Criteria for weak compactness
of vector-valued integration maps

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Abstract. Criteria are given for determining the weak compactness, or otherwise, of the integration map associated with a vector measure. For instance, the space of integrable functions of a weakly compact integration map is necessarily normable for the mean convergence topology. Results are presented which relate weak compactness of the integration map with the property of being a bicontinuous isomorphism onto its range. Finally, a detailed description is given of the compactness properties for the integration maps of a class of measures taking their values in \( \ell^1 \), equipped with various weak topologies.

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Introduction

The importance of vector measures in modern analysis is well established. An important aspect of the theory is the integration map. Associated with each \( X \)-valued measure \( \mu \), with \( X \) a locally convex space (briefly, lcs), is its integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) given by \( f \mapsto \int f \, d\mu \), for every \( f \in \mathcal{L}^1(\mu) \). Here \( \mathcal{L}^1(\mu) \) is the space of all \( \mathbb{C} \)-valued, \( \mu \)-integrable functions; it is a lcs for the mean convergence topology (see Section 1). Many classical operators, such as the Fourier transform in \( L^1(\mathbb{T}) \), certain integral operators (e.g. Volterra), representations for Boolean algebras of projections (arising from normal operators) and so on, can be viewed as integration maps \( I_\mu \) (or restrictions of such maps) for suitable measures \( \mu \) and spaces \( X \).

Properties of the operator \( I_\mu \), which is always linear and continuous, are closely related to the nature of the lcs \( \mathcal{L}^1(\mu) \). For \( X \) a Banach space, the compactness properties of \( I_\mu \) are investigated in detail in [5]. It turns out that such compactness results are not always a reliable guide as what to expect for \( X \) a lcs; the theory in such spaces (see [6]) is generally not attained from the Banach space case by simply replacing norms with seminorms. Genuinely new phenomena and difficulties occur.

Curiously though, all the examples exhibited in [6; §3] of compact or weakly compact (briefly, \( w \)-compact) integration maps \( I_\mu \) have the property that the lcs \( \mathcal{L}^1(\mu) \) is normable, although \( \mu \) itself takes its values in a non-normable lcs \( X \).
One of the aims of this note is to show that this is not a coincidence, but a general phenomenon. In particular, it provides a criterion for deciding about $w$-compactness of $I_{\mu}$; if $L^1(\mu)$ is not normable, then $I_{\mu}$ cannot be $w$-compact. Here, $w$-compactness is meant in the sense of Grothendieck, that is, some neighbourhood of zero is mapped into a relatively $w$-compact set. We also exhibit other criteria which are either necessary or sufficient for compactness (resp. $w$-compactness) of $I_{\mu}$. Several results are given which relate the $w$-compactness of $I_{\mu}$ with the property of $I_{\mu}$ being a bicontinuous isomorphism onto its range. For instance, if $X$ is a Fréchet space and $I_{\mu}$ is $w$-compact, then $I_{\mu}$ cannot be a bicontinuous isomorphism onto its range. Examples are given of a class of measures $\mu$ in $\ell^1$, considered not as a Banach space, but as a lcs equipped with one of the topologies $\sigma(\ell^1, c_0)$ or $\sigma(\ell^1, \ell^\infty)$, for which a complete description of the compactness properties of $I_{\mu}$ is possible.

1. Preliminaries

The continuous dual space of a locally convex Hausdorff space $X$ (briefly, lcHs) is denoted by $X'$. The set of all continuous seminorms on $X$ is denoted by $\mathcal{P}(X)$. The space $X$ equipped with its weak topology $\sigma(X, X')$ is denoted by $X_{\sigma(X,X')}$. The space $X'$ equipped with its weak-star topology $\sigma(X', X)$ is denoted by $X'_{\sigma(X',X)}$. We adopt the notation $\langle x', x \rangle = x'(x)$ for every $x \in X$ and $x' \in X'$. Given an $X$-valued set function $m$ on a $\sigma$-algebra of sets and $x' \in X'$, let $\langle x', m \rangle$ denote the set function given by $\langle x', m \rangle(E) = \langle x', m(E) \rangle$ for every set $E$ in the domain of $m$.

