On the conditional intensity of a random measure

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Abstract. We prove the existence of the conditional intensity of a random measure that is absolutely continuous with respect to its mean; when there exists an L^p-intensity, $p > 1$, the conditional intensity is obtained at the same time almost surely and in the mean.

Keywords: random measure, point process, conditional intensity, absolute continuity, martingales

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1. Introduction

Let $\mathcal{X}$ be a locally compact Hausdorff space with a countable topological basis, and d a distance such that the space $(\mathcal{X}, d)$ is Polish. Further, we denote by $\mathcal{B}$ the ring of relatively compact Borel subsets of $\mathcal{X}$, and by $\mathcal{M}$ the space of Borel non negative measures that are finite on $\mathcal{B}$, that is, the space of Radon measures, endowed with the vague topology. A random measure ξ is a measurable function defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ taking values on $\mathcal{M}$ endowed with the Borel σ-algebra associated with the vague topology. Finally, if $B \in \mathcal{B}$, we denote by $B\xi$ the random measure on B induced by ξ: $(B\xi)(A) = \xi(A \cap B)$, for every $A \in \mathcal{B}$.

Take a sequence $\{\Pi_n\}$, of $\mathcal{B}$-measurable partitions of $\mathcal{X}$ such that, for every $C \in \mathcal{B}$ and $n \geq 1$, the number of elements of the set $\{I \in \Pi_n : I \cap C \neq \emptyset\}$ is finite, and $\max_{I\in\Pi_n} \operatorname{diam}(I) \longrightarrow 0$ as $n \longrightarrow \infty$. Further, suppose that Π_{n+1} is a refinement of Π_n, for every $n \in \mathbb{N}$.

Let $K \in \mathcal{I} = \bigcup_{n=1}^{\infty} \Pi_n$ be such that $\mathbf{E}(\xi(K)) < \infty$. For every $n \geq 1$ define

$$\zeta_n(K) = \sum_{I\in\Pi_n\cap K} \mathbf{E}\left(\xi(I)|I^c\xi\right),$$

where I^c represents the complementary set of I. In [6], it is shown that $\zeta_n(K)$ converges almost surely and in mean to $\zeta(K)$, where ζ is a random measure, when ξ is a simple point process with finite second order moment. Moreover, Papangelou [6], [7] has given conditions for ζ to be almost surely diffuse and independent of the choice of the sequence of partitions. Kallenberg extended these results with

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a proof which enables a more accurate study of the limit random measure ζ, [2]. In the book [4, p. 160], it is remarked that this property may be generalized to any discrete random measure. In fact, the proofs depend essentially on the fact that the random measure ξ is discrete, as the following property is fundamental in that proof:

$$\forall_{I,J\in\mathcal{I},\,J\subset I},\ \mathbf{P}\{\cdot|J^c\xi\}=\frac{\mathbf{P}\{\cdot,\xi(I\setminus J)=0|I^c\xi\}}{\mathbf{P}\{\xi(I\setminus J)=0|I^c\xi\}}$$

with $\mathbf{P}\{\xi(I\setminus J)=0|I^c\xi\}>0$ a.s. on $\{\xi(I\setminus J)=0\}$.

In this paper we propose a quite different proof of the above mentioned convergences adapted to the case of a random measure almost surely absolutely continuous with respect to its mean measure (which excludes most discrete point processes!). In fact, to prove the almost sure convergence, we explicitly use Kallenberg's condition of absolute continuity: let $p>1$ and $\|\cdot\|_p$ be the norm of $\mathbf{L}^p(\Omega,\mathcal{F},\mathbf{P})$, and define the set function $\|\xi\|_p$ by

$$\|\xi\|_p(K)=\lim_{n\to\infty}\sum_{I\in\Pi_n\cap K}\|\xi(I)\|_p$$

for every $K\in\mathcal{I}$ (see [3] or [4, p. 23]). Suppose that, for every $K\in\mathcal{B}$, $\|\xi\|_p(K)<\infty$. If we put $\mu=\mathbf{E}\xi$ then, $\xi\ll\mu$ a.s. and $\|\xi\|_p$ is a measure verifying $\|\xi\|_p=\|X\mu\|_p=\|X\|_p\mu$, where $X=\frac{d\xi}{d\mu}$.

However, to prove the mean convergence, we follow an argument which is close to the proof of Theorem 1 in [7]. In what regards the mean convergence our result is somewhat weaker than the result proved by Papangelou [7], as the assumption of almost sure absolute continuity with respect to the mean measure implies the absolute continuity of the Campbell measure with respect to $\mu\otimes\mathbf{P}$ on the product σ-field which is stronger than the absolute continuity imposed by Papangelou in his theorem.

