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# Smooth points of the unit sphere in Musielak-Orlicz function spaces equipped with the Luxemburg norm

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*Abstract.* There is given a criterion for an arbitrary element from the unit sphere of Musielak-Orlicz function space equipped with the Luxemburg norm to be a point of smoothness. Next, as a corollary, a criterion of smoothness of these spaces is given.

*Keywords:* Musielak-Orlicz function, Musielak-Orlicz space, support functional, smooth point, smooth space

*Classification:* Primary 46E30; Secondary 45B20

## Introduction

In the following, $(T,\Sigma,\mu)$ denotes a non-atomic $\sigma$-finite measure space, $\mathbb{R}$ denotes the set of reals, $\mathbb{R}_+$ denotes the set of nonnegative reals, $\mathbb{N}$ denotes the set of natural numbers, $\chi_A$ stands for the characteristic function of a set $A \in \Sigma$, $X$ denotes a Banach space and $X^*$ denotes its dual space. Their unit balls and spheres are denoted by $B(X)$, $B(X^*)$ and $S(X)$, $S(X^*)$, respectively.

A map $\Phi : T \times \mathbb{R} \to [0,+\infty]$ is said to be a Musielak-Orlicz function if for $\mu$-a.e. $t \in T$, $\Phi(t,\cdot)$ is vanishing and continuous at zero, left-hand side continuous on the whole $\mathbb{R}_+$, not identically equal to zero, convex and even and if for any $u \in \mathbb{R}$, $\Phi(\cdot,u)$ is a $\Sigma$-measurable function.

For a given Musielak-Orlicz function $\Phi$ we define

$$a(t,\Phi) = \sup\{u > 0 : \Phi(t,u) < +\infty\}$$

for any $t \in T$.

We denote by $\Phi'_-$ and $\Phi'_+$ the left-hand side and the right-hand side derivatives of $\Phi$ with respect to the second variable, respectively. For any $u \in \mathbb{R}$ we define

$$\partial\Phi(t,u) = \begin{cases} [\Phi'_+(t,u), \Phi'_-(t,u)] & \text{if } -a(t,\Phi) < u < a(t,\Phi) \\ [\Phi'_-(t,u), +\infty) & \text{if } u = a(t,\Phi) \text{ and } \Phi'_-(t,a(t,\Phi)) < +\infty \\ (-\infty, \Phi'_+(t,u)] & \text{if } u = -a(t,\Phi) \text{ and } \Phi'_+(t,-a(t,\Phi)) > -\infty \\ \{+\infty\} & \text{if } u \ge a(t,\Phi) \text{ and } \Phi'_-(t,a(t,\Phi)) = +\infty \\ \{-\infty\} & \text{if } u \le -a(t,\Phi) \text{ and } \Phi'_+(t,-a(t,\Phi)) = -\infty \end{cases}$$

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for any $t \in T$ (cf. [GH]).

We have for any $u \in \mathbb{R}$:

$$\partial\Phi(t,u) = \{v \in \mathbb{R} : \Phi(t,u) + \Phi^*(t,v) = uv\}\,,$$

where $\Phi^*$ is the Musielak-Orlicz function complementary to $\Phi$ in the sense of Young, i.e.

$$\Phi^*(t,u) = \sup_{v>0}\{|u|v - \Phi(t,v)\}$$

for any $t \in T$ and $u \in \mathbb{R}$, under the convention $\Phi^*(t,\pm\infty) = +\infty$.

Denote by $L^0(\mu)$ the space of all ($\mu$-equivalence classes of) $\Sigma$-measurable real functions defined on $T$. Given a Musielak-Orlicz function $\Phi$ we can define on $L^0(\mu)$ a convex functional $I_\Phi$ by the formula

$$I_\Phi(x) = \int_T \Phi(t,x(t))\,d\mu\,.$$

The Musielak-Orlicz space generated by a Musielak-Orlicz function $\Phi$ is defined to be the set of all $x \in L^0(\mu)$ for which $I_\Phi(\lambda x) < +\infty$ for some $\lambda > 0$ depending on $x$ and it is denoted by $L^\Phi(\mu)$. This space endowed with the Luxemburg norm $\|\ \|_\Phi$ defined by

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\}$$

is a Banach space (cf. [M]). In the case when $\Phi(t_1,\cdot) = \Phi(t_2,\cdot)$ for $\mu$-a.e. $t_1, t_2 \in T$, $\Phi$ is a usual Orlicz function and $L^\Phi(\mu)$ is called an Orlicz space.