Let $\mathcal{S}$ be a $\sigma$-algebra of subsets of a non-empty set $\Omega$. Let $\mu : \mathcal{S} \to X$ be a vector measure, that is, a $\sigma$-additive set function. For every $x' \in X'$, the total variation measure of the scalar measure $\langle x', \mu \rangle$ is denoted by $|\langle x', \mu \rangle|$. Given $p \in \mathcal{P}(X)$, let $U^0_p = \{x' \in X' ; |\langle x', x \rangle| \leq 1, x \in p^{-1}([0,1])\}$. The $p$-semivariation of $\mu$ is the set function $p(\mu)$ given by

$$p(\mu)(E) = \sup\{|\langle x', \mu \rangle|(E); \ x' \in U^0_p\}, \quad E \in \mathcal{S}.$$

A scalar-valued, $\mathcal{S}$-measurable function $f$ on $\Omega$ is called $\mu$-integrable if it is $\langle x', \mu \rangle$-integrable, for every $x' \in X'$, and if there is a unique function $f \mu : \mathcal{S} \to X$ satisfying

$$\langle x', (f \mu)(E) \rangle = \int_E f \, d\langle x', \mu \rangle, \quad x' \in X', \ E \in \mathcal{S}.$$

In this case, $f \mu$ is also $\sigma$-additive by the Orlicz-Pettis lemma, and will be called the indefinite integral of $f$ with respect to $\mu$. We also use the classical notation

$$\int_E f \, d\mu = (f \mu)(E), \quad E \in \mathcal{S}.$$

The vector space of all $\mu$-integrable functions on $\Omega$ is denoted by $L^1(\mu)$. An element of $L^1(\mu)$ is called $\mu$-null if its indefinite integral is the zero measure.
The subspace of $L^1(\mu)$ consisting of all $\mu$-null functions is denoted by $N(\mu)$. For every $p \in P(X)$, the seminorm $f \mapsto p(f \mu)(\Omega)$, for $f \in L^1(\mu)$, is also denoted by $p(\mu)$. The space $L^1(\mu)$ is equipped with the lc-topology defined by the family of seminorms $p(\mu)$, $p \in P(X)$. This topology is called the mean convergence topology. The lcHs associated with $L^1(\mu)$ is the quotient space $L^1(\mu)/N(\mu)$.

The integration map $I_\mu : L^1(\mu) \to X$ is defined by
$$I_\mu(f) = (f \mu)(\Omega) = \int_{\Omega} f \, d\mu,$$
$f \in L^1(\mu)$.

It is clear that $I_\mu$ is linear and continuous.

**Definition 1.1.** The measure $\mu : S \to X$ is said to factor through a lcHs $Y$ if there exist a vector measure $\nu : S \to Y$ and a continuous linear map $j : Y \to X$ such that
\begin{enumerate}
\item $L^1(\mu) = L^1(\nu)$ as lcs,
\item $N(\mu) = N(\nu)$ as sets, and
\item $I_\mu = j \circ I_\nu$.
\end{enumerate}
In this case we say that $\mu$ factors through $Y$ (via $\nu$ and $j$); see [6; §1].

**Lemma 1.2.** Let $j$ be a continuous linear map from a lcHs $Y$ into a lcHs $X$ and $\nu : S \to Y$ be a vector measure. Let $\mu = j \circ \nu$. Suppose that $\mu$ factors through $Y$ via $\nu$ and $j$. Then, if the integration map $I_\nu : L^1(\nu) \to Y$ is w-compact (resp. compact, nuclear) so is the integration map $I_\mu : L^1(\mu) \to X$.

**Proof:** The statements for compact and w-compact maps are clear. For the case concerning nuclear maps see [8; Proposition 47.1].

**Remark 1.3.** It is shown in Section 3 (see Example 3.3) that the converse of Lemma 1.2 is not always valid.

**Lemma 1.4.** Let $Y$ be a lcHs and $\nu : S \to Y$ be a vector measure. Let $X = Y_{\sigma(Y',Y')}$ and $j : Y \to X$ be the identity map. Suppose that the measure $\mu = j \circ \nu$ factors through $Y$ via $\nu$ and $j$. Then the integration map $I_\mu : L^1(\mu) \to X$ is compact (= w-compact), if and only if, the integration map $I_\nu : L^1(\nu) \to Y$ is w-compact.