The results obtained are essentially convergence theorems, as there is no construction on our reasoning. Nevertheless, if $\xi\ll\mu$ a.s. and μ is diffuse, it is known that the conditional intensity measure is the same as the original random measure ξ. This is already known in the more general case of ξ being a.s. diffuse, cf. [8].

2. Mean convergence

(a) Let $K\in\mathcal{B}$ be fixed. The restriction μ_K of μ to $\mathcal{B}_K=\{B\in\mathcal{B}:\ B\subset K\}$ is a finite measure. Without loss of generality we may suppose that $\mu_K(K)=1$ in order to enable us to use martingale theory. For example, if $\Omega_0=\{\omega\in\Omega:\ \xi^\omega\ll\mu\}$ and $\omega\in\Omega_0$, the sequence

$$X_n(\omega)=\sum_{I\in\Pi_n\cap K}\frac{\xi^\omega(I)}{\mathbf{E}(\xi(I))}\mathbb{I}_I,\qquad n\geq 1$$

converges μ_K-almost everywhere on K to $X(\omega)$. As remarked by [4, p. 24], the sequence $\{X_n\}$ converges then to X $\mathbf{P}\otimes\mu_K$-almost everywhere on $\Omega\times K$, so we may suppose X measurable on $(\Omega\times K, \mathcal{F}\otimes\mathcal{B}_K)$.

For every $I\in\mathcal{I}\cap K$ and $t\in I$, put

$$\nu_t^I(A)=\int_A X(\cdot,t)\,d\mathbf{P},\qquad A\in\sigma(I^c\xi),$$

where $\sigma(I^c\xi)$ is the σ-algebra induced by the restriction of ξ to I^c. As the space $\mathcal{M}$ endowed with the vague topology is Polish, $\sigma(I^c\xi)$ admits a countable base. From a theorem of Doob ([5, p. 64]), there exists a $\sigma(I^c\xi)\otimes\mathcal{B}_K$-measurable function $u^I(\omega,t)$ on $\Omega\times K$ such that, for every $t\in K$

$$u^I(\cdot,t)=\frac{d\nu_t^I}{d\mathbf{P}}\,.$$

As $\mathbf{E}(X|I^c\xi)=u^I(\cdot,\cdot)$, we derive that

$$Y_n=\sum_{I\in\Pi_n\cap K}\mathbf{E}(X|I^c\xi)\mathbb{I}_I$$

is an $\mathcal{F}\otimes\mathcal{B}_K$-measurable function.

Define

$$Z_n=\sum_{I\in\Pi_n\cap K}\mathbf{E}\left(\frac{\xi(I)}{\mathbf{E}(\xi(I))}|I^c\xi\right)\mathbb{I}_I,$$

then

$$\int|Z_n-Y_n|\,d\mathbf{P}\otimes\mu_k=$$

$$=\int_K\mathbf{E}\left|\sum_{I\in\Pi_n\cap K}\mathbf{E}\left(\frac{\xi(I)}{\mathbf{E}(\xi(I))}|I^c\xi\right)\mathbb{I}_I\;-\sum_{I\in\Pi_n\cap K}\mathbf{E}\left(X|I^c\xi\right)\mathbb{I}_I\right|\,d\mu=$$

$$=\int_K\sum_{I\in\Pi\cap K}\mathbf{E}\left|\mathbf{E}\left(\frac{\xi(I)}{\mathbf{E}(\xi(I))}-X|I^c\xi\right)\right|\mathbb{I}_I\,d\mu\le$$

$$\le\int_K\sum_{I\in\Pi_n\cap K}\mathbf{E}\left(\mathbf{E}\left(\left|\frac{\xi(I)}{\mathbf{E}(\xi(I))}-X\right||I^c\xi\right)\right)\mathbb{I}_I\,d\mu=$$

$$=\int_K\sum_{I\in\Pi_n\cap K}\mathbf{E}\left|\frac{\xi(I)}{\mathbf{E}(\xi(I))}-X\right|\mathbb{I}_I\,d\mu=$$

$$=\int_K\mathbf{E}\left|\sum_{I\in\Pi_n\cap K}\frac{\xi(I)}{\mathbf{E}(\xi(I))}\mathbb{I}_I-X\right|\,d\mu=\int|X_n-X|\,d\mathbf{P}\otimes\mu_K.$$

We may now apply Scheffé's lemma ([1, p. 184]). In fact,

- $\mathbf{P}\otimes\mu_K$ is a finite measure;
- $\displaystyle\int X_n\, d\mathbf{P}\otimes\mu_K = \int X\, d\mathbf{P}\otimes\mu_K = \mathbf{E}(\xi(K)) < \infty$;
- $X_n \geq 0$, $n\in\mathbb{N}$, $X\geq 0$;
- $X_n \longrightarrow X$ $\mathbf{P}\otimes\mu_K$-almost everywhere.