We can define in $L^\Phi(\mu)$ another norm $\|\ \|_\Phi^0$, called the Orlicz norm, by the formula

$$\|x\|_\Phi^0 = \sup\{|\int_T x(t)y(t)\,d\mu| : I_{\Phi^*}(y) \le 1\}\,,$$

where $\Phi^*$ is the Musielak-Orlicz function complementary to $\Phi$ in the sense of Young (cf. [M] and in the case of Orlicz spaces also [KR], [L] and [RR]). The Amemiya formula for the Orlicz norm is the following:

$$\|x\|_\Phi^0 = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

(cf. [KR] and [RR]).

We say that a Musielak-Orlicz function $\Phi$ satisfies the $\Delta_2$-condition if there exist a constant $K \ge 2$, a set $T_0$ of measure zero and a $\Sigma$-measurable function $h : T \to \mathbb{R}_+$ such that $\int_T h(t)\,d\mu < +\infty$ and the inequality

$$\Phi(t,2u) \le K\Phi(t,u) + h(t)$$

holds for any $u \in \mathbb{R}$ and $t \in T \setminus T_0$ (cf.[K] and [M]).

Recall that a functional $x^* \in X^*$ is said to be a support functional at $x \in X$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. The set of all support functionals at $x$ is denoted by $\mathrm{Grad}(x)$. A point $x \in X$ is said to be smooth if $\mathrm{Card}(\mathrm{Grad}(x)) = 1$ (cf. [D] and [P]).

It is known (cf. [HY], [K] and in the case of Orlicz spaces also [A]) that for any finite-valued Musielak-Orlicz function $\Phi$, we have

$$(L^{\Phi}(\mu))^* = L^{\Phi^*}(\mu) \oplus S\,, \tag{1}$$

where $S$ is the space of all singular functionals over $L^{\Phi}(\mu)$, i.e. functionals which vanish on the subspace $E^{\Phi}(\mu)$ of $L^{\Phi}(\mu)$ defined by

$$E^{\Phi}(\mu) = \{x \in L^0(\mu) : I_{\Phi}(\lambda x) < +\infty \text{ for any } \lambda > 0\}\,.$$

Equality (1) means that every $x^* \in (L^{\Phi}(\mu))^*$ is uniquely represented in the form

$$x^* = T_v + \varphi\,, \tag{2}$$

where $\varphi \in S$ and $T_v$ is the functional generated by an element $v \in L^{\Phi^*}(\mu)$ by the following formula

$$T_v(y) = \int_T v(t)y(t)\,d\mu \quad (\forall y \in L^{\Phi}(\mu))\,. \tag{3}$$

Every functional $T_v$ of the form (3) is said to be a regular functional. It is well known that if $L^{\Phi}(\mu)$ is endowed with the Luxemburg norm, then for every $x^* \in (L^{\Phi}(\mu))^*$ we have

$$\|x^*\| = \|T_v\| + \|\varphi\|\,, \tag{4}$$

where $T_v$ and $\varphi$ are the regular and singular parts of $x^*$, respectively (cf. [K] and [A], [N]).

The set of all regular (singular) functionals from $\mathrm{Grad}(x)$ will be denoted by $\mathrm{RGrad}(x)$ (resp. $\mathrm{SGrad}(x)$).

It is worth to recall at this place that smoothness of Musielak-Orlicz sequence spaces was considered in [HY] and [PY]. Moreover, smoothness of Orlicz function spaces equipped with the Orlicz norm was characterized in [C].

## Results

We start with some auxiliary lemmas.