**Proof:** Follows from the fact that a subset $A$ of $Y$ is w-compact, if and only if, $j(A)$ is compact in $X$.

We conclude this section with a technical lemma needed later.

**Lemma 1.5.** Let $Z$ be a Banach space and $Z'$ be the dual Banach space. Let $j : Z' \to Z'_{\sigma(Z',Z)}$ be the identity map. A continuous linear map $T$ from a Banach space $W$ into $Z'$ is nuclear, if and only if, $j \circ T : W \to Z'_{\sigma(Z',Z)}$ is nuclear.

**Proof:** Since $Z'_{\sigma(Z',Z)}$ is quasicomplete, it follows from [8; Corollary 1, p. 482] that there exist a bounded sequence $\{w'_n\}_{n=1}^{\infty}$ in $W'$, a bounded sequence $\{z'_n\}_{n=1}^{\infty}$
in $Z'_{\sigma(Z',Z)}$ and an absolutely convergent series of scalars $\sum_{n=1}^{\infty}a_n$ such that

$$(j \circ T)w = \sum_{n=1}^{\infty} a_n \langle w_n', w \rangle z_n', \quad w \in W.$$ 

Since $\sum_{n=1}^{\infty} |a_n| \langle w_n', w \rangle \| z_n' \|$ is finite, we have

$$Tw = \sum_{n=1}^{\infty} a_n \langle w_n', w \rangle j^{-1}(z_n'), \quad w \in W.$$ 

Again by [8; Corollary 1, p. 482], $T$ is nuclear.

The converse statement is clear. \qed

2. $w$-Compactness criteria

In this section we present some general criteria which are sufficient to guarantee compactness and/or $w$-compactness of integration maps.

A lcs $Z$ is called seminormable if its topology is the same as that determined by a single seminorm. If $Z$ is Hausdorff then, of course, the single seminorm is a norm and we use the term normable. If, in addition, $Z$ is sequentially complete, then it must be complete for this norm, that is, $Z$ is a Banach space.

**Proposition 2.1.** Let $X$ be a lcHs and $\mu : S \to X$ be a vector measure. Then the following two statements are equivalent.

(i) There is a neighbourhood $V$ of 0 in $L^1(\mu)$ such that its image $I_{\mu}(V)$ is a bounded subset of $X$.

(ii) The lcs $L^1(\mu)$ is seminormable (i.e. the quotient space $L^1(\mu)/N(\mu)$ is normable).

If $X$ is sequentially complete, then either of (i) or (ii) is equivalent to the following statement.

(iii) The lcs $L^1(\mu)$ is a complete seminormed space (i.e. the quotient space $L^1(\mu)/N(\mu)$ is a Banach space).

**Proof:** The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear. So, suppose that (i) holds. Take a seminorm $p \in \mathcal{P}(X)$ satisfying

$$\{g \in L^1(\mu); \ p(\mu)(g) \leq 1\} \subseteq V.$$ 

Denote the left-hand-side of (1) by $V_p$. Let $q \in \mathcal{P}(X)$ be arbitrary. The boundedness of $I_{\mu}(V_p)$ implies that

$$I_{\mu}(V_p) \subseteq C_q \{x \in X; \ q(x) \leq 1\},$$

for some positive constant $C_q$. Let $g \in L^1(\mu)$. If $p(\mu)(g) \neq 0$, then it follows easily that

$$q(I_{\mu}g) \leq C_q p(\mu)(g).$$
If \( p(\mu)(g) = 0 \), then \( \alpha g \in V_p \subseteq V \) and so \( \alpha I_\mu g \in I_\mu(V_p) \), for all scalars \( \alpha \). Since \( I_\mu(V_p) \) is bounded, this forces \( I_\mu g = 0 \) and so again (2) holds. Accordingly, (2) holds for every \( g \in \mathcal{L}^1(\mu) \). It then follows from [2; Ch.I, Proposition 1.11] that

\[
q(\mu)(g) \leq 4 \sup_{E \in S} q \left( \int_E g \, d\mu \right) \leq 4Cq p(\mu)(g), \quad g \in \mathcal{L}^1(\mu).
\]

