On complemented copies of c_0 in spaces of operators, II

Giovanni Emmanuele*

Abstract. We show that as soon as c_0 embeds complementably into the space of all weakly compact operators from X to Y, then it must live either in X^* or in Y.

Keywords: spaces of weakly compact operators, complemented copies of c_0

Classification: 46B25, 46A32

Let X and Y be two infinite dimensional Banach spaces. It is well known (see for instance [E1], [E3], [E4], [EJ], [F], [H], [K]) that c_0 can embed into $K(X,Y)$, the space of all compact operators from X to Y equipped with the operator norm, even if it does not embed into X^* and Y; furthermore, such a copy of c_0 can be complemented in $K(X,Y)$ (see [E4], [E6]).

Recently ([E2], [E5]), we obtained some results proving that if c_0 embeds into either X^* or Y then it embeds complementably into some spaces of operators larger than $K(X,Y)$, for instance $W(X,Y)$, the space of all weakly compact operators from X to Y. The technique we used in order to construct the complemented copy of c_0 requires the presence of a copy of c_0 in either X^* or Y, because otherwise it does not work.

All the above facts lead us to the following natural question: *Is it possible to have a complemented copy of c_0 inside $W(X,Y)$ even when it does not embed into X^* and Y?*

In this short note (in which we continue the research started in [E5]) we want to show that the answer to this question is negative; indeed, we prove that as soon as c_0 embeds complementably into $W(X,Y)$, then it must live inside either X^* or Y. Actually, we shall prove a slightly more general result about the space $L_{w^*}(X^*,Y)$, i.e. the space of all weak*-weak continuous operators from X^* to Y equipped with the operator norm.

The announced result is the following

Theorem 1. *Let H be a complemented copy of c_0 in $L_{w^*}(X^*,Y)$. If T_n is a basis for H,then there is either a $x_0^* \in B_{X^*}$ or a $y_0^* \in B_{Y^*}$ and a subsequence (T_{n_k}) of (T_n) such that either the sequence $(T_{n_k}(x_0^*))$ spans a copy of c_0 in Y or the sequence $(T_{n_k}^*(y_0^*))$ spans a copy of c_0 in X.*

Proof: It is clear that for each $x^* \in B_{X^*}$ (resp. $y^* \in B_{Y^*}$) the series $\sum T_{n_k}(x^*)$ (resp. $\sum T_{n_k}^*(y^*)$) is weakly unconditionally converging in Y (resp. in X). It will

* Work performed under the auspices of GNAFA of CNR and partially supported by MURST of Italy (40%-1990).

be enough to show that there is either a $x_0^* \in B_{X^*}$ or a $y_0^* \in B_{Y^*}$ and a subsequence (T_{n_h}) of (T_n) such that either the series $\sum T_{n_h}(x_0^*)$ is not unconditionally converging in Y or the series $\sum T_{n_h}^*(y_0^*)$ is not unconditionally converging in X, because we can thus use a well known result due to Bessaga and Pelczynski ([BP]) to conclude our proof. By contradiction we assume that for each $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$ the series $\sum T_n(x^*)$ and $\sum T_n^*(y^*)$ are unconditionally converging in Y and X, respectively. So for any $\xi = (\xi_n) \in l_\infty$ and $x^* \in B_{X^*}$ the series $\sum \xi_n T_n(x^*)$ is unconditionally converging in Y. Define $T_\xi(x^*) = \sum \xi_n T_n(x^*)$ for all $x^* \in B_{X^*}$. We now show that T_ξ belongs to $L_{w^*}(X^*, Y)$. To this aim it will be enough to consider a w*-null net (x_α^*) in B_{X^*} and a y^* in B_{Y^*} and to prove that

$$\lim_\alpha |T_\xi(x_\alpha^*)(y^*)| = 0. \tag{1}$$

Since $\sum \xi_n T_n^*(y^*)$ is unconditionally converging in X by our assumption, we have

$$\lim_p \sup_{x^* \in B_{X^*}} |\sum_{n=p+1}^{\infty} \xi_n T_n^*(y^*)(x^*)| = 0. \tag{2}$$

Thanks to (2), given $\gamma > 0$ we can find a $\overline{p} \in N$ such that

$$\sup_\alpha |\sum_{n=\overline{p}+1}^{\infty} \xi_n T_n^*(y^*)(x_\alpha^*)| < \frac{\gamma}{2}. \tag{3}$$

On the other hand,

$$\lim_\alpha \sum_{n=1}^{\overline{p}} \xi_n T_n(x_\alpha^*)(y^*) = 0 \tag{4}$$

since $T_n \in L_{w^*}(X^*, Y)$, for all $n \in N$. (3) and (4) together give (1).

