Properties of function algebras in terms of their orthogonal measures

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Abstract. In the present note, we characterize the pervasive, analytic, integrity domain and the antisymmetric function algebras respectively, defined on a compact Hausdorff space X, in terms of their orthogonal measures on X.

Keywords: compact Hausdorff space X, the sup-norm algebra $C(X)$ of all complex-valued continuous functions on X, its closed subalgebras called function algebras, pervasive (analytic, integrity domain, antisymmetric) function algebra, measure orthogonal to a function algebra

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Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra, consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a <u>function algebra</u> on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X.

A function algebra A on X is said to be:
<u>pervasive</u> whenever for any nonvoid proper closed subset F of X, the algebra A/F of all restrictions of functions in A to the set F is dense in $C(F)$ with respect to $|\cdot|_F$, the sup-norm on F;
<u>analytic</u> whenever any function f in A which vanishes on a nonvoid open subset of X vanishes identically;
<u>integrity domain</u> whenever A has no nontrivial divisors of zero, i.e. whenever $fg = 0$ for f, g in A implies either $f = 0$ or $g = 0$;
<u>antisymmetric</u> provided it satisfies the following condition: any function in A, which is real-valued, is a constant one.

The mentioned notions are due to Helson, Quigley [1] and Hoffman, Singer [2].

Denote by $M(X)$ the space of all complex Borel regular measures on X, i.e. by the Riesz Representation Theorem, the dual space of $C(X)$.

Whenever A is a closed subspace of $C(X)$, let $A^\perp$ be the <u>annihilator</u> of A, or the set of all measures m in $M(X)$ such that $\int f\,dm = 0$ for any f in A. The dual space A' of A is then canonically isomorphic to the quotient space $M(X)/A^\perp$.

Now endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

Our aim here is to characterize the above-mentioned properties of a function algebra A by means of the properties of the measures in $A^\perp$.

We have dealt with a part of these problems in [3].

Theorem 1 (cf. [3])**.** *A function algebra A on X is pervasive if and only if the closed support of any nontrivial measure in $A^\perp$ is all X.*

Proof: Let A be pervasive. Fix an arbitrary m in $M(X)$ such that $\emptyset \neq \operatorname{spt} m = F \neq X$, and prove that m does not annihilate A. Without loss of generality, we may assume that $|m|$, the total variation of m on X, is equal to one. Take an f in $C(F)$ such that $\int_F f\,dm = 1$ and choose a g in A such that $|f-g|_F < 1$. Then

$$|1 - \int g\,dm| = |\int (f-g)\,dm| \leq |f-g|_F |m| < 1\,.$$

It follows that m does not annihilate g.

Let, on the contrary, A be not pervasive. Then there exists a closed nonvoid proper subset F of X and a function f in $C(F)$ which is not in the closure of A/F in the sup-norm on F. By the Hahn-Banach Separating Theorem there is an m in $M(F)$ such that $\int_F f\,dm \neq 0$ while $\int_F g\,dm = 0$ for any g in A. If $\mathbf{m}$ is the trivial extension of m from F to X, then $\mathbf{m}$ is in $A^\perp$. Indeed, $\int g\,d\mathbf{m} = \int_F g\,dm = 0$ for an arbitrary g in A. But $\operatorname{spt} \mathbf{m} = \operatorname{spt} m \subset F \neq X$ and, m being nontrivial, $\mathbf{m}$ is not trivial, too. □

Remark. Let us mention that the proof above does also work in case of A being merely a function space, not necessarily an algebra.

For a Borel set $G \subset X$, denote by $M(G)$ the set $M(X)/G$ of all restrictions of measures on X to the set G; for m in $M(G)$ let us not distinguish between m and its trivial extension to X.

For a closed set $F \subset X$, denote by I_F the ideal of all functions in $C(X)$ which vanish on F.

Now, let us search for the dual space I'_F of I_F.

Any measure on F annihilates I_F, this implies $I_F^\perp \subset M(F)$. F is closed in X and then $M(F)$ is weak-star closed in $M(X)$. Suppose that there is an m in $I_F^\perp$ which is not supported by F and take, by the Mazur Separation Theorem for locally convex spaces, an f in $C(X)$ such that $\int f\,dm = 1$ while $\int f\,dn = 0$ for any n in $M(F)$. But then f is in I_F, a contradiction.

So $I_F^\perp = M(F)$ and $m \to m + M(F)$ is a canonical isomorphism of $M(X-F)$ onto the quotient space $M(X)/M(F) = I'_F$. If we endow $M(X-F)$ with the w_F^+-topology, i.e. the weak-star topology generated by all functions in I_F, it becomes a locally convex space, and the ideal I_F becomes its dual space.