This shows that the mean convergence topology on \( \mathcal{L}^1(\mu) \) can be defined by the single seminorm \( p(\mu) \). In other words, (ii) holds. A further consequence is that there is a finite measure \( \lambda : \mathcal{S} \to [0, \infty) \) with respect to which the set functions \( q(\mu) \), for \( q \in \mathcal{P}(X) \), are absolutely continuous; that is, \( q(\mu)(E) \to 0 \) for all \( q \in \mathcal{P}(X) \) as \( \lambda(E) \to 0 \), \( E \in \mathcal{S} \). This follows from the fact that there is a finite measure on \( \mathcal{S} \) with respect to which the set function \( p(\mu) \) is absolutely continuous; see [4; Ch.II, Theorem 1.1], for example. It follows that the scalar measures \( \langle x', \mu \rangle \), for \( x' \in X' \), are absolutely continuous with respect to \( \lambda \).

Assume now that \( X \) is sequentially complete. Statement (iii) then follows from [4; Ch. IV, Theorem 7.3] and [7; Proposition 2.1].

Since \( w \)-compact sets are bounded, an immediate consequence is that the examples of \( w \)-compact integration maps \( I_\mu \) exhibited in [6], namely Examples 3.1 and 3.2 and Proposition 3.8, necessarily have normable spaces \( \mathcal{L}^1(\mu) \). Proposition 2.1 can also be used to check that an integration map is not \( w \)-compact. For instance, the lcHs \( X \) in Example 1.7 of [6] is quasicomplete and it was shown for the vector measure \( \mu : \mathcal{S} \to X \) given there, that \( \mathcal{L}^1(\mu) \) is not normable. So, by Proposition 2.1, the associated integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) is not \( w \)-compact.

We now consider the connection between \( w \)-compactness of \( I_\mu \) and the property of \( I_\mu \) being a bicontinuous isomorphism onto its range.

Let \( T \) be a continuous linear map from a lcHs \( U \) into a lcHs \( W \). We say that \( T \) factors through a lcHs \( Z \) if there exist continuous linear maps \( R : U \to Z \) and \( S : Z \to W \) such that \( T = S \circ R \).

**Remark 2.2.** (i) Let \( X \) be a non-reflexive Pták space. Let \( T \) be a bijective, continuous linear map from \( X \) onto a lcHs \( Y \). Then \( T \) does not factor through any reflexive Banach space, [6; Lemma 3.5]. We note that every Fréchet lcs is a Pták space and hence, in particular, Banach spaces are Pták spaces.

(ii) If a vector measure \( \mu : \mathcal{S} \to X \) factors through a lcHs \( Y \) (cf. Definition 1.1), then the associated integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) also factors through \( Y \). \( \square \)

For clarity of presentation, in the remainder of this paper the space \( \mathcal{L}^1(\mu) \) of all \( \mu \)-integrable functions, for a given vector measure \( \mu \), will be identified with its associated Hausdorff space \( \mathcal{L}^1(\mu)/\mathcal{N}(\mu) \).

**Proposition 2.3.** Let \( X \) be a Fréchet lcs and \( \mu : \mathcal{S} \to X \) be a vector measure such that \( \mathcal{L}^1(\mu) \) is a non-reflexive Fréchet space.

(i) If the integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) is an injective, continuous linear map with closed range, then \( I_\mu \) cannot be \( w \)-compact.
(ii) If $I_\mu : \mathcal{L}^1(\mu) \to X$ is $w$-compact, then $I_\mu$ cannot be a bicontinuous isomorphism onto its range.

Proof: (i) Suppose $I_\mu$ were $w$-compact, where we consider $\mu$ and $I_\mu$ as taking their values in the Fréchet lcs $X = I_\mu(\mathcal{L}^1(\mu))$. By Remark 2.5 of [6], applied to $X$, the measure $\mu$ would factor through a reflexive Banach space $Y$ and hence, the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ would factor through $Y$, by Remark 2.2(ii). This contradicts Remark 2.2(i).

(ii) If $I_\mu$ were a bicontinuous isomorphism onto its range $Z = I_\mu(\mathcal{L}^1(\mu))$, then $Z$ would be a Fréchet lcs and so, by part (i), $I_\mu : \mathcal{L}^1(\mu) \to Z$ could not be $w$-compact. This contradicts the hypothesis. □

We note that Proposition 2.3(ii) implies immediately that the $w$-compact integration map $I_\mu$ of Example 3.2 in [6] cannot be a bicontinuous isomorphism onto its range.