**Lemma 1.** *Let $\Phi$ be a Musielak-Orlicz function such that $\Phi(t,u)/u \to +\infty$ as $u \to +\infty$ for $\mu$-a.e. $t \in T$. Then there exists a constant $l > 0$ such that*

$$\|x\|_{\Phi}^0 = \frac{1}{l}(1 + I_{\Phi}(lx))\,. \tag{5}$$

Proof: It can be proceeded in an analogous way as the proof of Lemma 1 in [GH]. $\square$

**Lemma 2** (A.Kamińska [Ka]). *Let $\Phi$ be a Musielak-Orlicz function. Then there exists an increasing sequence $(T_i)$ such that $\mu(T_i) < +\infty$, $\mu(T \setminus \bigcup_{i=1}^{\infty} T_i) = 0$ and $\sup_{t \in T_i} \Phi(t,u) < +\infty$ for every $u \in \mathbb{R}_+$, $i \in \mathbb{N}$.*

**Lemma 3.** *Assume $\Phi$ is a finite-valued Musielak-Orlicz function, $x \in S(L^{\Phi}(\mu))$ and $I_{\Phi}(\lambda x) < +\infty$ for some $\lambda > 1$. Then every $x^* \in \mathrm{Grad}(x)$ must be regular.*

Proof: In virtue of Lemma 2, we can replay the proof of Lemma 2 in [GH]. □

**Lemma 4.** *Assume $\Phi$ is a finite-valued Musielak-Orlicz function and $I_{\Phi}(\lambda x/\|x\|_{\Phi}) < +\infty$ for some $\lambda > 1$. Then*

1° $\mathrm{RGrad}(x) \neq \emptyset$,

2° $x^* \in \mathrm{RGrad}(x)$ *if, and only if it is of the form*

$$x^*(y) = T_w(y) = \int_T w(t)y(t)\, d\mu \quad (\forall y \in L^{\Phi}(\mu))\,, \tag{8}$$

*where*

$$w(t) = z(t) \Big/ \int_T z(t)(x(t)/\|x\|_{\Phi})\, d\mu \tag{9}$$

*and*

$$z \text{ is a } \Sigma\text{-measurable function such that } z(t) \in \partial\Phi(t, x(t)/\|x\|_{\Phi}) \tag{10}$$

*for $\mu$-a.e. $t \in T$.*

Proof: We can repeat here the proof of Lemma 3 from [GH]. □

**Lemma 5.** *Let $\Phi$ be a finite-valued Musielak-Orlicz function, $x \in S(L^{\Phi}(\mu))$ and $I_{\Phi}(\lambda x) = +\infty$ for every $\lambda > 1$. Then there are sets $A, B \in \Sigma$ of positive measure such that $\mu(A \cap B) = 0$, $A \cup B = \operatorname{supp} x$ and*

$$\|x\chi_A\|_{\Phi} = \|x\chi_B\|_{\Phi} = 1\,.$$

Proof: Although Lemma 5 is an analogue of Lemma 6 from [GH], we cannot repeat its proof which was suitable only for Orlicz spaces which are rearrangement invariant spaces. We will present here a completely new proof.

Let $(T_n)_{n=1}^{\infty}$ be the sequence of sets from Lemma 2. Then we have

$$I_{\Phi}(u\chi_{T_n}) < +\infty \quad (\forall u > 0, n \in \mathbb{N})\,.$$

Let $\lambda_1 > \lambda_2 > \ldots$ and $\lambda_n \to 1$ as $n \to +\infty$. Since $I_{\Phi}(\lambda_1 x) = +\infty$, we can find $n_1 \in \mathbb{N}$ such that the set

$$A_1 = \{t \in T_{n_1} : |x(t)| \le n_1\}$$

satisfies the inequality $I_\Phi(\lambda_1 x\chi_{A_1}) \geq 2$.

We have $I_\Phi(\lambda_2 x\chi_{T\setminus A_1}) = +\infty$ because of $I_\Phi(\lambda_2 x\chi_{A_1}) < +\infty$, and we can find $n_2 \in \mathbb{N}$ such that defining

$$A_2 = \{t \in (T \setminus A_1) \cap T_{n_2} : |x(t)| \leq n_2\}$$

we get $A_1 \cap A_2 = \emptyset$ and $I_\Phi(\lambda_2 x\chi_{A_2}) \geq 2$.

