Ideals in selfdistributive groupoids

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Abstract. Products of (left) ideals in selfdistributive groupoids are studied.

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The purpose of this very short note is to complete some results from [1]. Other results on, comments about and aspects of left distributive groupoids (and further references as well) may be found in [2], [4] and [5].

1. Introduction

1.1. A groupoid is a non-empty set supplied with a binary operation.

Let G be a groupoid and let $\mathcal{P}(G)$ denote the set of all subsets of G. Then we define a binary operation on $\mathcal{P}(G)$ by $AB = \{ab; a \in A, b \in B\}$ for all $A, B \in \mathcal{P}(G)$. In this way, $\mathcal{P}(G)$ becomes a groupoid and we denote by $\mathcal{R}(G)$ the subgroupoid of $\mathcal{P}(G)$ generated by G. Clearly, $\mathcal{R}(G)$ is trivial iff $G = G^2$.

A non-empty subset I of G is said to be a left (right) ideal of G if $GI \subseteq I$ ($IG \subseteq I$). We denote by $\mathcal{I}_l(G)$ ($\mathcal{I}_r(G)$) the set of left (right) ideals of G.

A non-empty subset I of G is said to be an ideal if it is both a left and right ideal of G. We denote by $\mathcal{I}(G)$ the set of ideals of G.

1.2. Let G be a groupoid. We put $G^{\langle 1\rangle} = G$ and $G^{\langle n+1\rangle} = G \cdot G^{\langle n\rangle}$ for every $n \geq 1$. Further, $\mathcal{Q}(G) = \{G^{\langle n\rangle}; n \geq 1\} \subseteq \mathcal{R}(G)$.

Similarly, let $G^{\langle n,0\rangle} = G^{\langle n\rangle}$ and $G^{\langle n,m+1\rangle} = G^{\langle n,m\rangle} \cdot G$ for every $n \geq 1$ and every $m \geq 0$.

1.3. A groupoid G is said to be

- left distributive if $a \cdot bc = ab \cdot ac$ for all $a, b, c \in G$;
- right distributive if $bc \cdot a = ba \cdot ca$ for all $a, b, c \in G$;
- distributive if it is both left and right distributive;
- medial if $ab \cdot cd = ac \cdot bd$ for all $a, b, c, d \in G$.

2. Examples

2.1 Example. Let D_0 designate the set of ordered pairs (n, m), where n, m are integers, $n \geq 1$, $n \neq 2$ and $m \geq 0$. Now define a multiplication on D_0 as follows: $(n, m)(k, l) = (3, 0)$ if $l \geq 1$; $(n, m)(k, 0) = (k + 1, 0)$ if $k \geq 3$; $(n, m)(1, 0) = (n, m + 1)$. Then D_0 becomes a groupoid and it is easy to check that D_0 is a left distributive groupoid. Moreover, D_0 is medial, D_0 does not contain any idempotent element and $uv \cdot z \neq uz \cdot vz$ for all $u, v, z \in D_0$; in particular, D_0 is not right distributive. Further, notice that D_0 is generated by the element $(1, 0)$. Finally, define a relation $\leq_0$ on D_0 by $(n, m) \leq_0 (k, l)$ iff at least one of the following cases takes place: $k \leq n$, $m = l$; $3 \leq n$, $0 \leq m < l$; $3 \leq n$, $k = 1$; $k = 1$, $0 \leq l < m$. Then $\leq_0$ is a linear ordering of D_0 and this ordering is stable with respect to the operation of the groupoid D_0.

2.2 Example. Consider the following three-element groupoid G:

G	0	1	2
0	1	2	2
1	1	2	2
2	1	2	2

Then G is left distributive, $\mathcal{R}(G) = \mathcal{I}_l(G) = \{G^{\langle 1 \rangle}, G^{\langle 2 \rangle}, G^{\langle 3 \rangle}\}$ and $G^{\langle 3 \rangle}$ is not a right ideal.