Proposition. *Let A be a function algebra on X and let F be a closed nonvoid subset of X. Then the following two conditions are equivalent.*

(i) *The intersection $I_F \cap A$ consists of the zero function only;*

(ii) *The set $A^\perp/X-F$ is dense in $M(X-F)$with respect to the w_F^+-topology.*

Proof: Let (ii) be not valid. Take m in $M(X-F)$ but off $\overline{A^\perp/X-F}$, where the bar denotes the w_F^+-closure. Then there is an f in I_F such that $\int f\,dm = 1$ while $\int f\,dn = 0$ for all n in $A^\perp/X-F$.

Now take an arbitrary p in $A^\perp$. Then, f being equal to zero identically on F,

$$\int f\,dp = \int f\,d(p/X-F) = 0$$

and f is in A. But m does not annihilate f, hence $f \neq 0$. The condition (i) fails.

Let (i) be not valid. Take f in $I_F \cap A$ which does not vanish identically on X. Let m in $M(X/F)$ be such that $\int f\,dm = 1$ (e.g. a one-point mass). For any n in $A^\perp$, we have

$$0 = \int f\,dn = \int f\,d(n/X-F)\,,$$

so that f annihilates $A^\perp/X-F$, and, by the continuity, annihilates its closure, too. But f does not annihilate m, m is not in the closure of $A^\perp/X-F$, and the condition (ii) fails. $\square$

Now, let us formulate two simple consequences of the Proposition.

Theorem 2. *A function algebra A on X is analytic if and only if the following condition is fulfilled:*

For an arbitrary open G, which is not dense in X, the set $A^\perp/G$ is dense in $M(G)$ with respect to the w_{X-G}^+-topology.

Theorem 3. *A function algebra A on X is an integrity domain if and only if the following condition is fulfilled:*

Whenever G_1, G_2, is a disjoint couple of open non-dense subsets of X, then the space $A^\perp/G_i$ is $w_{X-G_i}^+$-dense in $M(G_i)$ either for $i=1$ or for $i=2$.

Now, denote by Re $C(X)$, Re $M(X)$, the space of all real parts of the elements in $C(X)$, $M(X)$, respectively. Then Re $M(X)$ is the real dual space of Re $C(X)$. Endow Re $M(X)$ with the weak-star topology. Re $M(X)$ becomes a real locally convex space with the real dual space Re $C(X)$.

Finally, denote by M_0 the set of all measures m in $M(X)$ such that $\int\,dm = 0$; M_0 is then the annihilator of constant functions on X and Re M_0 is the real annihilator of real (or all, the same) constant functions on X.

Theorem 4. *Let A be a function algebra on X. Then A is antisymmetric if and only if the set* Re $A^\perp = \{\mathrm{Re}\, m, m$ *in* $A^\perp\}$ *is dense in* Re M_0 *with respect to the weak-star topology.*

Proof: Let m be in $A^\perp$. Then

$$0 = \int 1\,dm = \int d\,\mathrm{Re}\;m + i\int d\,\mathrm{Im}\;m$$

and both Re m, Im m, are in Re M_0, so Re $A^\perp \subset$ Re M_0.

Suppose Re $A^\perp$ is not dense in Re M_0 and fix an m in Re M_0 but off the weak-star closure of Re $A^\perp$. Take, by the Hahn-Banach Separating Theorem (or Mazur Theorem, more precisely) an f in Re $C(X)$ such that

$$\int f\,dm \neq 0 \text{ and } \int f\,d\mathrm{Re}\, n = 0 \text{ for any } n \text{ in } A^\perp .$$

If n is in $A^\perp$, then both Re n, Im n, are in Re $A^\perp$, so that

$$\int f\,dn = \int f\,d\mathrm{Re}\, n + i\int f\,d\mathrm{Im}\, n = 0$$

and f annihilates $A^\perp$ and so f lies in A. But m annihilates constants and does not annihilate f, this implies that f is not constant, so A is not antisymmetric.

Conversely, let f be a non-constant real function in A. There is a measure m in Re $M(X)$ such that

$$\int f\,dm \neq 0 \text{ and } \int dm = 0 .$$

Then m is in M_0. For an arbitrary n in $A^\perp$,

$$\int f\,d\mathrm{Re}\, n = \mathrm{Re} \int f\,dn = 0 ,$$

so f annihilates Re $A^\perp$ and then, by a continuity reason, it annihilates the whole $\overline{\mathrm{Re}\ A^\perp}$ as well. But f does not annihilate m, thus m is not in Re $A^\perp$, consequently, Re $A^\perp$ is not dense in Re M_0, and Theorem 4 is proved. □

Remark Two. Both proofs, of Theorems 2 and 4, are valid also in the case of mere function spaces. Of course, the notion of "integrity domain" in case of a function space which is not an algebra does not make sense.

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