On powers of Lindelöf spaces

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Abstract. We present a forcing construction of a Hausdorff zero-dimensional Lindelöf space X whose square X^2 is again Lindelöf but its cube X^3 has a closed discrete subspace of size $\mathfrak{c}^+$, hence the Lindelöf degree $L(X^3) = \mathfrak{c}^+$. In our model the Continuum Hypothesis holds true.

After that we give a description of a forcing notion to get a space X such that $L(X^n) = \aleph_0$ for all positive integers n, but $L(X^{\aleph_0}) = \mathfrak{c}^+ = \aleph_2$.

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Introduction

It is well-known that a product of two Lindelöf spaces need not be Lindelöf. Indeed, the product of two Sorgenfrey lines has a closed discrete subspace of size $2^{\aleph_0} = \mathfrak{c}$. The general problem of the degree of non-productivity of the Lindelöf property is discussed in [2] and [5].

In 1978, Shelah, Hajnal and Juhasz proved that it is consistent that there is a Lindelöf space whose square has a closed discrete subspace of size $\mathfrak{c}^+ = \aleph_2$ (see [1], [2], [3]).

In 1990 we gave a consistent example of a Lindelöf space whose square has a closed discrete subspace of size $2^{\aleph_1}$, cardinal $2^{\aleph_1}$ arbitrarily large and does not depend on the size of the Continuum, see [4] and [5]. This is the best result up to this point. We conjecture that $2^{\aleph_1}$ is the true upper bound on the sizes of closed discrete subspaces of squares of Lindelöf spaces.

Definition. For a topological space X, $L(X)$ is the smallest cardinal κ such that every open cover of X has a subcover of size at most κ.

It is known about the higher powers of Lindelöf spaces that for each positive integer n, there is a space X such that X^n is Lindelöf, but $L(X^{n+1}) = \mathfrak{c}$, see [6].

The aim of the present paper is to show a consistent example of a space whose square is Lindelöf, but $L(X^3) = \mathfrak{c}^+$. It is an open problem whether $L(X^3) = 2^{\aleph_1} > \aleph_2$ is possible.

Our reference for forcing and basics is Kunen's book [7].

Theorem. Con$(ZF) \Longrightarrow$ Con$(ZFC+CH+$ *"There is a Lindelöf Hausdorff zero-dimensional space X with all points G_δ sets, $|X| = \omega_2 = \mathfrak{c}^+$, $L(X) = L(X^2) = \omega$, and $L(X^3) = \omega_2 = \mathfrak{c}^+$"*).

The proof will consist of the following:

A) Definitions,
B) Main Lemma, and
C) Facts, of which the last Corollary furnishes the space X mentioned in the theorem.

A) Definitions

0. Let $F\colon \omega_2 \times \omega_2 \longrightarrow \{0,1,2\}$ be fixed.
1. $D(f)$ denotes the domain of the function f and if $D(f) \subset \omega_2$, then $\mu(f) :=$ min $D(f)$ or ω_2, if $f = \phi$.
2. For $x \in \omega_2$ and $i \in 3$, let $A_x^i := \{y \in \omega_2 : \ y \neq x \text{ and } F(x,y) = i\}$.
3. $\forall s \in Fn(\omega_2, 3)\ U_s := \bigcap_{x \in D(s)} (A_x^{s(x)} \cup \{x\})$.
4. $\mathcal{U}_F := \{U_s : \ s \in Fn(\omega_2, 3)\}$.
5. F is *flexible* if $(\forall y \neq z \text{ in } \omega_2)\ (\forall i, j \in 3)\ (\exists x \in \omega_2 \backslash \{y,z\})\ F(x,y) = i$ and $F(x,z) = j$.
6. Define $\varphi\colon \omega_2 \times \omega_2 \longrightarrow \omega_2 + 1$ by letting
$$\begin{aligned} \varphi(y,y) &:= \omega_2, \text{ and for } y \neq z \\ \varphi(y,z) &:= \min\left(\{\delta \in y \cap z : \ F(\delta,y) \neq F(\delta,z)\} \cup \{y \cap z\}\right), \end{aligned}$$
i.e. the least $\delta \in \omega_2$ s.t. $(F(\delta,y) \neq F(\delta,z)$, or $\delta = y$ or $\delta = z)$.
7. We say that $\mathcal{U}_F \times \mathcal{U}_F$ is *sort-of-Lindelöf* if every "cover" $c\colon \omega_2^2 \longrightarrow \left(Fn(\omega_2,3)\right)^2$ satisfying $(\forall \langle y,z\rangle \in \omega_2^2)\ c(y,z) = \langle s,t\rangle \Longrightarrow$
(i) $\langle y,z\rangle \in U_s \times U_t$, and
(ii) $s \restriction \varphi(y,z) = t \restriction \varphi(y,z)$
has a countable "subcover", i.e. $\exists$ countable $A \subset \omega_2$ s.t.
$$\forall \langle y,z\rangle \in \omega_2^2\ \exists \langle a,b\rangle \in A^2$$
with $\langle y,z\rangle \in U_{c_1(a,b)} \times U_{c_2(a,b)}$ (where $c_1(a,b)$ is the left coordinate of c(a,b), and $c_2(a,b)$ is the right one).
We remark that (ii) simply means that $\forall y \in \omega_2([c(y,y) = \langle s,t\rangle \text{ and } y \in D(s) \cap D(t)] \Longrightarrow s(y) = t(y))$.
8. For an $S \subset \omega_2$, let $(S)^0 = S$ and $(S)^1 = \omega_2 \backslash S$.
9. For $k \in Fn(\omega_2, 2)$, let
$$\begin{aligned} V_k^0 &= \bigcap_{x \in D(k)} (A_x^0)^{k(x)}, \\ V_k^1 &= \bigcap_{x \in D(k)} (A_x^1)^{k(x)}, \\ V_k^2 &= \bigcap_{x \in D(k)} (A_x^2)^{k(x)}. \end{aligned}$$

10. Let τ^0, τ^1, τ^2 be topologies on ω_2 generated, respectively, by the following bases:
$$\{V_k^0 : k \in Fn(\omega_2, 2)\},$$
$$\{V_k^1 : k \in Fn(\omega_2, 2)\},$$
$$\{V_k^2 : k \in Fn(\omega_2, 2)\}.$$
So, e.g., τ^0 is generated on ω_2 by a subbasis $\{A_x^0, \omega_2 \backslash A_x^0 : x \in \omega_2\}$.

11. The definition of the forcing notion $(\mathbb{P}, \leq)$. $p \in \mathbb{P}$ iff $p = \langle A, f, T\rangle$, and
   (i) $A \subset \omega_2$ and $|A| \leq \omega$.
   (ii) $f\colon A^2 \longrightarrow 3$.
   (iii) $|T| \leq \omega$ and $(\forall B \in T)\ B \subset \big(Fn(A,3)\big)^2$ and $A^2 = \bigcup\{U_s \times U_t : \langle s,t\rangle \in B\} \cap A^2$.

(iv) $\forall B \in T$
$\forall \delta, \delta' \in A$
$\forall h \in Fn(A\backslash\delta, 3)$
$\forall h' \in Fn(A\backslash\delta', 3)$
$\forall y \in A\backslash\delta$
$\forall z \in A\backslash\delta'$

(a) $(\exists\langle s,t\rangle \in B)(\exists\langle s',t'\rangle \in B)$
  (α) $\langle y,z\rangle \in U_s \times U_{t\restriction\delta'}$ and $t \not\perp h'$,
  (β) $\langle y,z\rangle \in U_{s\restriction\delta} \times U_t$ and $s \not\perp h$

(b) $(\exists\langle s,t\rangle \in B)$
$$\langle y,z\rangle \in U_{s\restriction\delta} \times U_{t\restriction\delta'} \text{ and } s \not\perp h \text{ and } t \not\perp h'\,.$$

(c) If $y = z$, then $(\exists\langle s,t\rangle \in B)$ and $(\exists\langle s',t'\rangle \in B)$ s.t.
  (α) If $\delta \leq \mu(h')$, then
  $\langle z,z\rangle \in U_{s\restriction\delta} \times U_{t\restriction\delta}$
  and $h \cup s \cup \big(t\restriction\mu(h')\big) \in Fn$ and $t \not\perp h'$, and
  (β) if $\delta' \leq \mu(h)$, then
  $\langle z,z\rangle \in U_{s'\restriction\delta'} \times U_{t'\restriction\delta'}$ and
  $h' \cup t' \cup \big(s'\restriction\mu(h)\big) \in Fn$ and $s' \not\perp h$. □

Let $E^p(\delta,y,z) \overset{df}{\Longleftrightarrow} \delta, y, z \in A$ and $\delta \leq y, z$ and $(\forall x \in A \cap \delta)\ f^p(x,y) = f^p(x,z)$.
Let $q \leq p$ if, by definition, $A^q \supset A^p$, $f^q \supset f^p$, $T^q \supset T^p$ and $E^q \supset E^p$. □

B) Main Lemma

Let $V \models ZFC + CH$ and let $\mathbb{P}$ be defined in V by Definition 11. *Then $\mathbb{P}$ is ω_1-complete and has $\omega_2 - cc$.*

Let G be $\mathbb{P}$-generic over V, and let $F = \bigcup\{f^p : p \in G\}$. Then $F\colon \omega_2 \times \omega_2 \longrightarrow 3$ is a flexible total function and $\mathcal{U}_F \times \mathcal{U}_F$ is <u>*sort-of-Lindelöf*</u>.