2.3 Example. Consider the following four-element groupoid G:

G	0	1	2	3
0	0	0	0	0
1	0	0	3	0
2	0	0	1	0
3	0	0	3	0

Then G is left distributive, $\mathcal{R}(G) = \{G^{\langle 1,0 \rangle}, G^{\langle 1,1 \rangle}, G^{\langle 1,2 \rangle}, G^{\langle 3,0 \rangle}\} = \mathcal{I}(G) = \mathcal{I}_r(G) \neq \mathcal{I}_l(G) = \mathcal{R}(G) \cup \{A\}$, where $A = \{0, 1\}$ is a left ideal but not a right ideal; $\mathcal{I}_l(G)$ is not linearly ordered by inclusion.

2.4 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

Then G is distributive, $\mathcal{R}(G) = \{G^{\langle 1 \rangle}, G^{\langle 2 \rangle}\} \neq \mathcal{I}(G)$ and $\mathcal{I}(G)$ is not linearly ordered by inclusion.

2.5 Example. Consider the following three-element groupoid G:

G	0	1	2
0	1	2	0
1	1	2	0
2	1	2	0

Then G is left distributive and G is both left and right-ideal-free. Moreover, G is a left quasigroup but it is not a right quasigroup.

2.6 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	1	1	1
2	1	2	2

Then G is distributive and left-ideal-free. Moreover, G is neither a left nor a right quasigroup.

2.7 Remark. By [3, 5.10], every finite left and right-ideal-free distributive groupoid is a quasigroup.

3. First observations on ideals of left distributive groupoids.

3.1 Lemma. *Let I, J, K be left ideals of a left distributive groupoid G. Then:*

(i) *IJ is a left ideal and $IJ \subseteq J$.*
(ii) *$I \cdot JK = IJ \cdot IK$.*
(iii) *$I(J \cup K) = IJ \cup IK$ and $(J \cup K)I = JI \cup KI$.*
(iv) *If $J \subseteq K$, then $IJ \subseteq IK$ and $JI \subseteq KI$.*

3.2 Lemma. *Let G be a left distributive groupoid such that $G = G^2$.*

(i) *If I is a right ideal and J is an ideal of G, then IJ is a right ideal and $IJ \subseteq I \cap J$.*
(ii) *If I, J are ideals of G, then IJ is an ideal and $IJ \subseteq I \cap J$.*

3.3 Proposition. *Let G be a left distributive groupoid. Then:*

(i) *The set $\mathcal{I}_l(G)$ of left ideals of G is a subgroupoid of $\mathcal{P}(G)$ and $\mathcal{I}_l(G)$ is again a left distributive groupoid.*
(ii) *$\mathcal{R}(G)$ is a subgroupoid of $\mathcal{I}_l(G)$.*
(iii) *If $G = G^2$, then $\mathcal{I}(G)$ is a subgroupoid of $\mathcal{I}_l(G)$ and $\mathcal{I}(G)$ is a medial groupoid.*
(iv) *If G is idempotent, then $\mathcal{I}_l(G)$ is idempotent and $\mathcal{I}(G)$ is a semilattice.*

4. The groupoid $\mathcal{R}(G)$.

4.1 Lemma. *Let G be a left distributive groupoid and $A \in \mathcal{R}(G)$. Then:*

(i) $GA \subseteq A$.

(ii) *If $A \neq G$, then $G^{\langle n\rangle} \cdot A = GA$ for every $n \geq 1$.*

(iii) *There exists $m \geq 1$ such that $G^{\langle m\rangle} \subseteq A$.*

PROOF: (i) A is a left ideal by 3.3 (ii).

(ii) Let F be an absolutely free groupoid over a one-element set $\{x\}$ and let $f : F \to \mathcal{R}(G)$ be the uniquely determined homomorphism such that $f(x) = G$. Since $A \neq G$, we have $G \neq G^2$ and $A = f(r)$ for some $r \in F$, $l(r) \geq 2$; here, $l(r)$ means the length of r. Now, we shall proceed by induction on $l(r) + n$.

