On spaces with the property of weak approximation by points

Angelo Bella

Abstract. A sufficient condition that the product of two compact spaces has the property of weak approximation by points (briefly WAP) is given. It follows that the product of the unit interval with a compact WAP space is also a WAP space.

Keywords: weak approximation by points, product, semiradial, pseudo radial, compact
Classification: 54A25, 54B10, 54D55

Recently P. Simon [5] introduced the notion of space with the property of weak approximation by points (briefly WAP space). He observed that this property is in general not productive by exhibiting a Lindelöf WAP space whose product with the unit interval fails to be a WAP space. It is reasonable to wonder whether an example of the same sort can involve a compact WAP space. It turns out that this is not the case and we prove here that indeed the product of any compact WAP space with the unit interval is always a WAP space. In addition, the paper explores some links between pseudo radial and WAP spaces and also between radial and AP spaces.

Henceforth all spaces are assumed to be Hausdorff.

A space X has the property of weak approximation by points provided that for every non open set $M \subset X$ there exist a point $x \in M$ and a set $A \subset X \setminus M$ such that $\overline{A} \cap M = \{x\}$.

If the above condition is verified for every point $x \in M$, then the space X is said to have the property of approximation by points (briefly AP).

We say that a subset A of a space X is AP-closed, if for every $F \subset A$ the relation $|\overline{F} \setminus A| \neq 1$ holds.

It is clear that X is a WAP space if and only if every AP-closed subset of X is closed.

Recall that a subset A of a space X is κ-closed whenever $B \subset A$ and $|B| \leq \kappa$ imply $\overline{B} \subset A$.

A space X is called semiradial (see [1]) provided that for any non κ-closed set $A \subset X$ there exists a well ordered net $\{x_\xi : \xi \in \lambda\} \subset A$, with $\lambda \leq \kappa$, which converges to a point outside A.

More generally, a space X is called pseudo radial, if every non closed set $A \subset X$ contains a well ordered net (with no restriction on its length) which converges to a point outside A.

A space X is called radial, if every point in the closure of a subset A of X is the limit of a well ordered net contained in A.

The relations Radial $\rightarrow$ Semiradial $\rightarrow$ Pseudo radial always hold and in general the arrows cannot be reversed even for compact spaces.

For more details on pseudo radial and related spaces see [2].

Proposition 1. *Every semiradial space has the property of weak approximation by points.*

Proof: Let X be a semiradial space and M a non open subset of X. If κ is the minimum cardinal such that $X \setminus M$ is not κ-closed, then there exists a well ordered net $F = \{x_\xi : \xi \in \kappa\} \subset X \setminus M$ which converges to a point $x \in M$. Every point in $\overline{F} \setminus \{x\}$ must be in the closure of an initial segment of F and therefore, by the minimality of κ, it belongs to $X \setminus M$. This shows that $\overline{F} \cap M = \{x\}$ and hence X is a WAP space. $\square$

Notice that the above proposition cannot be reversed, i.e. there are WAP spaces which are not semiradial. For instance, in [6] there is described a ZFC example of a Hausdorff pseudo radial space of countable tightness which is not sequential. It is easy to check that such a space has the weak property approximation by points, but it is not semiradial because a semiradial space of countable tightness is actually sequential. A compact example of the same sort exists at least in a model of ZFC. The one point compactification of the Ostaszewski's space [4] has indeed this property.

Proposition 2. *Every compact space with the property of weak approximation by points is pseudo radial.*

Proof: Let X be a compact WAP space and let A be a non closed subset of X. By hypothesis, there exist a set $B \subset A$ and a point $x \in X \setminus A$ such that $\overline{B} \setminus A = \{x\}$. Select a minimal family of open sets $\{U_\xi : \xi \in \kappa\}$ in the subspace $\overline{B}$ satisfying $\bigcap_{\xi\in\kappa} \overline{U_\xi} = \{x\}$. By minimality, $\bigcap_{\nu\in\xi} \overline{U_\nu} \setminus \{x\} \neq \emptyset$ for every $\xi \in \kappa$. Picking a point $x_\xi \in \bigcap_{\nu\in\xi} \overline{U_\nu} \setminus \{x\}$ and taking into account the compactness of $\overline{B}$, it is easy to check that the well ordered net so obtained converges to x. By construction, this net lies in A and so the proof is complete. $\square$

The author does not know any example of a compact (or simply Hausdorff) pseudo radial non WAP space.