We have again $I_\Phi(\lambda_3 x\chi_{T\setminus(A_1\cup A_2)}) = +\infty$. Repeating this procedure by induction we can find a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint sets such that

$$I_\Phi(\lambda_n x\chi_{A_n}) \geq 2 \quad (n = 1, 2, \dots)\,.$$

We can now decompose every set $A_n$ into the sum

$$A_n = A_n' \cup A_n''$$

of disjoint and measurable sets such that

$$I_\Phi(\lambda_n x\chi_{A_n'}) = I_\Phi(\lambda_n x\chi_{A_n''}) = \frac{1}{2} I_\Phi(\lambda_n x\chi_{A_n}) \geq 1\,.$$

Define now disjoint sets

$$A = \bigcup_{n=1}^\infty A_n'\,, \quad B = \bigcup_{n=1}^\infty A_n''$$

and the functions

$$y = x\chi_A + \frac{1}{2} x\chi_{T\setminus(A\cup B)},$$
$$z = x\chi_B + \frac{1}{2} x\chi_{T\setminus(A\cup B)}\,.$$

Obviously, we have $x = y + z$ and we need to prove that $\|y\|_\Phi = \|z\|_\Phi = 1$. It is evident that $|y(t)| \leq |x(t)|$ $\mu$-a.e., therefore $I_\Phi(y) \leq I_\Phi(x) \leq 1$. Let us take an arbitrary $\lambda > 1$. We can find $m \in \mathbb{N}$, such that $\lambda \geq \lambda_m$. Hence

$$I_\Phi(\lambda y) \geq I_\Phi(\lambda_m y) \geq I_\Phi(\lambda_m x\chi_{A_m'}) \geq 1\,,$$

which yields together with $I_\Phi(y) \leq 1$ that $\|y\|_\Phi = 1$. In the same way we obtain that $\|z\|_\Phi = 1$. □

Now, we are ready to prove the main results of this paper.

**Theorem 6.** *Let $\Phi$ be a finite-valued Musielak-Orlicz function. A point $x \in S(L^\Phi(\mu))$ is smooth if and only if:*

(i) $I_\Phi(\lambda x) < +\infty$ *for some* $\lambda > 1$,
(ii) $\Phi$ *is smooth at* $x(t)$ *for* $\mu$*-a.e.* $t \in T$.

Proof: It follows by Lemmas 3, 4 and 5 in the same way as Theorem 8 in [GH]. □

**Theorem 7.** *Let $\Phi$ be a finite-valued Musielak-Orlicz function. $L^\Phi(\mu)$ is smooth if and only if*

(i) *$\Phi$ is smooth,*
(ii) *$\Phi$ satisfies the $\Delta_2$-condition.*

Proof: *Sufficiency.* Note that, in virtue of the $\Delta_2$-condition, $a(t,\Phi) = +\infty$ for $\mu$-a.e. $t \in T$ and $E^\Phi(\mu) = L^\Phi(\mu)$. Therefore, $\mathrm{Grad}(x) = \mathrm{RGrad}(x)$ for every $x \in S(L^\Phi(\mu))$. Thus, condition (i) implies that $\mathrm{Card}(\mathrm{Grad}(x)) = 1$ for every $x \in S(L^\Phi(\mu))$ (cf. Lemma 4), which means that $L^\Phi(\mu)$ is smooth.

*Necessity.* Assume that $\Phi$ does not satisfy condition (ii). Then there is $x \in S(L^\Phi(\mu))$ such that $I_\Phi(\lambda x) = +\infty$ for any $\lambda > 1$ (cf. [H]). Therefore, in view of Lemma 5, there exist $A, B \in \Sigma$ such that $\mu(A \cap B) = 0$, $A \cup B = \operatorname{supp} x$ and $\|x\chi_A\|_\Phi = \|x\chi_B\|_\Phi = 1$. Therefore, as it was shown on the occasion of the proof of Theorem 6, $x$ is not smooth.