Furthermore, using the Closed Graph Theorem we can prove easily that the linear map $\Psi : l_\infty \to L_{w^*}(X^*, Y)$ defined by $\Psi(\xi) = T_\xi$ is bounded. It is clear that $K = \Psi(l_\infty)$ contains H. If $P : L_{w^*}(X^*, Y) \longrightarrow H$ is the existing projection, the operator $P_{|K}\Psi : l_\infty \longrightarrow H$ is a quotient map of l_∞ onto c_0. This is a well known contradiction ([D]) that concludes our proof. □

Corollary 2. *Let c_0 embed complementably into $W(X, Y)$. Then c_0 embeds into either X^* or Y.*

Proof: It is enough to observe that $W(X, Y)$ is isomorphic with $L_{w^*}(X^{**}, Y)$.

With a proof similar to that of Theorem 1 we can prove the same result for the space $L(X, Y)$ of all bounded operators from X to Y. One could also consider the space $UC(X, Y)$ of all unconditionally converging operators from X to Y; in such a case we have been able to get just a slightly less precise result than Theorem 1; indeed, the same technique used for proving Theorem 1 shows that as soon as c_0 embeds complementably in $UC(X,Y)$, then either Y contains a copy of c_0 or there are a $y_0^* \in B_{Y^*}$ and a subsequence (T_{n_k}) of (T_n) so that the sequence $(T_{n_k}^*(y_0^*))$ spans a copy of c_0 in X^*, but in such a case we do not know how the copy of c_0 contained in Y is spanned.

At the end, we observe that in the paper [E5] we also considered other spaces of operators, such as spaces of Dunford-Pettis operators; we do not know if Theorem 1 can be extended to cover this case.

References

[BP] Bessaga C., Pelczynski A, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

[D] Diestel J., *Sequences and Series in Banach Spaces*, Graduate Text in Mathematics 97, Springer Verlag 1984.

[E1] Emmanuele G., *Dominated operators on* $C[0,1]$ *and the* (CRP), Collect. Math. **41** (1990), 21–25.

[E2] ———, *Remarks on the uncomplemented subspace* $W(E,F)$, J. Funct. Analysis **99** (1991), 125–130.

[E3] ———, *On the containment of* c_0 *by spaces of compact operators*, Bull. Sci. Math. **115** (1991), 177–184.

[E4] ———, *A remark on the containment of* c_0 *in spaces of compact operators*, Math. Proc. Cambridge Phil. Soc. **111** (1992), 331–335.

[E5] ———, *On complemented copies of* c_0 *in spaces of operators*, Comm. Math. **32** (1992), 29-32.

[E6] ———, *About the position of* $K_{w^*}(X^*,Y)$ *in* $L_{w^*}(X^*,Y)$, Atti Sem. Mat. Fisico Univ. Modena, to appear.

[EJ] Emmanuele G., John K., *Uncomplementability of spaces of compact operators in larger spaces of operators*, to appear.

[F] Feder M., *On subspaces of spaces with an unconditional basis and spaces of operators*, Illinois J. Math. **34** (1980), 196–205.

[H] Holub J.R., *Tensor product bases and tensor diagonals*, Trans. Amer. Math. Soc. **151** (1970), 563–579.

[K] Kalton N.J., *Spaces of compact operators*, Math. Annalen **208** (1974), 267–278.

Department of Mathematics, University of Catania Viale A. Doria 6, 95125 Catania, Italy

E-mail: emmanuele@mathct.cineca.it

(*Received* April 27, 1993)