First, let $l(r) = 2$. Then $A = G^2$ and $G^{\langle 3\rangle} = G^{\langle n\rangle} \cdot G^2 = (G^{\langle n\rangle}G)(G^{\langle n\rangle}G) = ((G^{\langle n\rangle}G)G^{\langle n\rangle})((G^{\langle n\rangle}G)G) \subseteq G^{\langle n+1\rangle} \cdot G^2$. The inclusion $G^{\langle n+1\rangle} \cdot G^2 \subseteq G^{\langle 3\rangle}$ is evident, and hence $G^{\langle n+1\rangle} \cdot G^2 = G^{\langle 3\rangle}$.

Next, let $r = sx$, $l(s) \geq 2$, $B = f(s)$. Then $GA = G^{\langle n\rangle} \cdot BG = (G^{\langle n\rangle}B)(G^{\langle n\rangle}G) = ((G^{\langle n\rangle}B)G^{\langle n\rangle})((G^{\langle n\rangle}B)G) \subseteq G^{\langle n+1\rangle} \cdot BG = G^{\langle n+1\rangle} \cdot A$, and so $GA = G^{\langle n+1\rangle} \cdot A$. Similarly, if $r = xs$.

Finally, let $r = st$, $l(s) \geq 2$, $l(t) \geq 2$, $B = f(s)$, $C = f(t)$. Then $G^{\langle n\rangle} \cdot A = (G^{\langle n\rangle}B)(G^{\langle n\rangle}C) = GB \cdot GC = G \cdot BC = GA$.

(iii) We can assume that $A = BC$ and that $G^{\langle n\rangle} \subseteq B \cap C$ for some $n \geq 2$. Then $G^{\langle n\rangle} \cdot G^{\langle n\rangle} \subseteq A$. However, by (ii), $G^{\langle n\rangle} \cdot G^{\langle n\rangle} = G^{\langle n+1\rangle}$. □

4.2 Lemma. *Let G be a left distributive groupoid. Then $G^{\langle n,m\rangle} \cdot G^{\langle k\rangle} = G^{\langle k+1\rangle}$ for all $n \geq 1$, $m \geq 0$ and $k \geq 2$.*

PROOF: We can assume that $G \neq G^2$. Now, for $m = 0$, our equality follows from 4.1 (ii).

Let $k = 2$. We shall proceed by induction on m. We have $G^{\langle 3\rangle} = G^{\langle n,m\rangle} \cdot G^2 = (G^{\langle n,m\rangle}G)(G^{\langle n,m\rangle}G) \subseteq G^{\langle n,m+1\rangle} \cdot G^2 \subseteq G^{\langle 3\rangle}$, and so $G^{\langle 3\rangle} = G^{\langle n,m+1\rangle} \cdot G^2$.

Let $k \geq 3$. Again, we shall proceed by induction on m. We have $G^{\langle k+1\rangle} = G^{\langle n,m\rangle} \cdot G^{\langle k\rangle} = G^{\langle n,m\rangle} \cdot (G \cdot G^{\langle k-1\rangle}) = (G^{\langle n,m\rangle}G)(G^{\langle n,m\rangle}G^{\langle k-1\rangle}) = G^{\langle n,m+1\rangle} \cdot G^{\langle k\rangle}$. □

4.3 Lemma. *Let G be a left distributive groupoid. Then $G \cdot G^{\langle n,m\rangle} = G^{\langle 3\rangle}$ for all $n \geq 1$, $m \geq 1$.*

PROOF: Assuming $G \neq G^2$, we shall proceed by induction on m. Now, $G \cdot G^{\langle n,m\rangle} = (G \cdot G^{\langle n,m-1\rangle}) \cdot G^2$. If $m \geq 2$, then $G \cdot G^{\langle n,m-1\rangle} = G^{\langle 3\rangle}$ by induction and $G^{\langle 3\rangle} \cdot G^2 = G^{\langle 3\rangle}$ by 4.2. If $m = 1$, then $G \cdot G^{\langle n,m-1\rangle} = G^{\langle n+1\rangle}$ and our result follows from 4.2 again. □

