On one class of solvable boundary value problems for ordinary differential equation of n-th order

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Abstract. New sufficient conditions of the existence and uniqueness of the solution of a boundary problem for an ordinary differential equation of n-th order with certain functional boundary conditions are constructed by the method of a priori estimates.

Keywords: boundary problem with functional conditions, differential equations of n-th order, method of a priori estimates, differential inequalities

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Introduction

In the paper we give new sufficient conditions for existence and uniqueness of the solution to the problem

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

$$\ell_i(u, u^{(1)}, \dots, u^{(k_0-1)}) = 0,\ i = 1, \dots, k_0 \tag{2_1}$$

$$\Phi_{0i}(u^{(i-1)}) = \Phi_i(u^{(k_0)}, u^{(k_0+1)}, \dots, u^{(n-1)}),\ i = k_0+1, \dots, n \tag{2_2}$$

where $f : \langle a, b\rangle \times R^n \to R$ satisfies the local Carathéodory condition and for each $i \in \{1, \dots, k_0\}$, $\ell_i : [C(\langle a, b\rangle)]^{k_0} \to R$ is a linear continuous functional and for each $i \in \{k_0+1 \dots n\}$, Φ_{0i} — the linear nondecreasing continuous functional on $C(\langle a, b\rangle)$ is concentrated on $\langle a_i, b_i\rangle \subseteq \langle a, b\rangle$, $(i = k_0+1, \dots, n)$ (i.e. the value of Φ_{0i} depends only on functions restricted to $\langle a_i, b_i\rangle$, and the segment can be degenerated to a point). Φ_i $(i = k_0+1, \dots, n)$ are continuous functionals on $[C(\langle a, b\rangle)]^{n-k_0}$. In general $\Phi_{0i}(1) = c_i$ $(i = k_0+1, \dots, n)$, without loss of generality we can suppose $\Phi_{0i}(1) = 1$ $(i = k_0+1, \dots, n)$.

Problem (1), (2) for $k_0 = 0$ is solved in paper [4].

Throughout the paper assume:

(3) Boundary value problem $u^{(k_0)} = 0$ possesses only the trivial solution

with condition (2_1).

Problem for differential equation (1) together with boundary condition

$$\sum_{j=1}^{k_0} a_{ij} \cdot u^{(j-1)}(a) + b_{ij} \cdot u^{(j-1)}(b) = 0 \qquad (i = 1, \dots, k_0)$$

$$u^{(i-1)}(t_i) = c_i \qquad (i = k_0 + 1, \dots, n)$$

is not the special case of problems in [1] and [4]. On the other hand, the boundary value problem with the same two groups of condition but in opposite order for $c_j = 0$ is the special case of problems, which were studied in [1].

Main result

We adopt the following notation:
$\langle a, b\rangle$ — a segment, $-\infty < a \le a_i \le b_i \le b < +\infty$ $(i = k_0 + 1, \dots, n)$, R^n — n-dimensional real space with points $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum_{i=1}^n | x_i |$,

$$R_+^n = \{x \in R^n : x_i \ge 0 \; i = 1, \dots, n\},$$

$C^{n-1}(\langle a, b\rangle)$ — the space of functions continuous together with their derivatives up to the order $n-1$ on $\langle a, b\rangle$ with the norm

$$\|u\|_{C^{n-1}(\langle a,b\rangle)} = \max\left\{\sum_{i=1}^n | u^{(i-1)}(t) |: a \le t \le b\right\},$$

$AC^{n-1}(\langle a, b\rangle)$ — a set of all functions absolutely continuous together with their derivatives to the $(n-1)$-order on $\langle a, b\rangle$, the space $L^p(\langle a, b\rangle)$ is the space of functions integrable on $\langle a, b\rangle$ in p-th power with a norm

$$\|u\|_{L^p} = \begin{cases} [\int_a^b |u(t)|^p dt]^{1/p} & \text{for } 1 \le p < \infty \\ vrai \max\{| x(t)| : a \le t \le b\} & \text{for } p = \infty, \end{cases}$$