Proof: The fact that $\mathbb{P}$ is ω_1-complete (i.e. that the naturally defined infimum of a countable descending sequence of conditions belongs to $\mathbb{P}$) is obvious, because "$p \in \mathbb{P}$" is a finitary property (i.e. if $p \notin \mathbb{P}$, then there is a finite collection of finite parts of p (as a structure) witnessing this). $\square$

We will prove 3 lemmas, of which Lemma 1 implies the totality, Lemma 2 implies the flexibility of F, and Lemma 3 establishes the ω_2-chain condition of $\mathbb{P}$. The final statement of the Main Lemma is proved last.

Lemma 1. *Let $p = \langle A, f, T\rangle \in \mathbb{P}$. Then*

$$\begin{aligned} &\forall \tilde{z} \in \omega_2 \backslash A \\ &\exists \tilde{g}:\ (A \cup \{\tilde{z}\})^2 \longrightarrow 3 \textit{ extending } f, \textit{ s.t.} \\ &q\ :=\ \langle A \cup \{\tilde{z}\}, \tilde{g}, T\rangle \in \mathbb{P} \textit{ and } q \leq p\,. \end{aligned}$$

Proof. Assume $A \backslash \tilde{z} \neq \phi$ (otherwise, use Lemma 2). So choose the least $a \in A\backslash \tilde{z}$. We will define by induction a partial function g: $A \longrightarrow 3$ such that, if $(\forall x \in A)\ \tilde{g}(x, \tilde{z}) = g(x)$, then $q \in \mathbb{P}$. g will be an increasing union $g = \bigcup_{i<\omega} g_i$. Let g_0: $A \cap \tilde{z} \longrightarrow 3$ be defined by $g_0(x) := f(x, a)$, for every $x \in A \cap \tilde{z} = A \cap a$.

Let

$$\begin{aligned} \mathcal{S}\ &=\ \{\langle B_i, \delta_i, \delta_i', h_i, h_i', y_i, z_i\rangle :\ i < \omega\} \\ &=\ \{B, \delta, \delta', h, h', y, z\rangle :\ B \in T,\ \delta, \delta' \in A \\ &\qquad h \in Fn(A\backslash\delta, 3),\ h' \in Fn(A\backslash\delta', 3), \\ &\qquad y \in (A \cup \{\tilde{z}\})\backslash\delta,\ z \in (A \cup \{\tilde{z}\}\backslash\delta',\ \textit{and } (y = \tilde{z} \text{ or } z = \tilde{z})\}\,. \end{aligned}$$

Step $i \geq 1$. Consider $\langle \cdots\rangle_i \in \mathcal{S}$.

Case 1. $y_i = z_i = \tilde{z}$.

0. By (iv)-c-α applied to $\langle a, a\rangle$, $\delta = \delta' = a$, $h = g_i\restriction_{A\backslash a}$ and $h' = \phi$, $\exists \langle s, t\rangle \in B$ s.t. $\langle a, a\rangle \in U_{s\restriction a} \times U_{t\restriction a}$ and $g_i \cup s \cup t \in F_n$. Let $g_{i+1}^0\ :=\ g_i \cup s \cup t$. This will guarantee that

$$\langle \tilde{z}, \tilde{z}\rangle \in U_s \times U_t \qquad \text{for iii}_q.$$

1. Apply (iv)-b to $\langle a, a\rangle$ with $\delta = a$, $h = g_{i+1}^0\restriction_{A\backslash a}$, $\delta' = \delta_i'$, $h' = h_i'$. Then $\exists \langle s, t\rangle \in B$, s.t. $\langle a, a\rangle \in U_{s\restriction a} \times U_{t\restriction \delta_i'}$ and $s \not\perp g_{i+1}^0\restriction_{A\backslash a}$ and $t \not\perp h_i'$.

Let $g_{i+1}^1 = g_{i+1}^0 \cup s$. This will guarantee that

$$\langle \tilde{z}, \tilde{z}\rangle \in U_s \times U_{t\restriction \delta_i'} \quad \text{and} \quad t \not\perp h_i'\,.$$

2. (iv)-b of q is obtained similarly from (iv)-b. We have g_{i+1}^2 at this stage. Note that (b) is automatic because of $E^q(\tilde{z}, \tilde{z}, a)$. And the same applies to (c). Let $g_{i+1}\ :=\ g_{i+1}^2$. $\square$

Case 2. $y_i = \tilde{z}$ and $z_i \neq \tilde{z}$, i.e. $z_i \in A$.

0. By (iv)-a-β applied to $\langle a, z_i\rangle$ with $\delta' = a$ and $h' = g_i\restriction_{A\setminus a}$, $\exists\langle s,t\rangle \in B$, s.t. $\langle a, z_i\rangle \in U_{s\restriction a} \times U_t$ and $s \not\perp g_i\restriction_{A\setminus a}$. Let $g^0_{i+1} := g_i \cup s$. This guarantees that

$$\langle \tilde z, z_i\rangle \in U_s \times U_t .$$

1. Apply (iv)-b to $\langle a, z_i\rangle$ and $\delta = a$, $h = g^1_{i+1}\restriction A\setminus a$, $\delta' = \delta'_i$, $h' = h'_i$. Get $\langle s,t\rangle \in B$ s.t. $\langle a, z_i\rangle \in U_{s\restriction a} \times U_{t\restriction\delta'}$ and $t \not\perp h'_i$ and $s \not\perp g^1_{i+1}\restriction_{A\setminus a}$. Set $g^2_{i+1} := g^1_{i+1} \cup s$, thus guaranteeing

$$\langle \tilde z, z_i\rangle \in U_s \times U_{t\restriction\delta'} \quad\text{and}\quad t \not\perp h'_i .$$

Note: (a)(β) of q for $\langle \tilde z, z_i\rangle$ is automatic because of $E^q(\tilde z, \tilde z, a)$. (b)$_q$ is automatic for the same reason and (c)$_q$ does not apply here. Set $g_{i+1} = g^1_{i+1}$.

□

Case 3. $y_i \in A$ and $z_i \in \tilde z$.

0. By (iv)-a-α, $\exists\langle s,t\rangle \in B$ s.t. $\langle y_i, a\rangle \in U_s \times U_{t\restriction a}$ and $t \not\perp g_i\restriction_{A\setminus a}$. Let $g^0_{i+1} = g_i \cup t$, guaranteeing

$$\langle y_i, \tilde z\rangle \in U_s \times U_t ,$$

i.e. (iii)$_q$ at $\langle y_i, \tilde z\rangle$.