With an argument as in Proposition 2, the next two propositions can also be established.

Proposition 3. *Every countably compact WAP space is sequentially compact.*

At least consistently this proposition cannot be reversed. For instance, in [3, Theorem 3], it is constructed in a model of $2^\omega = \omega_3$ a compact sequentially compact space X having a radially closed subset H which is the union of an increasing ω_1-type sequence of clopen subsets of X. The set H is not closed and therefore if X were WAP then we could find a set $A \subset H$ and a point $p \in X$ such that $\overline{A} \setminus H = \{p\}$. According with the structure of the set H, we see that the point p has in the subspace $\overline{A}$ a well ordered local base. This clearly permits to construct a well ordered net in $\overline{A} \setminus \{p\}$ converging to p — in contradiction with the fact that H was radially closed.

Proposition 4. *Every compact AP space is radial.*

It is easy to see that Proposition 4 cannot be reversed and in fact in [5] it is mentioned that the ordered space $\omega_1 + 1$ is an example of a compact radial space which has not the property of approximation by points. This can be checked directly or by referring to Corollary 1 below.

Proposition 5. *If X is a scattered AP space, then every accumulation point $p \in X$ can be included in a closed set $F \subset X$ such that p is the only accumulation point of the subspace F.*

PROOF: Let p be an accumulation point of X and let A be the set of all isolated points of X. Since p is in the closure of A there exists a set $B \subset A$ such that $\overline{B} \setminus A = \{p\}$. Clearly the set $F = B \cup \{p\}$ has the required properties. □

Corollary 1. *Every compact scattered AP space is Fréchet-Urysohn.*

PROOF: Let X be a compact scattered AP space. If A is a non closed subset of X and $p \in \overline{A} \setminus A$, then there exists a set $B \subset A$ such that $\overline{B} \setminus A = \{p\}$. Applying Proposition 5 to the space $\overline{B}$, we find a closed set $F \subset \overline{B}$ which has p as the only accumulation point. It is clear that every infinite countable subset of $F \setminus \{p\} \subset A$ is a sequence converging to p. Thus X is Fréchet-Urysohn. □

Now we come to the main result of the paper.

Theorem 1. *The product of a compact semiradial space and a compact WAP space is a WAP space.*

PROOF: Assume by contradiction that there exist a compact semiradial space X and a compact WAP space Y such that the product $X \times Y$ is not a WAP space. Then there is a AP-closed set $A \subset X \times Y$ which is not closed. Let κ be the minimum cardinal such that the set A is not κ-closed and choose a set $B \subset A$ satisfying $|B| = \kappa$ and $\overline{B} \setminus A \neq \emptyset$. Select a point $(x, y) \in \overline{B} \setminus A$. As $\{x\} \times Y$ is a WAP space and $A \cap \{x\} \times Y$ is AP-closed, there exists a closed neighbourhood V of (x, y) in $X \times Y$ such that $V \cap A \cap \{x\} \times Y = \emptyset$. Changing A with $A \cap V$, we can assume that $x \notin \pi_X(A)$. Since $x \in \overline{\pi_X(B)}$, it follows that $\pi_X(A)$ is not κ-closed. Now, X being semiradial, we can fix a well ordered net $\{x_\xi : \xi \in \lambda\} \subset \pi_X(A)$ which converges to a point $\hat{x} \in X \setminus \pi_X(A)$. Observe that the set $\pi_X(A)$ is $< \kappa$-closed and consequently $\lambda = \kappa$ and κ is a regular cardinal. For any $\xi \in \kappa$, choose y_ξ such that $(x_\xi, y_\xi) \in A$. Select a complete accumulation point $p \in Y$ of the set $\{y_\xi : \xi \in \kappa\}$. Since the point $(\hat{x}, p) \notin A$, we can assume as before that $p \notin \pi_Y(A)$. For any $\xi \in \kappa$, denote by C_ξ the closure in Y of the set $\{y_\nu : \nu \in \xi\}$ and put