Assume now that $\Phi$ satisfies condition (ii) and does not satisfy condition (i), i.e. $\Phi$ is not smooth. Thus, there exists a set $K \in \Sigma$, $\mu(K) > 0$, such that $\Phi(t,\cdot)$ have in $\mathbb{R}_+$ at least one point of nonsmoothness for any $t \in K$. Define a multifunction $\Gamma$ by

$$\Gamma(t) = \{u \in \mathbb{R}_+ : \Phi'_-(t,u) < \Phi'_+(t,u)\} \quad (\forall t \in K)\,.$$

The Carathéodory conditions for $\Phi$ imply the $\Sigma \times \mathcal{B}$-measurability of $\Phi$ and this implies the $\Sigma \times \mathcal{B}$-measurability of $\Phi'_-$ and $\Phi'_+$, where $\mathcal{B}$ denotes the $\Sigma$-algebra of Borel sets. Now, we can apply Theorem 5.2 from [Hi], because $\mathrm{Grad}\,\Gamma = \{(t,u) \in T \times \mathbb{R}_+ : u \in \Gamma(t)\} = \{(t,u) \in T \times \mathbb{R}_+ : \Phi'_-(t,u) < \Phi'_+(t,u)\} \in \Sigma \times \mathcal{B}$. We get a measurable function (selector) $a : K \to \mathbb{R}_+$, such that $a(t) \in \Gamma(t)$ for $\mu$-a.e. $t \in K$. Take $K_1 \in \Sigma$, $K_1 \subset K$ such that $I_\Phi(a\chi_{K_1}) \le 1$. Choose a function $b : T \setminus K_1 \to \mathbb{R}_+$ in such a manner that $I_\Phi(x) = 1$, whenever $x = a + b$. Now, we can take two measurable functions $c, d : K_1 \to \mathbb{R}_+$, $c(t), d(t) \in \partial\Phi(t, a(t))$, $c(t) \ne d(t)$ for $\mu$-a.e. $t \in K_1$ (we can put for example $c(t) = \Phi'_-(t,a(t))$, $d(t) = \Phi'_+(t,a(t))$, because the $\Sigma \times \mathcal{B}$-measurability of $\Phi'_-(t,u)$ and $\Phi'_+(t,u)$ implies the $\Sigma$-measurability of these functions). Take a $\Sigma$-measurable function such that $e : T \setminus K_1 \to \mathbb{R}_+$, $e(t) \in \partial\Phi(t, b(t))$ for $\mu$-a.e. $t \in T \setminus K_1$. Define two functionals:

$$x^*(y) = \frac{\int_T (c(t)\chi_{K_1}(t) + e(t)\chi_{T\setminus K_1}(t))y(t)\,d\mu}{\int_T (c(t)\chi_{K_1}(t) + e(t)\chi_{T\setminus K_1}(t))x(t)\,d\mu} \quad (\forall y \in L^\Phi(\mu))\,,$$

$$x_1^*(y) = \frac{\int_T (d(t)\chi_{K_1}(t) + e(t)\chi_{T\setminus K_1}(t))y(t)\,d\mu}{\int_T (d(t)\chi_{K_1}(t) + e(t)\chi_{T\setminus K_1}(t))x(t)\,d\mu} \quad (\forall y \in L^\Phi(\mu))\,,$$

belonging, in view of Lemma 4, to $\mathrm{RGrad}(x)$. Since $x^* \ne x_1^*$, $x$ is not smooth. Therefore, $L^\Phi(\mu)$ is not smooth, too. The theorem is proved. □

**Corollary 8.** *Let $\Phi$ be a finite-valued Musielak-Orlicz function. The space $E^\Phi(\mu)$ is smooth if and only if $\Phi$ is smooth.*

Proof: The sufficiency follows by the sufficiency part of the proof of Theorem 7 and the fact that the dual of $E^\Phi(\mu)$ consists of only regular functionals.