$L^p(\langle a, b\rangle, R_+) = \{u \in L^p(\langle a, b\rangle) : u(t) \ge 0, \; t \in \langle a, b\rangle\}$. If $x = (x_i(t))_{i=1}^n \in [C(\langle a, b\rangle)]^n$ and $y = (y_i(t))_{i=1}^n \in [C(\langle a, b\rangle)]^n$, then $x \le y$ if and only if $x_i(t) \le y_i(t)$ for all $t \in \langle a, b\rangle$ and $i = 1, \dots, n$. A functional $\Phi : [C(\langle a, b\rangle)]^n \to R_+$ is said to be homogeneous iff: $\Phi(\lambda x) = \lambda\Phi(x)$ for all $\lambda \in R_+$ $x \in [C(\langle a, b\rangle)]^n$ and nondecreasing if $\Phi(x) \le \Phi(y)$ for all $x, y \in [C(\langle a, b\rangle)]^n$, $x \le y$. Let us consider the problem (1), (2). Under the solution we understand the function with absolutely continuous derivatives up to the order $(n-1)$ on $\langle a, b\rangle$, which satisfies the equation (1) for almost all $t \in \langle a, b\rangle$ and fulfils the boundary condition (2).

To solve (1), (2) we specify a class of auxiliary functions

$$g, \ell_1, \ell_2 \dots \ell_{k_0}, \; h_{k_0+1} \dots h_n, \; \Psi_{k_0+1} \dots \Psi_n.$$

Definition. Let $\ell_i : [C(\langle a,b\rangle)]^{k_0} \to R$ $(i = 1,\dots,k_0)$ be the linear continuous functionals, $\Psi_i : [C(\langle a,b\rangle)]^{n-k_0} \to R_+$ $(i = k_0+1,\dots,n)$ the homogeneous continuous nondecreasing functionals and $g, h_i \in L^1(\langle a,b\rangle, R_+)$ $(i = k_0+1,\dots,n)$. If the system of differential inequalities

$$(4_1) \qquad |\varrho_i'(t)| \le |\varrho_{i+1}(t)| \quad t \in \langle a,b\rangle \ (i = 1,\dots,n-1)$$

$$(4_2) \qquad |\varrho_n'(t) - g(t)\cdot\varrho_n(t)| \le \sum_{j=k_0+1}^{n} h_j(t)|\varrho_j(t)|,\ t \in \langle a,b\rangle$$

with boundary conditions

$$(5_1) \qquad \ell_i(\varrho_1,\dots,\varrho_{k_0}) = 0 \quad (i = 1,\dots,k_0)$$

$$(5_2) \qquad \min\{|\varrho_i(t)| : a_i \le t \le b_i\} \le \Psi_i(|\varrho_{k_0+1}|,\dots,|\varrho_n|)\ (i = k_0+1,\dots,n)$$

has only the trivial solution, we say that

$$(6) \qquad \begin{gathered}(g,\ell_1,\ell_2,\dots,\ell_{k_0},h_{k_0+1},\dots,h_n,\Psi_{k_0+1},\dots,\Psi_n) \in \\ LN(\langle a,b\rangle, a_{k_0+1},\dots,a_n,b_{k_0+1},\dots,b_n).\end{gathered}$$

Remark. If $k_0 = 0$ we have

$$LN(\langle a,b\rangle, a_1,a_2,\dots,a_n,b_1,\dots,b_n) = Nic(\langle a,b\rangle, a_1,\dots,a_n,b_1,\dots,b_n)$$

from paper [4].