Consequently

$$\lim_{n\to\infty}\int |Z_n - Y_n|\, d\mathbf{P}\otimes\mu_K = 0.$$

(b) For every $x\in K$, let $I_n(x)$ be the element of Π_n that contains x and $\sigma_n(x) = \sigma(\xi(A),\ A\in I_n^c(x))$. The sequence of σ-algebras $\{\sigma_n(x)\}$, for fixed x, is increasing and $\sigma(x) = \sigma\left(\bigcup_{n=1}^{\infty}\sigma_n(x)\right) = \sigma(\{x\}^c\xi) = \sigma(\xi(A),\ x\notin A)$. In fact, if $A\in\mathcal{B}$ and $x\notin A$, $\xi(A) = \lim_{n\to\infty}\xi\left(A\cap I_n^c(x)\right)$.

On the other hand, $\mathbf{E}(\xi(K)) = \displaystyle\int_K \mathbf{E}(X)\, d\mu < \infty$, so $\mathbf{E}(X)<\infty$ μ_K-almost everywhere. Put $K_0 = \{x\in K:\ \mathbf{E}(X(\cdot,x))<\infty\}$. If $x\in K_0$, we have

$$\mathbf{E}\left(X(\cdot,x)|I_n^c\xi\right) \longrightarrow \mathbf{E}\left(X(\cdot,x)|\{x\}^c\xi\right) \quad \text{a.s.}.$$

The set $D = \{(\omega,x):\ Y_n(\omega,x)$ does not converge$\}$ is $\mathcal{F}\otimes\mathcal{B}_K$-measurable, and, for every $x\in K$, the set $D_x = \{\omega:\ Y_n(\omega,x)$ does not converge$\}$ is $\mathcal{F}$-measurable. As $\mathbf{P}(D_x)=0$ for $x\in K_0$, and $\mu(K\setminus K_0)=0$, it follows $\mathbf{P}\otimes\mu_K(D)=0$. We may then suppose that $Y = \mathbf{E}(X|\{\cdot\}^c\xi)$ is $\mathcal{F}\otimes\mathcal{B}_K$-measurable and $Y_n\longrightarrow Y$ $\mathbf{P}\otimes\mu_K$-almost everywhere. Also $Y_n\geq 0$, $Y\geq 0$ and

$$\int_K \mathbf{E}(Y_n)\, d\mu = \int_K \sum_{I\in\Pi_n}\mathbf{E}\left(\mathbf{E}(X|I^c\xi)\right)\mathbb{I}_I\, d\mu = \int_K \mathbf{E}(X)\, d\mu = \mathbf{E}(\xi(K))$$

$$\int_K \mathbf{E}(Y)\, d\mu = \int_K \mathbf{E}(X)\, d\mu = \mathbf{E}(\xi(K)).$$

Applying Scheffé's lemma as above, it follows $\displaystyle\int |Y_n - Y|\, d\mathbf{P}\otimes\mu_K \longrightarrow 0$.

(c) From (a) and (b)

$$\mathbf{E}\left|\zeta_n(K) - \int_K \mathbf{E}(X|\{\cdot\}^c\xi)\, d\mu\right| = \mathbf{E}\left|\int_K Z_n - Y\, d\mu\right| \leq \int |Z_n - Y|\, d\mathbf{P}\otimes\mu_K \longrightarrow 0.$$

Taking account of the $\mathcal{F}\otimes\mathcal{B}_K$-measurability of $\mathbf{E}(X|\{\cdot\}^c\xi)$, we may define a random measure on K by

$$\forall_{B\in\mathcal{B}\cap K}\ \zeta(B) = \int_B \mathbf{E}(X|\{\cdot\}^c\xi)\, d\mu,$$

which is easily extended to the whole space $\mathcal{X}$.

Remark that if μ is diffuse, then X_n is $\sigma(\{x\}^c)$-measurable, from what follows that X is $\sigma(\{x\}^c)$-measurable, so the conditional intensity measure coincides with the original random measure ξ.