To prove necessity we need to find a point $x \in S(E^{\Phi}(\mu))$ which is not smooth. Define

$$G_{i,n} = \{t \in T_i \cap K : \Phi(t, a(t)) \le n\} \quad i, n = 1, 2, \dots ,$$

where $T_i$ are the sets from Lemma 2.

There exist $i_0, n_0 \in \mathbb{N}$ and a measurable subset $G$ of $G_{i_0,n_0}$ such that

$$0 < I_\Phi(a\chi_G) = \int_G \Phi(t, a(t))\, d\mu < 1 .$$

Denote $I_\Phi(a\chi_G) = \varkappa$ and define

$$H_i = T_i \cap (T \setminus K) \quad i = 1, 2, \dots .$$

There exists $i_1 \in \mathbb{N}$ such that $\mu(H_{i_1}) > 0$. By Lemma 2 it follows that $b\chi_{H_{i_1}} \in E^{\Phi}(\mu)$ for every $b > 0$. The function $f(b) = I_\Phi(b\chi_{H_{i_1}})$ is convex and finite-valued, so it is continuous on $\mathbb{R}_+$. Moreover, $f(b) \to +\infty$ as $b \to +\infty$, whence it follows that Image$(f) = \mathbb{R}_+$. Thus, we can find $b_1 > 0$ satisfying $I_\Phi(b_1\chi_{H_{i_1}}) = 1 - \varkappa$. Defining $x$ by the formula

$$x(t) = a(t)\chi_G(t) + b_1\chi_{H_{i_1}}(t)$$

we have $I_\Phi(x) = 1$. The fact that $x$ is not smooth can be proved in the same way as in Theorem 7. □

## References


[A] Ando T., *Linear functionals on Orlicz spaces*, Nieuw Arch. Wisk. **3,8** (1960), 1–16.

[C] Chen S., *Smoothness of Orlicz spaces*, Comment. Math. (Prace Matem.) **27,1** (1987), 49–58.

[D] Diestel J., *Geometry of Banach spaces – selected topics*, Lecture Notes in Math. 481, Springer-Verlag, 1975.

[GH] Grzaslewicz R., Hudzik H., *Smooth points of Orlicz spaces equipped with the Luxemburg norm*, Math. Nachrichten **155** (1992), 31–45.

[Hi] Himmelberg C.J., *Measurable relations*, Fund. Math. **87,1** (1975), 53–72.

[H] Hudzik H., *On some equivalent conditions in Musielak-Orlicz spaces*, Comment. Math. (Prace Matem.) **24,1** (1984), 57–64.

[HY] Hudzik H., Ye Y., *Support functionals and smoothness in Musielak-Orlicz sequence spaces endowed with the Luxemburg norm*, Comment. Math. Univ. Carolinae **31,4** (1990), 661–684.

[Ka] Kamińska A., *Some convexity properties of Musielak-Orlicz spaces of Bochner type*, Supplemento ai Rendiconti del Circolo Mathematico di Palermo, Serie II **10** (1985), 63–73.

[K] Kozek A., *Convex integral functionals on Orlicz spaces*, Comment. Math. (Prace Matem.) **21,1** (1980), 109–135.

[KR] Krasnoselskiĭ M.A., Rutickiĭ Ya.B., *Convex functions and Orlicz spaces*, Groningen 1961 (translation).

[L] Luxemburg W.A.J., *Banach function spaces*, Thesis, Delft 1955.

[M] Musielak J., *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1034, Springer-Verlag 1983.

[N] Nowak M., *Linear functionals on Orlicz sequence spaces without local convexity*, Internat. J. of Math. and Math. Sciences 1992.

[P] Phelps R., *Convex functions, monotone operators and differentiability*, Lecture Notes in Math. 1364, Springer-Verlag, 1989.

[PY] Pluciennik R., Ye Y., *Differentiability of Musielak-Orlicz sequence spaces*, Comment. Math. Univ. Carolinae **30** (1989), 699–711.

[RR] Rao M.M., Ren Z.D., *Theory of Orlicz spaces*, Marcel Dekker Inc., New York-Basel-Hong Kong, 1991.

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