Theorem 1. *Let the condition* (6) *be satisfied and let the data* f, $\Phi_{k_0+1},\dots,\Phi_n$ *of* (1), (2) *satisfy the inequalities*

$$(7_1) \qquad \begin{gathered}[f(t,x_1,x_2,\dots,x_n) - g(t)\cdot x_n]\operatorname{sign} x_n \le \sum_{j=k_0+1}^{n} h_j(t)\cdot|x_j| + \omega(t,\sum_{j=1}^{n}|x_j|) \\ \text{for } t \in \langle a_n,b\rangle,\ x \in R^n\end{gathered}$$

$$(7_2) \qquad \begin{gathered}[f(t,x_1,x_2,\dots,x_n) - g(t)\cdot x_n]\operatorname{sign} x_n \ge -\sum_{j=k_0+1}^{n} h_j(t)|x_j| - \omega(t,\sum_{j=1}^{n}|x_j|) \\ \text{for } t \in \langle a,b_n\rangle,\ x \in R^n\end{gathered}$$

$$(8) \qquad \begin{gathered}|\Phi_i(u^{(k_0)},\dots,u^{(n-1)})| \le \Psi_i(|u^{(k_0)}|,\dots,|u^{(n-1)}|) + r \\ \text{for } (i = k_0+1,\dots,n),\end{gathered}$$

where $r \in R_+$, $\omega : \langle a,b\rangle \times R_+ \to R_+$ *and* $\omega(\cdot,\varrho) \in L(\langle a,b\rangle, R_+)$ $\forall\varrho \in R_+$, $\omega(t,\cdot)$ *is nondecreasing for all* $t \in \langle a,b\rangle$ *and*

$$(9) \qquad \lim_{\varrho\to+\infty} \frac{1}{\varrho}\int_a^b \omega(t,\varrho)\,dt = 0.$$

Then the problem (1), (2) *has at least one solution.*

To prove Theorem 1 the following lemma is suitable.

Lemma 1. *Let the condition* (6) *be satisfied. Then there exists a nonnegative constant* $\varrho > 0$ *such that the estimate*

$$\|u\|_{C^{n-1}(\langle a,b\rangle)} \le \varrho(r + \|h_0\|_{L^1(\langle a,b\rangle)}) \tag{10}$$

holds for each constant $r \ge 0$, $h_0 \in L^1(\langle a,b\rangle, R_+)$ *and for each solution* $u \in AC^{n-1}(\langle a,b\rangle)$ *of the differential inequalities*

$$[u^{(n)}(t) - g(t)\cdot u^{(n-1)}(t)]\cdot \operatorname{sign} u^{(n-1)}(t) \le \sum_{j=k_0+1}^{n} h_j(t)|u^{(j-1)}(t)| + \\ +h_0(t) \quad \text{for} \quad a_n \le t \le b \tag{11_1}$$

$$[u^{(n)}(t) - g(t)\cdot u^{(n-1)}(t)]\cdot \operatorname{sign} u^{(n-1)}(t) \ge -\sum_{j=k_0+1}^{n} h_j(t)|u^{(j-1)}(t)| - \\ -h_0(t) \quad \text{for} \quad a \le t \le b_n \tag{11_2}$$

with boundary condition (2_1) *and*

$$\min\{|u^{(i-1)}(t)| : a_i \le t \le b_i\} \le \Psi_i(|u^{(k_0)}|, \dots |u^{(n-1)}|) + r \\ (i = k_0+1, \dots, n). \tag{12}$$

Proof: Let us denote by M the set of all 3-tuples (u, h_0, r) such that $u \in AC^{n-1}(\langle a,b\rangle)$, $h_0 \in L^1(\langle a.b\rangle)$, $r \ge 0$ and the relations (2_1), (11_1), (11_2) and (12) are satisfied. It is easy to verify that $(u, h_0, r) \in M$ if and only if the 3-tuple $(u^{(k_0)}, h_0, r)$ fulfils the assumptions of Lemma 1 in [4] (with $n - k_0$ in the place of n). Hence there exists $\varrho_1 > 0$ such that

$$\|u^{(k_0)}\|_{C^{n-k_0}(\langle a,b\rangle)} \le \varrho_1(r + \|h_0\|_{L^1(\langle a,b\rangle)}) \tag{13}$$

holds for all $(u, h_0, r) \in M$. Furthermore, by the assumption (3) there exists the Green function $G(t,s)$ of the boundary value problem $u^{(k_0)} = 0$, (2_1). Consequently, for any $(u, h_0, r) \in M$, the relations