4.4 Lemma. *Let G be a left distributive groupoid. Then $G^{\langle n,m\rangle} \cdot G^{\langle k,l\rangle} = G^{\langle 3\rangle}$ for all $n \geq 1$, $m \geq 0$, $k \geq 1$, $l \geq 1$.*

PROOF: Using 4.1, 4.2 and 4.3, the result follows easily by induction on l. □

4.5 Proposition ([1]). *Let G be a left distributive groupoid. Then:*

(i) $G^{\langle n,m\rangle} \cdot G^{\langle k,l\rangle} = G^{\langle 3\rangle}$ *for all* $n \geq 1$, $m \geq 0$, $k \geq 1$, $l \geq 1$.
(ii) $G^{\langle n,m\rangle} \cdot G^{\langle k,0\rangle} = G^{\langle k+1,0\rangle}$ *for all* $n \geq 1$, $m \geq 0$, $k \geq 2$.
(iii) $G^{\langle n,m\rangle} \cdot G^{\langle 1,0\rangle} = G^{\langle n,m+1\rangle}$ *for all* $n \geq 1$, $m \geq 0$.

Proof: See the preceding lemmas. □

4.6 Corollary. *Let G be a left distributive groupoid. Then:*

(i) $\mathcal{R}(G) = \{G^{\langle n,m\rangle}; n \geq 1, m \geq 0\}$.
(ii) *If* $G \neq G^2$, *then* $\mathcal{Q}(G) - \{G\} = \{G^{\langle k\rangle}; k \geq 2\}$ *is a left ideal of* $\mathcal{R}(G)$.

4.7 Theorem. *Let G be a left distributive groupoid. Define a mapping $f : D_0 \to \mathcal{R}(G)$ by $f(n,m) = G^{\langle n,m\rangle}$. Then*

(i) *f is a projective homomorphism of the left distributive groupoids.*
(ii) *If $(n,m), (k,l) \in D_0$ and $(n,m) \leq_0 (k,l)$, then $G^{\langle n,m\rangle} \subseteq G^{\langle k,l\rangle}$.*

Proof: (i) See 4.5 and 2.1.

(ii) First, let $k \geq n$, $m = 1$. We have $G^{\langle n\rangle} = (G \dots (G \cdot G^{\langle k\rangle}))$, where G appears $(n-k)$-times, and hence $G^{\langle n\rangle} \subseteq G^{\langle k\rangle}$, since $G^{\langle k\rangle}$ is a left ideal. This also implies that $G^{\langle n,m\rangle} \subseteq G^{\langle k,l\rangle}$.

Next, let $3 \leq n$ and $0 \leq m < l$. If $m = 0$, then $G^{\langle n,0\rangle} \subseteq G^{\langle 3\rangle} = G \cdot G^{\langle k,l\rangle} \subseteq G^{\langle k,l\rangle}$. If $m \geq 1$, then $G^{\langle n,0\rangle} \subseteq G^{\langle k,l-m\rangle}$, and therefore $G^{\langle n,m\rangle} = ((G^{\langle n,0\rangle} \cdot G) \dots)G \subseteq ((G^{\langle k,l-m\rangle} \cdot G) \dots)G = G^{\langle k,l\rangle}$.

Finally, let $3 \leq n$ and $k = 1$. With respect to the preceding case, we can assume that $l \leq m$. Now, $G^{\langle n,m\rangle} = ((G^{\langle n,m-l\rangle} \cdot G) \dots)G \subseteq ((GG) \dots)G = G^{\langle 1,l\rangle}$. Similarly, if $k = 1$ and $0 \leq l < m$. □

4.8 Corollary. *Let G be a left distributive groupoid. Then $\mathcal{R}(G)$ is a medial left distributive groupoid which is linearly ordered by inclusion; this ordering is stable.*

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