1. Note that (a)(α) is automatic. Let $\langle s,t\rangle \in B$ be s.t. $\langle y_i, a\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction a}$ and $s \not\perp h_i$ and $t \not\perp g^0_{i+1}\restriction_{A\setminus a}$ (by (iv)-b). Set $g^1_{i+1} := g^0_{i+1} \cup t$, thus guaranteeing that

$$\langle y_i, \tilde z\rangle \in U_{s\restriction\delta_i} \times U_t \quad\text{and}\quad s \not\perp h_i .$$

2. Note again that (b) is automatic and (c) does not apply. Let $g_{i+1} := g^1_{i+1}$.

□

Lemma 2. *Let $p = \langle A, f, T\rangle \in \mathbb{P}$. Let $\gamma \in A$, $r \in Fn(A\setminus\gamma, 3)$, $\bar z \in A\setminus\gamma$ and $\tilde z \in \omega_2\setminus \sup^+ A$. Then $\exists\tilde g$: $(A\cup\{\tilde z\})^2 \longrightarrow 3$ extending f s.t. $q := \langle A\cup\{\tilde z\}, \tilde g, T\rangle \in \mathbb{P}$, $q \le p$, $\tilde z \in U_r$ and $E^q(\gamma, \bar z, \tilde z)$.*

Proof: Let

$$\begin{aligned}
\mathcal{S} &= \{\langle B_i, \delta_i, \delta'_i, h_i, h'_i, y_i, z_i\rangle :\ i < \omega\}\\
&= \{\langle B, \delta, \delta', h, h', y, z\rangle :\ B \in T,\ \delta, \delta' \in A,\ h \in Fn(A\setminus\delta, 3),\\
&\qquad h' \in Fn(A\setminus\delta', 3),\ y \in (A \cup \{\tilde z\})\setminus\delta,\ z \in (A \cup \{\tilde z\})\setminus\delta' \text{ and}\\
&\qquad (y = \tilde z \text{ or } z = \tilde z)\} .
\end{aligned}$$

As in the proof of Lemma 1, let $\forall x \in A \cap \gamma$ $g_0(x) = f(x, z)$ and $\forall x \in D(r)$ $g_0(x) = r(x)$, so g_0: $(A \cap \gamma) \cup D(r) \longrightarrow 3$. This guarantees at once that $\tilde z \in U_r$.

Step $i > 0$. Consider $\langle \cdots \rangle_i \in \mathcal{S}$.

Case 1. $y_i = z_i = \tilde z$.

0. By (iv)-c applied to $\langle \bar{z}, \bar{z}\rangle$ with $h = g_i \restriction A\backslash\gamma$, $h' = \phi$, $\delta = \delta' = \gamma$, $\exists\langle s,t\rangle \in B$ s.t. $\langle \bar{z}, \bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $g_i\restriction_{A\backslash\gamma} \cup s \cup t \in Fn$.

Let $g^0_{i+1} = g_i \cup s \cup t$. Thus

$$\langle \tilde{z}, \tilde{z}\rangle \in U_s \times U_t\,,$$

and so is covered by B_i.

1. If $\delta'_i \leq \gamma$, by (iv)-b, $\exists\langle s,t\rangle \in B_i$, s.t. $\langle \bar{z}, \bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp g^0_{i+1}\restriction_{A\backslash\gamma}$ and $t \not\perp h'_i$.

Let $g^1_{i+1} = g^0_{i+1} \cup s$, guaranteeing

$$\langle \tilde{z}, \tilde{z}\rangle \in U_s \times U_{t\restriction\delta'_i} \text{ and } t \not\perp h'_i\,.$$

If $\delta'_i > \gamma$, by (iv)-c-α, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle \bar{z}, \bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g^0_{i+1}\restriction_{A\backslash\gamma}) \cup s \cup t\restriction\delta'_i \in Fn$ and $t \not\perp h'_i$.

Set $g^0_{i+1} = g^0_{i+1} \cup s \cup t\restriction\delta'_i$, implying

$$\langle \tilde{z}, \tilde{z}\rangle \in U_s \times U_{t\restriction\delta'_i} \text{ and } t \not\perp h'_i\,.$$

2. Symmetrically we obtain g^2_{i+1} guaranteeing (a)(β) (i.e. that $\langle \tilde{z}, \tilde{z}\rangle \in U_{s\restriction\delta_i} \times U_t$ and $s \not\perp h_i$, for some $\langle s,t\rangle$ in B_i).

3. Note that if $\delta_i, \delta'_i \leq \gamma$, then (b) for $\langle \tilde{z}, \tilde{z}\rangle$ follows automatically from (b) for $\langle \bar{z}, \bar{z}\rangle$. If one of δ_i, δ'_i is $\leq \gamma$, then e.g. in $\delta'_i \leq \gamma < \delta_i$ case, by (b), $\exists\langle s,t\rangle \in B_i$, s.t. $\langle \bar{z}, \bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp (g^2_{i+1}\restriction\delta_i\backslash\gamma) \cup h_i$ and $t \not\perp h'_i$. Let $g^3_{i+1} = g^2_{i+1} \cup s\restriction\delta_i$, guaranteeing

$$\langle \tilde{z}, \tilde{z}\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i} \text{ and } s \not\perp h_i \text{ and } t \not\perp h'_i\,.$$

Similarly for the symmetric case of $\delta_i \leq \gamma < \delta'_i$.

If $\gamma < \delta_i, \delta'_i$, then if $\delta_i \leq \delta'_i$ use $(c)(b)$, and if $\delta_i > \delta'_i$, use (c)(α), e.g. if $\delta_i \leq \delta'_i$, then $\gamma \leq \mu(h_i) \geq \delta_i$. So by $(c)(\beta)$, $\exists\langle s,t\rangle \in B_i$, s.t. $\langle \bar{z}, \bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g^2_{i+1}\restriction\delta'_i\backslash\gamma) \cup h'_i \cup t \cup s\restriction\delta_i \in Fn$ and $s \not\perp h_i$. Let $g^3_{i+1} = g^2_{i+1} \cup (t\restriction\delta'_i) \cup (s\restriction\delta_i)$, guaranteeing

$$\langle \tilde{z}, \tilde{z}\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i} \text{ and } s \not\perp h_i \text{ and } t \not\perp h'_i\,.$$

4. Suppose $\gamma < \delta_i \leq \mu(h'_i)$. By (c)($\alpha$), $\exists\langle s,t\rangle \in B$ s.t. $\langle \bar{z}, \bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $s \cup [(g^3_{i+1}\restriction\delta_i\backslash\gamma) \cup h_i] \cup (t\restriction\mu(h'_i)) \in Fn$ and $t \not\perp h'_i$. Set $g^4_{i+1} = g^3_{i+1} \cup (s\restriction\delta_i) \cup (t\restriction\delta_i)$, guaranteeing

$$\langle \tilde{z}, \tilde{z}\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i}\,,$$

$$h_i \cup s \cup t\restriction\mu(h'_i) \in Fn \text{ and } t \not\perp h'_i\,.$$

5. Suppose now that also $\gamma < \delta'_i \leq \mu(h_i)$. Then, by (c)($\beta$) $\exists\langle s,t\rangle \in B$ s.t. $\langle \bar{z},\bar{z}\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $[(g^4_{i+1}\restriction\delta'_i\backslash\gamma) \cup h'_i] \cup t \cup (s\restriction\mu(h_i)) \in Fn$ and $s \not\perp h_i$. Let $g^5_{i+1} = g^4_{i+1} \cup (t\restriction\delta'_i) \cup (s\restriction\delta'_i)$, guaranteeing

$$\langle \tilde{z},\tilde{z}\rangle \in U_{s\restriction\delta'_i} \times U_{t\restriction\delta'_i}\,,$$
$$h'_i \cup t \cup (s\restriction\mu(h_i)) \in Fn \text{ and } \; s \not\perp h_i\,.$$

Finally, let $g_{i+1} = g^5_{i+1}$.

Case 2. $y_i = \tilde{z}$ and $z_i \in A$.

0. By (a)(β), $\langle \bar{z}, z_i\rangle \in U_{s\restriction\gamma} \times U_t$ with $s \not\perp g_i\restriction_{A\backslash\gamma}$. Let $g^0_{i+1} = g_i \cup s$, guaranteeing

$$\langle \tilde{z}, z_i\rangle \in U_s \times U_t\,.$$

1. By (b), $\langle \bar{z}, z_i\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp g^0_{i+1}\restriction_{A\backslash\gamma}$ and $t \not\perp h'_i$. Let $g'_{i+1} = g^0_{i+1} \cup s$. Then

$$\langle \tilde{z}, z_i\rangle \in U_s \times U_{t\restriction\delta_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

2. W.l.o.g., $\gamma < \delta_i$. $\langle z, z_i\rangle \in U_{s\restriction\gamma} \times U_t$ and $s \not\perp (g'_{i+1}\restriction\delta_i\backslash\gamma) \cup h_i$. Let $g^2_{i+1} = g'_{i+1} \cup (s\restriction\delta_i)$. Then

$$\langle \tilde{z}, z_i\rangle \in U_{s\restriction\delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

3. $\langle z, z_i\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp (g^2_{i+1}\restriction\delta_i\backslash\gamma) \cup h_i$ and $t \not\perp h'_i$. Let $g^3_{i+1} = g^2_{i+1} \cup (s\restriction\delta_i)$. Then

$$\langle \tilde{z}, z_i\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i}\,, s \not\perp h_i \quad \text{and} \quad t \not\perp h'_i\,.$$

Finally, let $g_{i+1} = g^3_{i+1}$.

Case 3. $y_i \in A$ and $z_i = \tilde{z}$. This is symmetric to Case 2.