$$u^{(i-1)}(t) = \int_a^b \frac{\partial^{(i-1)} G(t,s)}{\partial t^{(i-1)}} u^{(k_0)}(s)\, ds, \quad t \in \langle a,b\rangle, \quad i = 1, 2, \dots, k_0 \tag{14}$$

are true. Putting

$$\varrho_2 = \max_{a \le t \le b} \sum_{i=1}^{k_0} \int_a^b \Big| \frac{\partial^{(i-1)} G(t,s)}{\partial (t)^{(i-1)}} \Big|\, ds,$$

we obtain the relation

$$\|u\|_{C^{k_0}(\langle a,b\rangle)} \le \varrho_1\varrho_2(r + \|h\|_{L^1(\langle a,b\rangle)}) \tag{15}$$

holds for all $(u, h_0, r) \in M$. We put $\varrho = \varrho_1 + \varrho_1 \cdot \varrho_2$, then (10) follows from (13) by (15). □

Proof of Theorem 1: Let $\varrho > 0$ be the constant from Lemma 1. According to (9) there exists constant $\varrho_0 > 0$ such that

$$\varrho(r + \int_a^b \omega(t, \varrho_0)\,dt) \le \varrho_0. \tag{16}$$

Putting

$$\chi(s) = \begin{cases} 1 & \text{for } |s| \le \varrho_0 \\ 2 - \frac{|s|}{\varrho_0} & \text{for } \varrho_0 \le |s| \le 2\varrho_0, \\ 0 & \text{for } |s| > 2\varrho_0 \end{cases} \tag{17}$$

$$\tilde f(t, x_1, x_2, \dots, x_n) = \chi(\|x\|)[f(t, x_1, x_2, \dots, x_n) - g(t)\cdot x_n], \tag{18}$$

$$\tilde\Phi_i(u^{(k_0)}, \dots, u^{(n-1)}) = \chi(\|u\|_{C^{n-1}\langle a,b\rangle})\Phi_i(u^{(k_0)}, \dots, u^{(n-1)}) \quad (i = k_0+1, \dots, n). \tag{19}$$

We consider the problem

$$u^{(n)}(t) = g(t)u^{(n-1)}(t) + \tilde f(t, u(t), \dots, u^{(n-1)}(t)) \tag{20}$$

with condition (2_1) and

$$\Phi_{0i}(u^{(i-1)}) = \tilde\Phi_i(u^{(k_0)}, \dots, u^{(n-1)}) \quad (i = k_0+1, \dots, n). \tag{21}$$

The relations (18), (19) immediately imply that $\tilde f : \langle a,b\rangle \times R^n \to R$ satisfies the local Carathéodory conditions, $\tilde\Phi_i : [C(\langle a,b\rangle)]^{(n-k_0)} \to R$ $(i = k_0+1, \dots, n)$ are continuous functionals,

$$f_0(t) = \sup\{|\tilde f(t, x_1, \dots, x_n)| : (x_i)_{i=1}^n \in R^n\} \in L^1(\langle a,b\rangle) \tag{22}$$

and

$$r_i = \sup\{|\tilde\Phi_i(u^{(k_0)}, \dots, u^{(n-1)})| : u \in C^{n-1}(\langle a,b\rangle)\} < +\infty. \tag{23}$$

Now we want to show that the homogeneous problem

$$u^{(n)} = g(t)\cdot u^{(n-1)}(t) \tag{20_0}$$

with conditions (2_1) and

$$(21_0) \qquad \Phi_{0i}(u^{(i-1)}) = 0 \ (i = k_0 + 1, \dots, n)$$

has only trivial solution. Let u be an arbitrary solution of this problem. Then

$$u^{(n-1)}(t) = c \cdot w(t)$$

where $c = \text{const}$ and $w(t) = \exp[\int_a^t g(s)\, ds]$.

