nu}{u} du \rightarrow 0$$

uniformly in \mathbb{K} , as $N \rightarrow \infty$, $i = 1, 2, 3$. Therefore, by Lemma 2.1, $\Theta_N(x) \rightarrow \phi(x)$ in $\mathcal{E}(\mathbf{I})$, and the lemma is proved. \square

Now, we establish an inversion formula on the subspace $\mathcal{E}'(\mathbf{I})$ of the distributions of compact support which is a subspace of $U'_{a,\mu,\alpha}$.

Theorem 4.1. *Let $f \in \mathcal{E}'(\mathbf{I})$ be and set*

$$F(\tau) = \langle f(t), \mathbf{F}(\mu, \alpha, \tau, t) \rangle.$$

Then, for every $\phi \in \mathcal{D}(\mathbf{I})$

$$(4.21) \quad \langle f, \phi \rangle = \lim_{N \rightarrow \infty} \left\langle \int_0^N S(\mu, \tau) \mathbf{G}(\mu, \alpha, \tau, t) F(\tau) d\tau, \phi(t) \right\rangle$$

with $R_e\alpha > 0, R_e\mu > 0, \frac{1}{8} < R_e(\mu - \alpha) < \frac{1}{4}$ and $R_e(\frac{\mu}{2} - \alpha) < -\frac{1}{2}$.

PROOF: Let $\phi \in \mathcal{D}(\mathbf{I})$ be. We shall show that

$$(4.22) \quad \left\langle \int_0^N S(\mu, \tau) \mathbf{G}(\mu, \alpha, \tau, t) F(\tau) d\tau, \phi(t) \right\rangle$$

tends to $\langle f, \phi \rangle$ as $N \rightarrow \infty$. From the analyticity of $F(\tau)$ and the fact that the support of $\phi(t)$ is a compact subset of \mathbf{I} , it follows that (4.22) is really a repeated integral in (t, τ) having a continuous integrand on a closed bounded domain of integration. Thus, we may change the order of integration to obtain from (4.22):

$$\int_0^N \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle d\tau \int_0^\infty \phi(t) S(\mu, \tau) \mathbf{G}(\mu, \alpha, \tau, t) dt.$$

By Lemma 4.3, this is equal to

$$(4.23) \quad \left\langle f(x), \int_0^N \mathbf{F}(\mu, \alpha, \tau, x) d\tau \int_0^\infty \phi(t) S(\mu, \tau) \mathbf{G}(\mu, \alpha, \tau, t) dt \right\rangle.$$

Then, $f \in \mathcal{E}'(\mathbf{I})$, and according to Lemma 4.4, the testing function inside (4.23) converges in $\mathcal{E}(\mathbf{I})$ to $\phi(x)$ as $N \rightarrow \infty$, and this completes the proof. □

An immediate consequence of the above inversion theorem is the following uniqueness theorem:

Theorem 4.2. *Let $F(\tau) = {}_2\mathcal{F}_1(f)$ and $G(\tau) = {}_2\mathcal{F}_1(g)$ with $f, g \in \mathcal{E}'(\mathbf{I})$ and assume that $F(\tau) = G(\tau)$ for all $\tau > 0$. Then $f = g$.*

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(Received July 7, 1992, revised May 5, 1993)