End of the i-th induction step. □

Lemma 3. *$\mathbb{P}$ has $\omega_2 - cc$.*

PROOF: Let $\mathbb{Q} \subset \mathbb{P}$ with $|\mathbb{Q}| \geq \omega_2$. By CH and the Δ-system lemma, we may assume that there are $p \neq p'$ in $\mathbb{Q}$, $p = \langle A, f, T\rangle$, $p' = \langle A', f', T'\rangle$ such that

$$A \cap A' \;=:\; \Delta < A\backslash\Delta < A'\backslash\Delta\,,$$

$tp\, A = tp\, A'$ and f, f' are "typewise the same", so $f\restriction\Delta^2 = f'\restriction\Delta^2$, and $z \in A$ and $z' \in A'$ with $tp(A \cap z) = tp(A \cap z')$ implies $(\forall x \in A)\; f(x,z) = f(x,z')$.

Let $\gamma :=$ the least ordinal in $A\backslash\Delta$, and let z' denote the member of A' corresponding to $z \in A$. (So $tp(A \cap z) = tp(A' \cap z)$ and $A' \cap \gamma' = \Delta$).

We want to extend $(f \cup f')$ to $g\colon (A \cup A')^2 \longrightarrow 3$ so that $q := \langle A \cup A', g, T \cup T' \rangle \in \mathbb{P}$ and $q \leq p, p'$. First we will define g on $(A \backslash \Delta) \times (A' \backslash \Delta)$. For every $z \in A \backslash \Delta$, let

$$g^z_{-1} \;=\; \phi\,.$$

By induction in ω steps, we will extend every $g^z_{-1} (z \in A \backslash \Delta)$ to a partial function $g^z\colon A \backslash \Delta \longrightarrow 3$ s.t., in the process,

(i) $(\forall i)\ g^z_i$ will all be finite.

(ii) $E^p(\gamma, y, z) \Longrightarrow (\forall i)\ g^z_i = g^y_i$.

Let

$$\begin{aligned} \mathcal{S} = \quad & \{\langle B_i, \delta_i, \delta'_i, h_i, h'_i, y_i, z_i \rangle :\ i < \omega\} \\ r = \quad & \{\langle B, \delta, \delta', h, h', \tilde{y}, \tilde{z} \rangle :\ B \in T,\ \delta, \delta' \in A,\ h \in Fn(A \backslash \delta, 3), \\ & h' \in Fn(A \backslash \delta', 3),\ \tilde{y}, \tilde{z} \in (A \cup A') \backslash \Delta,\ \delta \leq \tilde{y},\ \delta' \leq \tilde{z}\}\,. \end{aligned}$$

Step $i \geq 0$. Consider $\langle \cdots \rangle_i \in \mathcal{S}$.

There are 3 relevant cases (for the future pairs involving $y, z \in A \backslash \Delta$):

(1) $\tilde{y}_i \in A$ and $\tilde{z}_i = z' \in A'$

(2) $\tilde{y}_i = y' \in A'$ and $\tilde{z}_i \in A$

(3) $\tilde{y}_i = y' \in A'$ and $\tilde{z}_i = z' \in A'$.

Case 1. $\tilde{y}_i = y \in A$ and $\tilde{z}_i = z' \in A'$.

0. By (a)(α), $\exists \langle s, t \rangle \in B_i$ s.t. $\langle y, z \rangle \in U_s \times U_{t \restriction \gamma}$ and $t \not\perp g^z_{i-1}$.

Let $g^{z^0}_i := g^z_{i-1} \cup t\restriction_{A \backslash \gamma}$. This will guarantee that

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_s \times U_t\,.$$

(*) Let also $g^{x^0}_i := g^{z^0}_i$ for every $x \in A \backslash \gamma$ with $E^p(\gamma, x, z)$.

Let $g^{x^0}_i := g^x_{i-1}$ for all other $x \in A \backslash \gamma$.

1. By (a)(β) of p, $\exists \langle s, t \rangle \in B_i$ s.t. $\langle y, z \rangle \in U_s \times U_{t \restriction \gamma}$ and $t \not\perp (g^{z^0}_i \restriction \delta'_i) \cup h'_i$. Let $g^{z^1}_i := g^{z^0}_i \cup (t \restriction (\delta'_i \backslash \gamma))$, guaranteeing

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_s \times U_{t \restriction \delta'_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

Then $(*)$-update, i.e. let $g^{x^1}_i = g^{z^1}_i$ for every $x \in A \backslash \gamma$ with $E^p(\gamma, x, z)$, and $g^{x^1}_i = g^{x^0}_i$ for all other $x \in A \backslash \gamma$.

2. Concerning (a)(β): Similarly, by (b) of p get $\langle s, t \rangle \in B_i$ s.t. $\langle y, z \rangle \in U_{s \restriction \delta_i} \times U_{t \restriction \gamma}$ and $s \not\perp h_i$ and $t \not\perp g^{z^1}_i$.

Let $g^{z^2}_i = g^{z^1}_i \cup (t\restriction_{(A \backslash \gamma)})$, guaranteeing that

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_{s \restriction \delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

Then $(*)$-update.

3. (b) Assume w.l.o.g. that $\gamma < \delta'_i$. Here $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$ and $s \not\perp h_i$ and $t \not\perp g_i^{z^2}\restriction\delta'_i \cup h'_i$. Let $g_i^{z^3} := g_i^{z^2} \cup t\restriction(\delta'_i\backslash\gamma)$. This will guarantee that

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta'_i}\,, \quad s \not\perp h_i \quad \text{and} \quad t \not\perp h'_i\,.$$

Then $(*)$-update all γ-twins of z.

Finally, let $\forall x \in A\backslash\Delta$, $g_i^x := g_i^{x^3}$ $\square$

Case 2. Is entirely symmetric.

Case 3. $\tilde{y}_i = y' \in A'$ and $\tilde{z}_i = z' \in A'$
Subcase 3a. $E^p(\gamma,y,z)$. So $g_{i-1}^z = g_{i-1}^y$.

0. By (c)(α) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $g_{i-1}^y \cup s \cup t \in Fn$ (i.e. $h' = \phi$ and $h = g_{i-1}^y$ here).

Let $g_i^{y^0} := g_{i-1}^y \cup (s\cup t)\restriction(A\backslash\gamma)$. Also $(*)$-update g_{i-1}^x's, i.e. for every $x \in A\backslash\gamma$ s.t. $E^p(\gamma,y,x)$, set $g_i^{x^0} := g_i^{y^0}$, and for every other $x \in A\backslash\Delta$, set $g_i^{x^0} = g_{i-1}^x$. This will guarantee that

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_t\,.$$

1. (a)(α) If $\delta'_i \le \gamma$, then by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta'_i}$ and $s \not\perp g_i^{y^0}$ and $t \not\perp h'_i$. Let $g_i^{y^1} := g_i^{y^0} \cup s\restriction(A\backslash\gamma)$, and $(*)$-update. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_{t\restriction\delta'_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

If $\delta'_i > \gamma$, then by (c)(α), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma}\times U_{t\restriction\gamma}$ and $g_i^{y^0}\cup s\cup t\restriction\delta'_i \in Fn$ and $t \not\perp h'_i$. Let $g_i^{y^1} := g_i^{y^0} \cup (s\restriction(A\backslash\gamma)) \cup (t\restriction\delta'_i)$, and $(*)$-update. Then

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_s \times U_{t\restriction\delta'_i} \quad \text{and} \quad t \not\perp h'_i\,.$$

2. Concerning $(a)(\beta)$: Similarly to **1**, get $\langle s,t\rangle \in B_i$ and update to $g_i^{y^2}$, guaranteeing

$$\langle\tilde{y}_i,\tilde{z}_i\rangle \in U_{s\restriction\gamma} \times U_t \quad \text{and} \quad s \not\perp h_i\,.$$

3. Concerning (b). There are 4 possibilities here:

(1) $\delta_i \le \gamma$ and $\delta'_i \le \gamma$
(2) $\delta_i \le \gamma$ and $\delta'_i > \gamma$
(3) $\delta_i > \gamma$ and $\delta'_i \le \gamma$
(4) $\delta_i > \gamma$ and $\delta'_i > \gamma$, 4(a) $\delta_i \le \delta'_i$, 4(b) $\delta'_i < \delta_i$.

If (1) — there is nothing to do: make $g_i^{x^3} = g_i^{x^2}$ for all $x \in A\backslash A$.

If (2), then by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$ and $s \not\perp h_i$ and $t \not\perp (g_i^{y^2}\restriction\delta'_i) \cup h'_i$.