According to (21_0) and the character of functional Φ_{0n} we get

$$\Phi_{0n}(u^{(n-1)}) = 0 = c \cdot \Phi_{0n}(w).$$

From $\Phi_{0n}(w) \ge \exp(-\int_a^b |g(t)|\, dt) \cdot \Phi_{0n}(1) > 0$ it follows that $c = 0$ and $u^{(n-1)} = 0$. Similarly we have $u^{(n-2)} \equiv 0, \dots, u^{(k_0)} \equiv 0$, therefore u is a solution of the differential equation $u^{(k_0)} = 0$ with condition (2_1). By hypothesis (3) we have $u \equiv 0$. Using 2.1 from [3], we obtain that the condition (22), (23) and the unicity of trivial solution of each problem (20_0), (21_0), (2_1) guarantees the existence of solutions of the problem (20), (21), (2_1). Let u be the solution of problem (20), (21), (2_1). We want to show that

$$(24) \qquad \|u\|_{C^{n-1}(\langle a,b\rangle)} \le \varrho_0.$$

From (18) and (7) we have

$$\begin{aligned}
&[u^{(n)}(t) - g(t)u^{(n-1)}(t)] \cdot \operatorname{sign} u^{(n-1)}(t) = \\
&= \tilde{f}(t, u(t), \dots, u^{(n-1)}(t)) \cdot \operatorname{sign} u^{(n-1)}(t) = \\
&= \chi(\sum_{i=1}^n |u^{(i-1)}(t)|)[f(t, u, \dots, u^{(n-1)}) - g(t)u^{(n-1)}(t)] \cdot \operatorname{sign} u^{(n-1)}(t) \le \\
&\le \chi(\sum_{j=1}^n |u^{(j-1)}(t)|)[\sum_{j=k_0+1}^n h_j(t)|u^{(j-1)}(t)| + \omega(t, \sum_{j=1}^n u^{(j-1)}(t)|)] \le \\
&\le \sum_{j=k_0+1}^n h_j(t)|u^{(j-1)}(t)| + \omega(t, 2\varrho_0) \quad \text{for} \quad t \in \langle a_n, b\rangle.
\end{aligned}$$

Similarly

$$\begin{aligned}
&[u^{(n)}(t) - g(t)u^{(n-1)}(t)] \cdot \operatorname{sign} u^{(n-1)}(t) \ge \\
&\qquad \ge -\sum_{j=k_0+1}^n h_j(t)|u^{(j-1)}(t)| - \omega(t, 2\varrho_0) \quad \text{for} \quad t \in \langle a, b_n\rangle.
\end{aligned}$$

From (8) and the character of functionals Φ_{0i} $(i = k_0 + 1, \dots, n)$ imply that

$$\begin{aligned}
\min\{|u^{(i-1)}(t)| : a_i \le t \le b_i\} &\le |\Phi_{0i}(u^{(i-1)})| \le \\
&\le \Psi_i(u^{(k_0)}, \dots, u^{(n-1)}) + r.
\end{aligned}$$

Therefore by Lemma 1 and by (16), (24) holds. Then $\chi(\sum_{j=1}^n |u^{(j-1)}(t)|) = 1$ and hence by (18), (19) u is a solution of problem (1), (2). □

Theorem 2. *Let the condition* (6) *be satisfied and let the data* $f, \Phi_{k_0+1}, \dots, \Phi_n$ *of* (1), (2) *satisfy the inequalities*

$$\begin{gathered}\{[f(t,x_{11},\dots,x_{1n})-f(t,x_{21},\dots,x_{2n})]-g(t)[x_{1n}-x_{2n}]\}\times\\ \times\operatorname{sign}[x_{1n}-x_{2n}]\le\sum_{j=k_0+1}^{n}h_j(t)|x_{1j}-x_{2j}|\\ \text{for }\ t\in\langle a_n,b\rangle, x_1,x_2\in R^n,\end{gathered}\tag{25_1}$$

$$\begin{gathered}\{[f(t,x_{11},\dots,x_{1n})-f(t,x_{21},\dots,x_{2n})]-g(t)[x_{1n}-x_{2n}]\}\times\\ \times\operatorname{sign}[x_{1n}-x_{2n}]\ge-\sum_{j=k_0+1}^{n}h_j(t)|x_{1j}-x_{2j}|\\ \text{for }\ t\in\langle a,b_n\rangle,\ x_1,x_2\in R^n,\end{gathered}\tag{25_2}$$

$$\begin{gathered}[\Phi_i(u^{(k_0)},\dots,u^{(n-1)})-\Phi_i(v^{(k_0)},\dots,v^{(n-1)})]\le\\ \le\Psi_i(|u^{(k_0)}-v^{(k_0)}|,\dots,|u^{(n-1)}-v^{(n-1)}|)\\ \text{for }\ u,v\in C^{n-1}(\langle a,b\rangle)\ (i=k_0+1,\dots,n).\end{gathered}\tag{26}$$