Remark 2.4. A slight variation of Proposition 2.3(i) is as follows: Let $\mu : S \to X$ be a vector measure with values in a non-normable lcHs $X$ such that its integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is a bicontinuous isomorphism of $\mathcal{L}^1(\mu)$ onto $X$. Then $I_\mu$ cannot be $w$-compact.

For, otherwise $X$ would have a bounded neighbourhood of 0, which would force $X$ to be normable. □

Example 2.5. Let $\mathbb{N}$ denote the natural numbers. Let $X = \mathbb{C}^{\mathbb{N}}$, equipped with the seminorms given by

$$q_n : x \mapsto \max_{1 \leq r \leq n} |x_r|, \quad x = (x_j)_{j=1}^\infty \in X,$$

for each $n = 1, 2, \ldots$. Then $X$ is a separable, reflexive Fréchet space. Let $S = 2^{\mathbb{N}}$ and $\mu(E) = \chi_E$, for each $E \in S$. Then $I_\mu$ is a bicontinuous isomorphism of $\mathcal{L}^1(\mu)$ onto $X$ (see Remark 2.7 below) and hence, $I_\mu$ is not $w$-compact (by Remark 2.4). This is despite the fact that $X$ is reflexive; for reflexive Banach spaces $X$ this cannot happen as $I_\mu$ is always $w$-compact in such spaces. □

We now exhibit a class of measures $\mu$ for which the criterion given by Remark 2.4 is especially effective; Example 2.5 is a particular case of such a measure $\mu$.

Let $X$ be a lcHs and $L(X)$ be the space of all continuous linear operators of $X$ into $X$. With respect to the topology of pointwise convergence in $X$ (i.e. the strong operator topology), $L(X)$ is also a lcHs; it is denoted by $L_s(X)$. For the definition of a spectral measure $P : S \to L_s(X)$ we refer to [3]. These are generalizations of the resolution of the identity for normal operators in Hilbert space. A spectral measure $P$ is called equicontinuous if its range $P(S)$ is an equicontinuous subset of $L(X)$. Given $x \in X$, the cyclic space $P(S)[x]$ generated by $x$ with respect to $P$ is defined to be the closed linear span of the set $\{P(E)x ; E \in S\}$. For each $x \in X$, let $Px : S \to X$ denote the $X$-valued measure $E \mapsto P(E)x$, $E \in S$. 


Proposition 2.6. Let $X$ be a quasicomplete lcHs such that $L_s(X)$ is sequentially complete and $P : S \to L_s(X)$ be an equicontinuous spectral measure with range $P(S)$ a closed subset of $L_s(X)$.

(i) For each $x \in X$, the integration map $I_{P \cdot x} : L^1(P \cdot x) \to X$ is a bicontinuous isomorphism of $L^1(P \cdot x)$ onto the cyclic space $P(S)[x]$.

(ii) If the cyclic space $P(S)[x]$ is non-normable, then the integration map $I_{P \cdot x}$ is not $w$-compact.

Proof: Part (i) is just [3; Proposition 2.1], while part (ii) follows from (i) and Remark 2.4.

We note that the condition of the range $P(S)$ being closed in $L_s(X)$ is automatically satisfied in separable Fréchet spaces, [3], [7].

Remark 2.7. The claim made in Example 2.5 that the integration map $I_\mu$ given there is a bicontinuous isomorphism onto $X = C^N$ follows from Proposition 2.6. For, in the notation of Example 2.5, given a subset $E$ of $\mathbb{N}$ define the projection $P(E)$ by $P(E) x = \chi_E \cdot x$ (coordinatewise multiplication), for each $x \in X$. Since $X$ is barrelled, the spectral measure is necessarily equicontinuous. Moreover, as $X$ is a separable Fréchet space, $P(S)$ is a closed subset of $L_s(X)$. In addition, the element $1 1 \in X$ (consisting of 1 in every co-ordinate) is a cyclic vector for $P$, that is, $P(S)[1 1] = X$. Since $\mu = P(1 1)$, we can apply Proposition 2.6.