Let $g_i^{y^3} = g_i^{y^2} \cup t{\restriction}(\delta_i'\backslash\gamma)$ and $(*)$-update. Then

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i'}\,, \quad s \not\leq h_i \quad \text{and} \quad t \not\leq h_i'\,.$$

If (3), act similarly.

If 4(a), then by (c)(α), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^2}{\restriction}\delta_i) \cup h_i \cup s \cup t{\restriction}\delta_i' \in Fn$.

Let $g_i^{y^3} = g_i^{y^2} \cup s{\restriction}(\delta_i\backslash\gamma) \cup t{\restriction}(\delta_i'\backslash\gamma)$ and $(*)$-update. Then

$$\langle \tilde{y}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i'}\,, \quad s \not\leq h_i \quad \text{and} \quad t \not\leq h'\,.$$

If 4(b), then by (c)(β), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^2}{\restriction}\delta_i') \cup h_i' \cup t \cup s{\restriction}\delta_i \in Fn$.

Let $g_i^{y^3} = g_i^{y^2} \cup t{\restriction}(\delta_i'\backslash\gamma) \cup s{\restriction}(\delta_i\backslash\gamma)$ and $(*)$-update. Then the same formula as in 4(a) holds.

4. Concerning (c)(α): If $y = z$ and $\delta_i \leq \mu(h_i)$, then w.l.o.g. $\gamma < \delta_i$ and, by (c)(α) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^3}{\restriction}\delta_i) \cup h_i \cup s \cup (t{\restriction}\mu(h_i')) \in Fn$, and $t \not\leq h_i'$.

Let $g_i^{y^4} = g_i^{y^3} \cup s{\restriction}(\delta_i\backslash\gamma) \cup t{\restriction}(\delta_i\backslash\gamma)$, and $(*)$-update. Then

$$\langle \tilde{z}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i}\,, \quad h_i \cup s \cup (t{\restriction}\mu(h_i')) \in Fn \quad \text{and} \quad t \not\leq h_i'\,.$$

5. Concerning (c)(β): If $y = z$ and $\delta_i' \leq \mu(h_i)$, then w.l.o.g. $\gamma < \delta_i'$, and by (c)(β) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,y\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $(g_i^{y^4}{\restriction}\delta_i') \cup h_i' \cup t \cup (s{\restriction}\mu(h_i)) \in Fn$ and $s \not\leq h_i$.

Let $g_i^{y^5} = g_2^{y^4} \cup t{\restriction}(\delta_i'\backslash\gamma) \cup s{\restriction}(\delta_i'\backslash\gamma)$ and $(*)$-update. Then

$$\langle \tilde{z}_i, \tilde{z}_i \rangle \in U_{s\restriction\delta_i'} \times U_{t\restriction\delta_i'}\,,$$
$$h_i' \cup t \cup (s{\restriction}\mu(h_i)) \in Fn \quad \text{and} \quad s \not\leq h_i\,.$$

□

Subcase 3b. Not — $E^p(\gamma, y, z)$.

0. By (b) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$ and $s \not\leq g_{i-1}^y$ and $t \not\leq g_{i-1}^z$.

Let $g_i^{y^0} = g_{i-1}^y \cup s{\restriction}(A\backslash\gamma)$ and $g_i^{z^0} = g_{i-1}^z \cup t{\restriction}(A\backslash\gamma)$. Then

$$\langle \tilde{y}_i, \tilde{y}_i \rangle \in U_s \times U_t\,.$$

Then **($*$)**-update, i.e.

(a) for every $x \in A\backslash\Delta$ s.t. $E^p(\gamma, x, y)$, set $g_i^{x^0} = g_i^{y^0}$,
(b) for every $x \in A\backslash\Delta$ s.t. $E^p(\gamma, x, z)$, set $g_i^{x^0} = g_i^{z^0}$ and
(c) for every other $x \in A\backslash\Delta$, set $g_i^{x^0} = g_{i-1}^x$.

1. Concerning (a)(α): Again, if $\delta_i' \leq \gamma$, then, by (b) of p, $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta_i'}$, $s \not\perp g_i^{y^0}$ and $t \not\perp h_i'$.

Let $g_i^{y^1} = g_i^{y^0} \cup s\restriction(A\backslash\gamma)$. Then

$$\langle \tilde{y}_i, \tilde{z}_i\rangle \in U_s \times U_{t\restriction\delta_i'} \quad \text{and} \quad t \not\perp h_i' .$$

Then $(*)$-update, i.e. all $x \in A\backslash\delta$ with $E^p(\gamma,x,y)$ will get $g_i^{x^1} = g_i^{y^0}$.

If $\gamma < \delta_i'$, then $\exists\langle s,t\rangle \in B_i$, s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$, $s \not\perp g_i^{y^0}$ and $t \not\perp (g_i^{z^0}\restriction\delta_i') \cup h_i'$.

Let $g_i^{y^1} = g_i^{y^0} \cup s\restriction(A\backslash\gamma)$ and $g_i^{z^1} = g_i^{z^0} \cup t\restriction(\delta_i'\backslash\gamma)$, and $(**)$-update, as (mutatis mutandis) in **0.**

2. Re (a)(β). If $\delta_i \leq \gamma$, then $\exists\langle s,t\rangle \in B$, s.t. $\langle y,z\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$, $s \not\perp h_i$ and $t \not\perp g_i^{z^1}$.

Let $g_i^{z^2} = g_i^{z^1} \cup (t\restriction(A\backslash\gamma))$ and $g_i^{y^2} = g_i^{y^1}$ and $(**)$-update, as in **0**. Then

$$\langle \tilde{y}_i, \tilde{z}_i\rangle \in U_{s\restriction\delta_i} \times U_t \quad \text{and} \quad s \not\perp h_i .$$

If $\gamma < \delta_i$, then $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$, $s \not\perp (g_i^{y^1}\restriction\delta_i) \cup h_i$ and $t \not\perp g_i^{z^1}$.

Let $g_i^{y^2} = g_i^{y^1} \cup (s\restriction(\delta_i\backslash\gamma))$ and $g_i^{z^2} = g_i^{z^1} \cup (t\restriction(A\backslash\gamma))$ and $(**)$-update. Then the formula above holds.

3. (b) Again, there are 4 possibilities here:

(1) $\delta_i \leq \gamma$ and $\delta_i' \leq \gamma$,
(2) $\delta_i \leq \gamma$ and $\delta_i' > \gamma$,
(3) $\delta_i > \gamma$ and $\delta_i' \leq \gamma$,
(4) $\delta_i > \gamma$ and $\delta_i' > \gamma$.

If (1), do nothing.

If (2), then by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\gamma}$, $s \not\perp h_i$ and $t \not\perp (g_i^{z^2}\restriction\delta_i') \cup h_i'$.

Let $g_i^{y^3} = g_i^{y^2}$ and $g_i^{z^3} = g_i^{z^2} \cup (t\restriction(\delta_i'\backslash\gamma))$. Then $(**)$-update. Then

$$\langle \tilde{y}_i, \tilde{z}_i\rangle \in U_{s\restriction\delta_i} \times U_{t\restriction\delta_i'}, \quad s \not\perp h_i \quad \text{and} \quad t \not\perp h_i' .$$

If (3), then, by (b), $\exists\langle s,t\rangle \in B_i$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\delta_i'}$, $s \not\perp g_i^{y^2}\restriction\delta_i \cup h_i$ and $t \not\perp h_i'$.

Let $g_i^{y^3} = g_i^{y^2} \cup (s\restriction(\delta_i\backslash\gamma))$ and $g_i^{z^3} = g_i^{z^2}$. Then $(**)$-update.

If (4), then $\exists\langle s,t\rangle \in B$ s.t. $\langle y,z\rangle \in U_{s\restriction\gamma} \times U_{t\restriction\gamma}$, $s \not\perp g_i^{y^2}\restriction\delta_i \cup h_i$ and $t \not\perp g_i^{z^2}\restriction\delta_i' \cup h_i'$.

Let $g_i^{y^3} = g_i^{y^2} \cup (s\restriction(\delta_i\backslash\gamma))$ and $g_i^{z^3} = g_i^{z^2} \cup (t\restriction(\delta_i'\backslash\gamma))$. Then $(**)$-update.
End of the Subcase 3b and of Case 3. $\square$

For every $z \in A\backslash\Delta, g_i^z$ is defined as the most recent value.
End of the i-th induction step. $\square$

At the end of induction, let for every $z \in A\backslash\Delta$

$$g^z = \bigcup_{i<\omega} g_i^z .$$

Finally, define g on $(A\backslash\Delta) \times (A'\backslash\Delta)$ by the following rule:

$$g(y, z') = \begin{cases} g^z(y), & \text{if } y \in \operatorname{dom}(g^z) \\ 0 & \text{otherwise.} \end{cases}$$

The extension procedure for g on $(A'\backslash\Delta) \times (A\backslash\Delta)$ and the condition p' is the same. (We do not have to take care there of γ'-twins, but we may).