Then the problem (1), (2) *has unique solution.*

Proof: Let us put $\omega(t,\varrho)=|f(t,0\dots 0)|$, $r=\max\limits_{i=k_0+1,\dots,n}|\Phi_i(0,\dots,0)|$. From (25), (26) and Theorem 1 follows that problem (1), (2) has a solution. We shall prove its uniqueness.

Let u and v be arbitrary solutions of the problem (1), (2). Put

$$\varrho_i(t)=u^{(i-1)}(t)-v^{(i-1)}(t)\ (i=1,\dots,n).$$

From (25) follows that

$$|\varrho'_n(t)-g(t)\cdot\varrho_n(t)|\le\sum_{j=k_0+1}^{n}h_j|\varrho_j|.\tag{27}$$

From (26) and the character of ℓ_i $(i=k_0+1,\dots,n)$ and Φ_{0i} $(i=k_0+1,\dots,n)$ we have

$$\begin{gathered}\min\{|\varrho_i(t)|:a_i\le t\le b_i\}=\Phi_{0i}(\min\{|\varrho_i(t)|:a_i\le t\le b_i\})\le\\ \le|\Phi_{0i}(\varrho_i)|\le\Psi_i(|\varrho_{k_0+1}|,\dots,|\varrho_n|)\ (i=k_0+1,\dots,n)\\ \ell_i(\varrho_1,\dots,\varrho_{k_0})=0\ \text{ for }\ i=1,\dots,k_0.\end{gathered}\tag{28}$$

Therefore by (6) we have $\varrho_i(t)\equiv 0$ $(i=1,\dots,n)$, i.e. $u(t)\equiv v(t)$. □

Effective criteria

Theorem 3. *Let the inequalities*

$$f(t,x_1,\dots,x_n)\cdot \operatorname{sign} x_n \le \sum_{j=k_0+1}^{n} h_j(t)|x_j| + \omega(t,\sum_{j=1}^{n}|x_j|) \tag{29_1}$$
$$\text{for } \ t\in\langle a_n,b\rangle,\ x\in R^n,$$

$$f(t,x_1,\dots,x_n)\cdot \operatorname{sign} x_n \ge -\sum_{j=k_0+1}^{n} h_j(t)|x_j| - \omega(t,\sum_{j=1}^{n}|x_j|) \tag{29_2}$$
$$\text{for } \ t\in\langle a,b_n\rangle,\ x\in R^n,$$

$$|\Phi_i(u^{(k_0)},\dots,u^{(n-1)})| \le \sum_{j=k_0+1}^{n} r_{ij}\|u^{(j-1)}\|_{L^q\langle a,b\rangle} + r \tag{30}$$
$$\text{for } \ u\in C^{n-1}(\langle a,b\rangle)\ (i=k_0+1,\dots,n)$$

hold, where $r, r_{ij}\in R_+$ $(i,j=k_0+1,\dots,n)$, $\omega:\langle a,b\rangle\times R_+\to R_+$ *is a measurable function nondecreasing in the second variable satisfying* (9), $h_i\in L^p(\langle a,b\rangle,R_+)$, $p\ge 1$*;* $1/p+2/q=1$,

$$\begin{aligned} s_i = \sum_{m=k_0+1}^{n} \{(b-a)^{1/q} \times \sum_{j=i}^{n} [\frac{2(b-a)}{\pi}]^{\frac{2}{q}(j-i)} (\prod_{k=i}^{j-1}\triangle_k) r_{jm} + \\ +[\frac{2(b-a)}{\pi}]^{\frac{2}{q}(n+1-i)} (\prod_{k=i}^{n-1}\triangle_k) h_{0m}\} < 1 \ (i=k_0+1,\dots,n), \end{aligned} \tag{31}$$

where

$$\triangle_k = \max\{(b-a_k)^{1-\frac{2}{q}}, (b_k-a)^{1-\frac{2}{q}}\} \quad (k=k_0+1,\dots,n),$$
$$h_{0m} = \max\{\|h_m\|_{L^p(\langle a,b_m\rangle)},\ \|h_m\|_{L^p(\langle a_m,b\rangle)}\} \quad (m=k_0+1,\dots,n).$$