3. Examples

In this section we exhibit some examples of measures in lc-spaces which arise from Banach spaces with their weak or weak-star topologies. For the particular Banach space $\ell^1$ quite detailed information is available. The dual operator to a continuous linear operator $T$ between lc-spaces is denoted by $T'$.

Proposition 3.1. Let $j$ be an injective, continuous linear map from the Banach space $\ell^1$ into a lcHs $X$ such that $(j')^{-1}(\{f_1\}) \neq \phi$, where $f_1 = (1, 0, 0, \ldots)$ is considered as an element of $\ell^\infty$. Let $\lambda : S \to [0, \infty)$ be a finite measure. Let $g_1 = 1 1$ be the function constantly equal to 1 and $g_n \in L^\infty(\lambda)$, $n = 2, 3, \ldots$, satisfy

$$\sum_{n=1}^\infty |\langle g_n, f \rangle| < \infty, \quad f \in L^1(\lambda).$$

Let $e_n$, $n \in \mathbb{N}$, be the standard basis vectors of $\ell^1$ and $\nu : S \to \ell^1$ be the vector measure given by

$$\nu(E) = \sum_{n=1}^\infty \langle g_n, \chi_E \rangle e_n, \quad E \in S.$$

Finally, let $\mu = j \circ \nu$. Then the following statements hold.

(i) The measure $\mu : S \to X$ factors through $\ell^1$ via $\nu$ and $j$. 

(ii) If \( \{g_n\}_{n=1}^{\infty} \) is unconditionally summable in \( \mathcal{L}^\infty(\lambda) \), then the integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) is compact.

(iii) If \( \{g_n\}_{n=1}^{\infty} \) is absolutely summable in \( \mathcal{L}^\infty(\lambda) \), then the integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) is nuclear.

**Proof:** (i) The continuity of \( j \) implies that \( \mathcal{L}^1(\nu) \subseteq \mathcal{L}^1(\mu) \). Choose a vector \( x' \in X' \) such that \( j'(x') = f_1 \), in which case

\[
\langle x', \mu \rangle = \langle x', j \circ \nu \rangle = \langle j'(x'), \nu \rangle = \langle f_1, \nu \rangle = \lambda,
\]

and so \( \mathcal{L}^1(\mu) \subseteq \mathcal{L}^1(\lambda) = \mathcal{L}^1(\nu) \). Accordingly, \( \mathcal{L}^1(\mu) = \mathcal{L}^1(\nu) = \mathcal{L}^1(\lambda) \) as vector spaces. By (4) we conclude that \( \mathcal{L}^1(\mu) \) and \( \mathcal{L}^1(\lambda) \) are isomorphic. The identity \( I_\mu = j \circ I_\nu \) is a consequence of the fact that the \( \mathcal{S} \)-simple functions are dense in both \( \mathcal{L}^1(\mu) \) and \( \mathcal{L}^1(\nu) \). The equality \( \mathcal{N}(\mu) = \mathcal{N}(\nu) \) follows from the injectivity of \( j \). Hence, (i) holds.

Statements (ii) and (iii) follow from part (i), Lemma 1.2 and [5; Proposition 3.6].

Special choices of the space \( X \) in Proposition 3.1 give a way of producing integration maps with specific properties.

**Corollary 3.1.1.** Let \( X = \ell^1_{(\ell^1, c_0)} \) and \( j : \ell^1 \to X \) be the identity map. Let the measure \( \lambda \), the sequence \( \{g_n\}_{n=1}^{\infty} \) in \( \mathcal{L}^\infty(\lambda) \) and the vector measure \( \nu \) be as in Proposition 3.1. Let \( \mu = j \circ \nu \).

(i) The measure \( \mu : \mathcal{S} \to X \) factors through the Banach space \( \ell^1 \) via \( \nu \) and \( j \).

(ii) The integration map \( I_\mu : \mathcal{L}^1(\mu) \to X \) is compact (= \( w \)-compact).

(iii) \( I_\mu \) is nuclear, if and only if, \( I_\nu \) is nuclear.

(iv) If the Banach space \( \mathcal{L}^1(\lambda) \) is infinite-dimensional, then the integration map \( I_\mu \) is not a bicontinuous isomorphism onto its range.