3. Almost sure convergence

(a) Using the same notations as in Section 2, put, for every $m \in \mathbb{N}$

$$U_m = \inf_{n \geq m} X_n \qquad V_m = \sup_{n \geq m} X_n$$

and suppose that $\|\xi\|_p(K) < \infty$. Let $q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then, from [4, 2.19, p. 24],

$$\mathbf{E}\left(\sup_n \int_K X_n^{1+\frac{1}{q}}\, d\mu\right) < \infty.$$

So, for almost every ω

$$\sup_n \int_K X_n^{1+\frac{1}{q}}(\omega)\, d\mu < \infty$$

which means that the martingale $\{X_n(\omega)\}$ is bounded in $\mathbf{L}^{1+\frac{1}{q}}(K, \mathcal{B}_K, \mu_K)$. Then, from [5, p. 55]

$$\| \sup_n X_n(\omega)\|_{1+\frac{1}{q}} \leq (q+1) \sup_n \|X_n(\omega)\|_{1+\frac{1}{q}},$$

from which follows

$$\mathbf{E}\left(\int_K \sup_n X_n^{1+\frac{1}{q}}\, d\mu\right) \leq (q+1)^{1+\frac{1}{q}} \mathbf{E}\left(\sup_n \int_K X_n^{1+\frac{1}{q}}\, d\mu\right) < \infty.$$

Then

$$\int \sup_n X_n\, d\mathbf{P} \otimes \mu_K < \infty, \tag{1}$$

from which we derive, using the dominated convergence theorem,

$$\int V_m - U_m\, d\mathbf{P} \otimes \mu_K \longrightarrow 0.$$

We remark that it is the condition (1) that is essential for the rest of the proof.

(b) The proof follows now the proof of a theorem by Hunt ([6, p. 66]): for $n \geq m$

$$\sum_{I \in \Pi_n \cap K} \mathbf{E}(U_m | I^c\xi)\mathbb{I}_I \leq \sum_{I \in \Pi_n \cap K} \mathbf{E}\left(\frac{\xi(I)}{\mathbf{E}(\xi(I))} | I^c\xi\right)\mathbb{I}_I \leq \sum_{I \in \Pi_n \cap K} \mathbf{E}(V_m | I^c\xi)\mathbb{I}_I.$$

Analogously to (b) of Section 2, noting that, according to (a), $U_m(x)$ and $V_m(x)$ are, for μ_K-almost all x, $\mathbf{P}$-integrable, it follows that $\mathbf{P} \otimes \mu_K$-almost everywhere

$$\sum_{I \in \Pi_n \cap K} \mathbf{E}(U_m | I^c\xi)\mathbb{I}_I \longrightarrow \mathbf{E}(U_m | \{\cdot\}^c\xi)$$

$$\sum_{I \in \Pi_n \cap K} \mathbf{E}(V_m | I^c\xi)\mathbb{I}_I \longrightarrow \mathbf{E}(V_m | \{\cdot\}^c\xi)$$

as $n \longrightarrow \infty$. So, $\mathbf{P} \otimes \mu_K$-almost everywhere

$$\mathbf{E}(U_m|\{\cdot\}^c\xi) \leq \liminf Z_n \leq \limsup Z_n \leq \mathbf{E}(V_m|\{\cdot\}^c\xi).$$

From Fatou's lemma

$$\int_K \mathbf{E}(U_m|\{\cdot\}^c\xi)\, d\mu \leq \liminf \zeta_n(K).$$

On the other hand, as V_m is $\mathbf{P} \otimes \mu_K$-integrable, $\mathbf{E}(V_m|\{\cdot\}^c\xi)$ is almost surely μ_K-integrable, so from Fatou-Lebesgue's theorem

$$\limsup \zeta_n(K) \leq \int_K \mathbf{E}(V_m|\{\cdot\}^c\xi)\, d\mu, \quad \mathbf{P}\text{-a.s.}.$$

Finally, $\mathbf{E}\left(\int_k \mathbf{E}\left(V_m - U_m|\{\cdot\}^c\xi\right)\ d\mu\right) = \int V_m - U_m\ d\mathbf{P} \otimes \mu_K$, for every $m \in \mathbb{N}$, so it follows

$$\liminf \zeta_n(K) = \limsup \zeta_n(K) \quad \mathbf{P}\text{-a.s.}.$$

4. Conclusion

(a) The simple procedure used is unfortunately specific to the case $\xi \ll \mathbf{E}\xi$ a.s. and the following counterexample shows that it is not applicable to point processes: put $\xi = \delta_u$ where u is a uniform random variable on $[0,1]$, then $X_n \longrightarrow 0$ $\mathbf{P} \otimes \mu$-almost everywhere, $\int X_n\ d\mathbf{P} \otimes \mu = 1$ for every $n \in \mathbb{N}$ and $\int \sup_n X_n\ d\mathbf{P} \otimes \mu = \infty$, so

$$\sum_{I \in \Pi_n} \mathbf{E}(\xi(I)|I^c\xi) = 1 \quad \mathbf{P}\text{-a.s.}.$$

(b) We remark that the result does not change if we take the σ-algebras $\sigma\left(\xi(J) : J \in \Pi_n,\ J \neq I\right)$ in place of $\sigma(I^c\xi)$. In fact, it would be interesting to know if this substitution is possible when ξ is a point process.

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