Then the problem (1), (2) *has a solution.*

Theorem 4. *Let the inequalities*

$$\begin{aligned} &[f(t,x_{11},\dots,x_{1n}) - f(t,x_{21},\dots,x_{2n})]\operatorname{sign}[x_{1n}-x_{2n}] \le \\ &\le \sum_{j=k_0+1}^{n} h_j(t)|x_{1j}-x_{2j}| \end{aligned} \tag{32_1}$$
$$\text{for } \ t\in\langle a_n,b\rangle,\ x_1,x_2\in R^n,$$

$$(32_2)\qquad [f(t,x_{11},\dots,x_{1n})-f(t,x_{21},\dots,x_{2n})]\,\mathrm{sign}\,[x_{1n}-x_{2n}]\ge$$
$$\ge-\sum_{j=k_0+1}^{n}h_j(t)|x_{1j}-x_{2j}|$$
$$\text{for}\ \ t\in\langle a,b_n\rangle,\ x_1,x_2\in R^n,$$

$$(33)\qquad |\Phi_i(u^{(k_0)},\dots,u^{(n-1)})-\Phi_i(v^{(k_0)},\dots,v^{(n-1)})|\le$$
$$\le\sum_{j=k_0+1}^{n}r_{ij}\|u^{(j-1)}-v^{(j-1)}\|_{L^q(\langle a,b\rangle)}$$
$$\text{for}\ \ u,v\in C^{n-1}(\langle a,b\rangle)\ (i=k_0+1,\dots,n)$$

hold, where the functions h_i and constants r_{ij} and s_i satisfy the assumptions of Theorem 3. Then the problem (1), (2) *has unique solution.*

We consider the differential equation

$$(34)\qquad u''=f(t,u,u')$$

with boundary condition

$$(35_1)\qquad \ell(u)=\int_a^b p(t)\cdot u(t)\,dt+\xi u(t_0)=0$$

$$(35_2)\qquad \Phi_{02}(u')=\Phi_2(u')$$

where $f:\langle a,b\rangle\times R^2\to R$ satisfies the local Carathéodory condition and $p(t)\in C(\langle a,b\rangle)$, $\xi\in R$, $t_0\in\langle a,b\rangle$, Φ_{02} — the linear non-decreasing continuous functional on $C(\langle a,b\rangle)$ is concentrated on $\langle a_2,b_2\rangle\subset\langle a,b\rangle$ (e.g.

$$\Phi_{02}(u')=\int_{a_2}^{b_2}q(t)\cdot u'(t)\,dt,$$

$q(t)\in C(\langle a,b\rangle,R_+))$.
$\Phi_2:C(\langle a,b\rangle)\to R$ is a continuous functional.

Theorem 5. *Let the inequalities*

$$(36_1)\qquad f(t,x_1,x_2)\cdot\mathrm{sign}\ x_2\le h(t)\cdot|x_2|+\omega(t,\sum_{i=1}^{2}|x_i|)$$

for $a_2\le t\le b$, $(x_1,x_2)\in R^2$,

$$(36_2)\qquad f(t,x_1,x_2)\cdot\mathrm{sign}\ x_2\ge -h(t)\cdot|x_2|-\omega(t,\sum_{i=1}^{2}|x_i|)$$

for $a \leq t \leq b_2$, $(x_1, x_2) \in R^2$.