Since the construction has, as in side remarks, a verification of the conditions (iii) and (iv) of q, we are done. $\square$

The proof that in $V[G]$ $\mathcal{U}_F \times \mathcal{U}_F$ is sort-of-Lindelöf:

1. Let $c \in V[G]$ be as in the definition (7), and let σ be a $\mathbb{P}$-name for it.

2. It is enough to show that, for every $p \in \mathbb{P}$ with

$$(*) \qquad p \Vdash \text{“definition (7) for } \sigma\text{”} ,$$

there are $p^* \leq p$ and a countable $A^* \subset \omega_2$ s.t.

$$p^* \Vdash \check{\omega}_2^2 = \bigcup \{\dot{U}_{\sigma_1(y,z)} \times \dot{U}_{\sigma_2(y,z)} : \langle y, z\rangle \in \check{A}^* \times \check{A}^*\}.$$

3. Note that $\forall q \in \mathbb{P}$ with $q \Vdash$ “$(*)$”, $\forall \langle y, z\rangle \in \omega_2^2\ \exists r = r(q, y, z) \leq q$ and $\exists \langle s, t\rangle \in (Fn(\omega_2, 3))^2$ s.t. $r \Vdash \sigma(\check{y}, \check{z}) = \langle \check{s}, \check{t}\rangle$ and $D(s) \cup D(t) \subset A^r$.

4. Let φ: $\omega \longrightarrow \omega \times \omega$ be a bijection s.t. $(\forall n \in \omega)$ $(\varphi(n) = \langle i, j\rangle \Rightarrow n \geq i)$.

5. We will construct an ω-sequence of conditions $p_0 \geq p_1 \geq \cdots \geq p_n \geq \cdots$, $n < \omega$ by induction, starting with $p_0 = p$.

6. If $p_i = \langle A_i, f_i, T_i\rangle$ has been already constructed, we fix an ω-enumeration

$$\begin{aligned} \mathcal{S}^i &= \{\langle \delta_j^i, y_j^i, z_j^i, h_j^i\rangle : j < \omega\} \\ &= \{\langle \delta, y, z, h\rangle : \delta, y, z \in A_i, \ \delta \leq z, \ h \in Fn(A_i\backslash\delta, 3)\}. \end{aligned}$$

7. *Step* $n+1$, *for* $n \geq 0$. How to choose p_{n+1}?

(1) Find $\varphi(n) = \langle i, j \rangle$, $i \leq n$.
(2) Consider $\langle \delta^i_j, y^i_j, z^i_j, h^i_j \rangle \in \mathcal{S}^i$ and pick $z_n \in \omega_2 \setminus \sup^+ A_n$.
(3) Apply Lemma 2 to p_n and z_n, to get $q_n \leq p_n$ such that

$$\begin{aligned} &z_n \in A^{q_n} \\ &E^{q_n}(\delta^i_j, z^i_j, z_n) \text{ holds, and} \\ &z_n \in U_{h^i_j}. \end{aligned}$$

(4) Apply note in **3.** to get $p_{n+1} = r(q_n, y^i_j, z^i_j)$.

8. So $p_{n+1} \Vdash \sigma(\check{y}^i_j, \check{z}^i_j) = \langle \check{s}_n, \check{t}_n \rangle$, for some s_n, t_n in $Fn(A_{n+1}, 3)$. Also, for every μ,

$$E^{p_{n+1}}(\mu, y^i_j, z^i_j) \longrightarrow s_n \restriction \mu \;=\; t_n \restriction \mu\,.$$

(Because here $p_{n+1} \Vdash \dot{\varphi}(\check{y}, \check{z}) \geq \check{\mu}$).

9. Let $q^* := \langle A^*, f^*, T^* \rangle$, where $A^* = \bigcup\limits_i A_i$, $f^* = \bigcup\limits_i f_i$, $T^* = \bigcup\limits_i T_i$. Then $q^* \in \mathbb{P}$, because $\mathbb{P}$ is ω_1-complete.

10. Let

$$\begin{aligned} B^* &:= \{\langle s_n, t_n \rangle :\; n < \omega\} \\ &= \{\langle s, t \rangle \in (Fn(A^*, 3))^2 :\; (\exists \langle y, z \rangle \in A^* \times A^*) \\ &\qquad (q^* \Vdash \sigma(\check{y}, \check{z}) = \langle \check{s}, \check{t} \rangle)\}\,. \end{aligned}$$

Let $p^* := \langle A^*, f^*, T^* \cup \{B^*\} \rangle$.

11. Claim $p^* \in \mathbb{P}$.

Regarding (iii) of p^* at B^*.

$< y, z > \in \; A^* \times A^* \Rightarrow (\exists n \in \omega)$ s.t. $\varphi(n) = < i, j >$ and $< y, z > = < y^i_j, z^i_j >$. Then $p_{n+1} \Vdash \sigma(\check{y}, \check{z}) = \langle \check{s}_n, \check{t}_n \rangle$, as remarked in **8**. Then $q^* \Vdash (\langle \check{y}, \check{z} \rangle \in \dot{U}_{\check{s}_n} \times \dot{U}_{\check{t}_n})$, because $q^* \leq p$ and $\leq p_{n+1}$. Then $\langle y, z \rangle \in U^{q^*}_{s_n} \times U^{q^*}_{t_n}$, by absoluteness (because $D(s_n) \cup D(t_n) \subset A^*$). □

Regarding (iv) of p^* at B^*

Suppose $\delta, \delta' \in A$, $h \in Fn(A \setminus \delta, 3)$, $h' \in Fn(A \setminus \delta', 3)$, $y \in A \setminus \delta$, $z \in A \setminus \delta'$.

(a)(α). Find $n \in \omega$ s.t. $\varphi(u) = \langle i, j \rangle$ and $z = z^i_j$, $h' = h^i_j$ and $\delta' = \delta^i_j$.

By (iii) p^* already checked, $\exists k \in \omega$ s.t. $\langle y, z_n \rangle \in U_{s_k} \times U_{t_k}$ and, by choice in **7**, $E^{q^*}(\delta', z, z_n)$ and $z_n \in U_{h'} \setminus D(h')$, so $t_k \not\perp h'$. Then

$$\langle y, z \rangle \in U_{s_k} \times U_{t_k \restriction \delta'} \quad \text{and} \quad t_k \not\perp h'\,.$$

(a)(β). Similarly, find $n \in \omega$ s.t. $\varphi(n) = \langle i, j\rangle$, $y = z^i_j$, $h = h^i_j$, $\delta = \delta^i_j$. Then, by (iii) of q^*, $\exists k \in \omega$ s.t. $\langle z_n, z\rangle \in U_{s_k} \times U_{t_k}$ and $E^{q^*}(\delta, y, z_n)$ and $z_n \in U_h \setminus D(h)$. Then

$$\langle y, z\rangle \in U_{s_k \restriction \delta} \times U_t \quad \text{and} \quad s_k \not\leq h\,.$$

(b). Find $n_1, n_2 \in \omega$ s.t. $\varphi(n_1) = \langle i_1, j_1\rangle$, $\varphi(n_2) = \langle i_2, j_2\rangle$, and $y = z^{i_1}_{j_1}$, $\delta = \delta^{i_1}_{j_1}$, $h = h^{i_1}_{j_1}$ and $z = z^{i_2}_{j_2}$, $\delta' = \delta^{i_2}_{j_2}$, $h' = h^{i_2}_{j_2}$. Then $E^{q^*}(\delta, y, z_{n_1})$, $z_{n_1} \in U_h \setminus D(h)$, $E^{q^*}(\delta', z, z_{n_2})$ and $z_{n_2} \in U_{h'} \setminus D(h')$, by construction.