$C = \bigcup_{\xi\in\kappa} C_\xi$. As $p_Y(A)$ is $< \kappa$-closed, it follows that $C \subset \pi_Y(A)$. Moreover, since $p \in \overline{C} \setminus \pi_Y(A)$, it follows that C is not closed in Y. Thus there exists a set $D \subset C$ and a point $\hat{y} \notin C$ such that $\overline{D} \setminus C = \{\hat{y}\}$. Clearly we can write $\overline{D} = \{\hat{y}\} \cup (\bigcup_{\xi\in\kappa} \overline{D} \cap C_\xi)$. For every $\xi \in \kappa$ choose a closed neighbourhood U_ξ of $\hat{y}$ in the subspace $\overline{D}$ satisfying $U_\xi \cap \overline{D} \cap C_\xi = \emptyset$ and let $V_\xi = \bigcap_{\nu\in\xi} U_\nu$. Since the subspace $\overline{D}$ is compact and $\overline{D} \setminus \{\hat{y}\}$ is $< \kappa$-closed, it follows that every $V_\xi \setminus \{\hat{y}\}$ is not empty. Now, picking a point $y'_\xi \in V_\xi \setminus \{\hat{y}\}$ for every $\xi \in \kappa$, we obtain a well ordered net converging to $\hat{y}$. Next, fix a function $f : \kappa \to \kappa$ such that $y'_\xi \in \overline{\{y_\nu : \xi \in \nu \in f(\xi)\}}$ for any ξ. Using the fact that A is $< \kappa$-closed, we can select a point $x'_\xi \in \overline{\{x_\nu : \xi \in \nu \in f(\xi)\}}$ in such a way that $(x'_\xi, y'_\xi) \in A$ for any $\xi \in \kappa$. To finish, observe that the sequence $F = \{(x'_\xi, y'_\xi) : \xi \in \kappa\}$ must converge to the point $(\hat{x}, \hat{y}) \notin A$. Every point in the closure of F and distinct from $(\hat{x}, \hat{y})$ is actually in the closure of an initial segment of F and hence in A. Thus we have $\overline{F} \setminus A = (\hat{x}, \hat{y})$, in contradiction with the fact that A is AP-closed. □

Corollary 2. *The product of a compact WAP space with the unit interval is a WAP space.*

Using Proposition 4 we also have:

Corollary 3. *The product of a compact WAP space and a compact AP space is a WAP space.*

From Corollary 3 we see that the product of finitely many compact AP spaces is a WAP space and we may conjecture that even the product of countably many compact AP spaces should be a WAP space.

To finish, observe that there are two main questions left open here. The first is to find the precise relationship between compact WAP spaces and compact pseudo radial spaces (see Proposition 2) and the second is to check whether it is true or not that the product of two compact WAP spaces is always a WAP space.

References

[1] Bella A., Gerlits J., *On a condition for the pseudo radiality of a product*, Comment. Math. Univ. Carolinae **33** (1992), 311–313.

[2] Nyikos P.J., *Convergence in Topology*, in Recent Progress in General Topology, Hušek and van Mill ed., North Holland, Amsterdam, 1992.

[3] Juhász I., Szentmiklóssy Z., *Sequential compactness versus pseudo radiality in compact spaces*, Topology Appl. **50** (1993), 47–53.

[4] Ostaszewski A.J., *On countably compact perfectly normal spaces*, J. London Math. Soc. **14** (1976), 505–516.

[5] Simon P., *On accumulation points*, to appear.

[6] Simon P., Tironi G., *Two examples of pseudo radial spaces*, Comment. Math. Univ. Carolinae **27** (1986), 155–161.

Department of Mathematics, University of Catania, Città Universitaria, viale A. Doria 6, 95125 Catania, Italy

(*Received* February 18, 1994)