(37) $$|\Phi_2(u')| \leq m.\|u'\|_{L^2(\langle a,b\rangle)} + r$$

hold, where $m, r \in R_+$, $h(t) \in L^2(\langle a,b\rangle, R_+)$,

$$\sqrt{b-a}(m + \|h\|_{L^2(\langle a,b\rangle)}) < 1, \int_a^b p(t)\,dt + \xi \neq 0,$$

$\omega : \langle a,b\rangle \times R_+ \to R_+$ *is a measurable function nondecreasing in the second variable satisfying* (9).
Then the problem (34), (35) *has at least one solution.*

Proof: We put

$$g(t) \equiv 0;\ \psi_2(|x_2|) = m \cdot \|x_2\|_{L^2(\langle a,b\rangle)}$$

for $x_2 \in C(\langle a,b\rangle)$.
By Theorem 1 we must prove that the data (g, h, ℓ, ψ_2) are of the class $LN(\langle a,b\rangle, a_2, b_2)$. Let the vector $(\varrho_1(t), \varrho_2(t))$ be the solution of the problem (38),

(38_1) $$|\varrho_1'(t)| \leq |\varrho_2(t)| \quad a \leq t \leq b$$

(38_2) $$|\varrho_2'(t)| \leq h(t)|\varrho_2(t)| \quad a \leq t \leq b$$

with boundary condition

(39_1) $$\ell(\varrho_1) = \int_a^b p(t) \cdot \varrho_1(t)\,dt + \xi \cdot \varrho_1(t_0) = 0$$

(39_2) $$\min\{|\varrho_2(t)| : a_2 \leq t \leq b_2\} \leq m\|\varrho_2\|_{L^2(\langle a,b\rangle)}.$$

We shall prove that this solution is zero. Let us choose $\tau_0 \in \langle a_2, b_2\rangle$ so that

$$|\varrho_2(\tau_0)| = \min\{|\varrho_2(t)| : a_2 \leq t \leq b_2\}.$$

Then integrating relation (38_2) and using Hölder inequality we obtain

$$\begin{aligned}|\varrho_2(t)| &\leq |\varrho_2(\tau_0)| + \left|\int_{\tau_0}^t h(s)|\varrho_2(s)|\,ds\right| \\ &\leq m\|\varrho_2\|_{L^2(\langle a,b\rangle)} + \left|\int_{\tau_0}^b h(s)|\varrho_2(s)|\,ds\right|\end{aligned}$$

and

$$\begin{aligned}\|\varrho_2\|_{L^2(\langle a,b\rangle)} \leq \sqrt{b-a}&(m + \|h\|_{L^2(\langle a,b\rangle)})\times \\ &\times\|\varrho_2\|_{L^2(\langle a,b\rangle)}.\end{aligned}$$

Since $\sqrt{b-a} \cdot (m + \|h\|_{L^2(\langle a,b\rangle)}) < 1$, it follows that $\varrho_2(t) \equiv 0$.
From (38_1) we have

$$\varrho_1(t) \equiv C = \text{ const.}$$

The relation (39_1) implies that $\varrho_1(t) \equiv 0$, because $\int_a^b p(t)\,dt + \xi \neq 0$. □

Theorem 6. *Let the inequalities*

$$[f(t,x_{11},x_{12})-f(t,x_{21},x_{22})]\cdot\operatorname{sign}[x_{12}-x_{22}]\le$$
$$\le h(t)|x_{12}-x_{22}|$$

for $a_2\le t\le b$; $(x_{11},x_{12}),(x_{21},x_{22})\in R^2$,

$$[f(t,x_{11},x_{12})-f(t,x_{21},x_{22})]\cdot\operatorname{sign}[x_{12}-x_{22}]\ge$$
$$\ge -h(t)|x_{12}-x_{22}|$$

for $a\le t\le b$, $(x_{11},x_{12}),(x_{21},x_{22})\in R^2$,

$$|\Phi_2(u')-\Phi_2(v')|\le m\|u'-v'\|_{L^2(\langle a,b\rangle)}$$

for $u,v\in C^1(\langle a,b\rangle)$ *hold, where the functionals* h *and* m *satisfy the assumptions of Theorem* 5. *Then the problem* (34), (35) *has unique solution.*

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