By (iii) of p^*, $\exists k \in \omega$ s.t. $\langle z_{n_1}, z_{n_2}\rangle \in U_{s_k} \times U_{t_k}$, implying that

$$\langle y, z\rangle \in U_{s_k \restriction \delta} \times U_{t_k \restriction \delta'} \quad \text{and} \quad s_k \not\leq h \text{ and } t_k \not\leq h'\,.$$

(c)(α). Suppose $y = z$ and $\delta \leq \mu(h')$. Find $n_1, n_2 \in \omega$ s.t. $\varphi(n_1) = \langle i_2, j_1\rangle$, $\varphi(n_2) = \langle i_2, j_2\rangle$, $z = z^{i_1}_{j_1}$, $\delta = \delta^{i_1}_{j_1}$, $h = h^{i_1}_{j_1}$ and $z_{n_1} = z^{i_2}_{j_2}$, $\mu(h') = \delta^{i_2}_{j_2}$, $h' = h^{i_2}_{j_2}$. Then $E^{q^*}(\delta, z, z_{n_1})$, $z_{n_1} \in U_h \setminus D(h)$ and $E^{q^*}(\mu(h'), z_{n_1}, z_{n_2})$, $z_{n_2} \in U_{h'} \setminus D(h')$.

By (iii) of p^*, pick $k \in \omega$ s.t. $\langle z_{n_1}, z_{n_2}\rangle \in U_{s_k} \times U_{t_k}$. This implies that

$$\langle z, z\rangle \in U_{s_k \restriction \delta} \times U_{t_k \restriction \delta}\,, h \cup s_k \cup (t_k \restriction \mu(h')) \in Fn \quad \textit{and} \quad t \not\leq h'$$

because $s_k \restriction \mu(h') = t_k \restriction \mu(h')$, by **8.**

(c)(β). Similarly, assuming $y = z$ and $\delta' \leq \mu(h)$, find $n_1, n_2 \in \omega$, $\varphi(n_1) = \langle i_1, j_1\rangle$, $\varphi(n_2) = \langle i_2, j_2\rangle$ s.t. $z = z^{i_1}_{j_1}$, $\delta' = \delta^{i_1}_{j_1}$, $h' = h^{i_1}_{j_1}$, $z_{n_1} = z^{i_2}_{j_2}$, $\mu(h) = \delta^{i_2}_{j_2}$, $h = h^{i_2}_{j_2}$. Then $E^{q^*}(\delta', z, z_{n_1})$, $z_{n_1} \in U_{h'} \setminus D(h')$ and $E^{q^*}(\mu(h), z_{n_1}, z_{n_2})$, $z_{n_2} \in U_h \setminus D(h)$. Let $\langle z_{n_2}, z_{n_1}\rangle \in U_{s_k} \times U_{t_k}$ for some $k \in \omega$. Then

$$\langle z, z\rangle \in U_{s_k \restriction \delta'} \times U_{t_k \restriction \delta'},\ h' \cup t_k \cup (s_k \restriction \mu(h)) \in Fn \quad \text{and} \quad s_k \not\leq h$$

because $s_k \restriction \mu(h) = t_k \restriction \mu(h)$, by **8.** $\square$

12. Finally, $p^* \in \mathbb{P} \Rightarrow p^* \leq p$ and

$$\begin{aligned} p^* \Vdash \text{“}\omega_2^2 &= \bigcup\{\dot U_s \times \dot U_t :\ \langle s, t\rangle \in B^*\} \\ &= \bigcup\{\dot U_{\sigma_1(y,z)} \times \dot U_{\sigma_2(y,z)} :\ \langle y, z\rangle \in \check A^* \times \check A^*\}\text{”}\,. \end{aligned}$$

(The first line is a consequence of Lemma 1 and $p^* \Vdash$ “$\check A^* \times \check A^* = \bigcup\{\dot U_s \times U_t : \langle s, t\rangle \in B^*\}$”.) As required. $\square$

This concludes the proof of the Main Lemma.

C) Facts about F in V[G]

Fact 1. $\mathcal{U}_F$ is a Lindelöf family, i.e. every $\mathcal{U}_F$-cover of ω_2 has a countable subcover.

Proof: Let c: $\omega_2 \longrightarrow Fn(\omega_2, 3)$ be a $\mathcal{U}_F$-cover of ω_2, i.e. $\forall y \in \omega_2$ $y \in U_{c(y)}$. If $z \in \omega_2$ and $\varphi(y,z) = \delta$, then $z \in U_{c(y)\restriction\delta}$. Define d: $\omega_2^2 \longrightarrow (Fn(\omega_2,3))^2$ by

$$d(y,z) \;=\; \langle c(y), c(y)\restriction\varphi(y,z)\rangle\,.$$

Then d is as in Definition (7). By Main Lemma, $\exists$ countable $A \subset \omega_2$ s.t. $d''A^2$ covers ω_2^2. But then $c''A$ covers ω_2. [$y \in \omega_2 \Rightarrow \langle y, 0\rangle \in U_s \times U_t$, where $\langle s,t\rangle = d(a,b)$ for some $\langle a,b\rangle \in A^2 \Rightarrow y \in U_s = U_{c(a)}$ by definition of d]. $\square$

Fact 2. Each of τ^0, τ^1, τ^2 is a Lindelöf topology.

Proof: Let $\mathcal{C}$ be a cover of ω_2 by τ^0-basic open sets, i.e.

$$\omega_2 \;=\; \bigcup\{V_k^0 :\; k \in \mathcal{C}\}\,, \quad \mathcal{C} \subset Fn(\omega_2, 2)\,.$$

$\forall z \in \omega_2$ pick $k_z \in \mathcal{C}$ s.t. $z \in V_{k_z}^0$. Let s_z: $D(k_z) \longrightarrow 3$ be defined by

$$s_z(x) \;=\; \begin{cases} i \in 3 \text{ s.t. } s \in A_x^i\,, & \text{if } z \neq x \\ 1 \text{ (or 2)} & \text{if } z = x \end{cases}$$

Then $z \in U_{s_z}$ and $\exists F_z \subset D(k_z)$ s.t.

$$z \in U_{s_z} \backslash F_z \;\subset V_{k_z}^0.$$

$$[\text{Indeed, } \forall x \in D(k_z) \quad A_x^{s_z(x)} \subset (A_x^0)^{k_z(x)}$$
$$\Downarrow$$
$$\bigcap_{x \in D(k_z)} A_x^{s_z(x)} \subset V_{k_z}^0\,.$$

Let $F_z = (\bigcap\limits_{x \in D(k_z)} (A_x^{s_z(x)} \cup \{x\})) \backslash \bigcap_{x \in D(k_z)} A_x^{s_z(x)} \subset D(k_z)$.

Then $U_{s_z} := \bigcap_{x \in D(k_z)} (A_x^{s_z(x)} \cup \{x\}) \subset V_{k_z}^0 \cup F_z$.]

So $\{s_z :\; z \in \omega_2\}$ is a $\mathcal{U}_F$-cover of ω_2, hence, by Fact 1, there is a countable subcover $\{s_{z_i} :\; i \in \omega\}$. But then $\bigcup\{V_{k_{z_i}}^0 \;:\; i < \omega\}$ is co-countable in ω_2, and hence τ^0 is a Lindelöf topology. $\square$

Fact 3. Each of τ^0, τ^1, τ^2 is a points G_δ topology.

PROOF: For τ^0. Fix $z \in \omega_2$. By flexibility of F, $\forall y \neq z\ \exists x \in \omega_2 \backslash \{y, z\}$ s.t.

$$F(x,y) \;=\; 0 \quad \text{and} \quad F(x,z) \;=\; 1 \text{ (2 is equally possible).}$$

Let $K = \{x \in \omega_2 \backslash \{z\} :\ F(x,z) = 1\}$. Then $\omega_2 = \bigcup_{x \in K} (A^0_x \cup \{x\}) \cup (A^0_z \cup \{z\})$.

By Lindelöfness of $\mathcal{U}_F$, $\exists$ countable $K_0 \subset K$ s.t.

$$\omega_2 \;=\; \bigcup_{x \in K_0} (A^0_x \cup \{x\}) \cup (A^0_z \cup \{z\})\,.$$

Consequently, we have

$$\omega_2 \backslash \{z\} \;=\; \bigcup_{x \in K_0} (A^0_x \cup \{x\}) \cup A^0_z\,,$$

so $\omega \backslash \{z\}$ is a countable union of τ^0-closed (points are closed by flexibility of F) sets, and so $\{z\}$ is a G_δ of τ^0. □

Fact 4. Each of $\tau^i \times \tau^j$, $i, j \in 3$, is a Lindelöf topology on ω_2^2.

PROOF: For $\tau^0 \times \tau^1$.

A. Suppose $\langle y, z\rangle \in V^0_k \times V^1_\ell$. Then, as in Fact 2, define 2 functions s, t: $D(k) \cup D(\ell) \longrightarrow 3$ as follows:

$$s(x) \;=\; \begin{cases} i \in 3 \text{ s.t. } y \in A^i_x, & \text{if } y \neq x \\ 2, & \text{if } y = x. \end{cases}$$

$$t(x) \;=\; \begin{cases} i \in 3 \text{ s.t. } z \in A^i_x, & \text{if } z \neq x \\ 2, & \text{if } z = x. \end{cases}$$

Then $s{\restriction}\varphi(y,z) = t{\restriction}\varphi(y,z)$. (Indeed, if $y = z$, then by observation that definitions of s and t coincide. If $y \neq z$ and $x < \varphi(y,z)$, then $F(x,y) = F(x,z)$, and $y \in A^i_x \leftrightarrow z \in A^i_x$, $(y \neq x \neq z)$.) Also, as in Fact 3, $\exists$ finite $F, G \subset D(k) \cup D(\ell)$ s.t. $y \in U_s \backslash F \subset V^0_k$ and $z \in U_t \backslash G \subset V^1_\ell$, so $\langle y, z\rangle \in U_s \backslash F \times U_t \backslash G \subset V^0_k \times V^1_\ell$, and $U_s \times U_t \subset (V^0_k \cup F) \times (V^1_\ell \cup G)$.

B. Let $\mathcal{C}$ be a $\tau^0 \times \tau^1$ cover of ω_2^2, and let $\mathcal{D} \subset \mathcal{U}_F \times \mathcal{U}_F$ be its refinement, obtained, for each point as in **A**, point by point. Since, by the Main Lemma, $\mathcal{U}_F \times \mathcal{U}_F$ is sort-of-Lindelöf and $\mathcal{D}$ satisfies Definition (7), there is a countable subcover of $\mathcal{D}$, say $\{U_{S_i} \times U_{t_i} :\ i < \omega\} \subset \mathcal{D}$. Then

$$\begin{aligned} \omega_2^2 \;&=\; \bigcup_{i<\omega} (U_{S_i} \times U_{t_i}) \;=\; \bigcup_{i<\omega} [(V^0_{k_i} \cup F_i) \times (V^1_{\ell_i} \cup G_i)] \\ &=\; \bigcup_{i<\omega} [(V^0_{k_i} \times V^1_{\ell_i}) \cup (V^0_{k_i} \times G_i) \cup (F_i \times V^1_{\ell_i}) \cup (F_i \times G_i)]\,. \end{aligned}$$

Since $V^0_{k_i}$ is Lindelöf in τ^0, $V^1_{l_i}$ in τ^1 by Fact 2, and F_i and G_i are finite, $\mathcal{D}$ has a countable subco $\tau^0 \times \tau^1$ is Lindelöf. □

(Other cases of $\langle i, j\rangle \in 3 \times 3$ are similar.)

Fact 5. In $\tau^0 \times \tau^1 \times \tau^2$, ω_2^3 has a closed discrete diagonal.

PROOF: Closed by the flexibility of F, and $\langle x, x, x\rangle \in (A_x^0)^c \times (A_x^1)^c \times (A_x^2)^c$ witnesses the discreteness. $\square$

Corollary. *Let, in $V[G]$, $S := (\omega_2, \tau^0) \oplus (\omega_2, \tau^1) \oplus (\omega_2, \tau^2)$. Then S and S^2 are Lindelöf points G_δ 0-dimensional spaces, and $L(S^3) = \mathfrak{c}^+ = \omega_2$.* $\square$

This finishes the proof of our theorem. $\square$

We conclude with a sketch of the forcing notion $\mathbb{P}$ to get a zero-dimensional space X such that,for all finite n, $L(X^n) = \aleph_0$, but $L(X^{\aleph_0}) = \mathfrak{c}^+ = \aleph_2$. $p \in \mathbb{P}$ iff $p = \langle A, f, \vec{\mathcal{B}}\rangle$, where

(i) $A \in [\omega_2]^{\leq\omega}$
(ii) $f : A^2 \to \omega$
(iii) $\vec{\mathcal{B}} = \langle \mathcal{B}_n : n \in \omega\backslash 1\rangle$ and $\forall n\ |\mathcal{B}_n| \leq \omega$ and $\forall B \in \mathcal{B}_n\ B \subset (Fn(A, \omega))^n$ (and $A^n = \bigcup\{U_{s_0} \times ... \times U_{s_{n-1}} : \vec{s} \in B\} \cap A^n$; this follows from (iv)).
(iv) $\forall n \in \omega\backslash 1$
$\forall B \in \mathcal{B}_n$
$\forall \vec{z} \in A^n$
$\forall$ partition of n, $N \cup \tilde{N} = n$
$\forall \vec{\delta} \in A^{\tilde{N}}$
$\forall \vec{h} \in (Fn(A, \omega))^{\tilde{N}}$ s.t. $\vec{h} \geq \vec{\delta}$ (i.e. $D(h_i) \geq \delta_i,\ \forall i$).
$\forall$ assignment $\Big[\Big[$ for $\forall z \in ran(\vec{z} \restriction \tilde{N})$,
of $\bullet$**1** a finite tree $(T^z, \preceq)$ s.t.

$$(t \in T^z \Rightarrow t = \langle \delta_t, h_t\rangle),$$
$$\delta_t \in A\ \&\ h_t \in Fn(A\backslash\delta_t, \omega),$$
$$\&\ (s \prec t \Rightarrow \delta_s < \delta_t)$$
$$\&\ (t \in Lev_0(T) \Rightarrow \delta_t \leq z);$$

and
of $\bullet$**2** an identification map $m^Z : N^z \to T$, where $N^z := \{i \in \tilde{N} : z_i = z\}$, s.t.
$(m^z(i) = t \Rightarrow (\delta_i = \delta_t \& h_i = h_t))\Big]\Big]$
$\exists \vec{s} \in B$ such that
(a) $(\forall i, j \in N)\ [z_i \in \mathcal{U}_{s_i}\ \&\ (z_i = z_j \Rightarrow s_i = s_j)]$
and
(b) $\forall z \in ran(\vec{z} \restriction \tilde{N})$
$\forall i, j \in N^z$
(1) $m^z(i) = m^z(j) \Rightarrow s_i = s_j$;
(2) if $m^z(i) \prec m^z(j)$ and $t \in T^z$ is the immediate successor of $m^z(i)$ in the chain of

T^z leading to $m^z(j)$, then $s_i \restriction \delta_t = s_j \restriction \delta_t$;
(3) $s_i \not\perp h_i$;
(4) if $t \in Lev_0(T)$ & $t \preceq m^z(i)$,
then
$$z \in \mathcal{U}_{s_i} \restriction \delta_t.$$

Let E^p be defined by

$$E^p(\delta, y, z) \Leftrightarrow \begin{cases} \delta, y, z \in A^p \\ \delta \leq y, z \\ \forall x \in (A^p \cap \delta) \ f^p(x, y) = f^p(x, z). \end{cases}$$

We define $q \leq p$ iff $A^q \supset A^p$, $f^q \supset f^p$, $(\forall n \in \omega - 1)$ $\mathcal{B}_n^q \supset \mathcal{B}_n^p$, and $E^q \supset E^p$. End of definition.

It is worth observing that if T^z contains a chain of the form

$$\delta_0 \quad h_0 \quad \delta_1 \quad h_1 \quad \delta_2 \quad h_2 \quad \delta_3 \quad h_3$$

then
$$h_0 \cup s_0 \cup (s_1 \restriction \delta_1) \cup (s_2 \restriction \delta_1) \cup (s_3 \restriction \delta_1) \in Fn$$

&
$$h_1 \cup s_1 \cup (s_2 \restriction \delta_2) \cup (s_3 \restriction \delta_2) \in Fn$$

&
$$h_2 \cup s_2 \cup (s_3 \restriction \delta_3) \in Fn$$

&
$$h_3 \cup s_3 \in Fn.$$

$\mathbb{P}$ preserves cardinals and CH, (it is ω_1-complete & ω_2-cc). If G is $\mathbb{P}$-generic over V and$V \models CH$, then $\exists X \in V[G]$, s.t.

X is a Lindelöf Hausdorff 0-dimensional space of size $\aleph_2$, and

$$\forall n < \omega \ L(X^n) = \aleph_0$$
$$\text{and } L(X^\omega) = \mathfrak{c}^+ = \aleph_2.$$

With slightly simpler partial orders, we can set for every $n < \omega$ a Hausdorff Y_n s.t.
$$L(Y_n^n) = \aleph_0 \ \& \ L(Y_n^{n+1}) = \mathfrak{c}^+ = \aleph_2.$$

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