**Proof:** (i) Let \( f_1 \in \ell^\infty \) be as in Proposition 3.1. Since \( j'(f_1) = f_1 \), Proposition 3.1(i) implies (i).

(ii) Since \( I_\mu = j \circ I_\nu \) with \( j \) compact, it follows that \( I_\nu \) is compact.

(iii) See Lemma 1.5.

(iv) By the proof of Proposition 3.1, the spaces \( \mathcal{L}^1(\mu) \) and \( \mathcal{L}^1(\lambda) \) are isomorphic Banach spaces; in particular, \( \mathcal{L}^1(\mu) \) is non-reflexive. Statement (iv) follows from (ii).

**Corollary 3.1.2.** Let \( X = \ell^1_{(\ell^1, \ell^\infty)} \) and \( j : \ell^1 \to X \) be the identity map. Let the measure \( \lambda \), the sequence \( \{g_n\}_{n=1}^{\infty} \) in \( \mathcal{L}^\infty(\lambda) \) and the measure \( \nu \) be as in Proposition 3.1. Let \( \mu = j \circ \nu \).

(i) The measure \( \mu : \mathcal{S} \to X \) factors through the Banach space \( \ell^1 \) via \( \nu \) and \( j \).
(ii) The integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is compact (= w-compact), if and only if, the integration map $I_\nu : \mathcal{L}^1(\nu) \to \ell^1$ is compact.

(iii) The integration map $I_\mu$ is nuclear, if and only if, the integration map $I_\nu$ is nuclear.

(iv) If the Banach space $\mathcal{L}^1(\lambda)$ is infinite-dimensional, then the integration map $I_\mu$ is not a bicontinuous isomorphism onto its range.

**Proof:** Part (i) follows as in the proof of Corollary 3.1.1 (i). Part (ii) is a consequence of part (i) and Lemma 1.4.

(iii) Let $Z = \ell^1_{\sigma(\ell^1,c_0)}$ and $k : X \to Z$ be the identity map. Then the measure $k \circ \mu : S \to Z$ factors through $X$ via $\mu$ and $k$ so that $I_{k\circ\mu} = k \circ I_\mu$. By part (i), we have $j \circ I_\nu = I_\mu$, and hence, $I_{k\circ\mu} = k \circ I_\mu = (k \circ j) \circ I_\nu$. Therefore, if $I_\mu$ is nuclear, then so is $I_{k\circ\mu}$ and hence, $I_\nu$ is nuclear by Corollary 3.1.1 (iii). The converse implication is clear.

(iv) If $I_\mu$ were a bicontinuous isomorphism then, on the infinite-dimensional linear subspace $I_\nu(\mathcal{L}^1(\lambda)) = j^{-1}(I_\mu(\mathcal{L}^1(\mu)))$ of $\ell^1$, the norm topology and the weak topology would coincide, which is a contradiction. □

**Corollary 3.1.3.** Let $X$ be the Fréchet space $\mathbb{C}^N$ and $j : \ell^1 \to X$ be the natural injection. Let the measure $\lambda$, the sequence $\{g_n\}_{n=1}^\infty$ in $L^\infty(\lambda)$ and the measure $\nu$ be as in Proposition 3.1. Let $\mu = j \circ \nu$.

(i) The measure $\mu : S \to X$ factors through the Banach space $\ell^1$ via $\nu$ and $j$.

(ii) The integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is compact (= w-compact).

(iii) The integration map $I_\mu$ is nuclear, if and only if, the integration map $I_\nu$ is nuclear.

(iv) If the Banach space $\mathcal{L}^1(\lambda)$ is infinite-dimensional, then the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is not an isomorphism onto its range.

**Proof:** (i) The arguments in the proof of Corollary 3.1.1 (i) apply.

(ii) Since $X$ is a Montel space, the map $j$ is compact. Hence, $I_\mu = j \circ I_\nu$ is compact and thus, also w-compact.

(iii) Since $\mathcal{L}^1(\lambda) = \mathcal{L}^1(\mu)$ is barrelled and $X$ is complete, statement (iii) can be proved as in Corollary 3.1.1 (iii) by using the analogue of Lemma 1.5 with $Z = \mathbb{C}^N$; again apply [8; Corollary 1, p. 482].

(iv) Use the same argument as in the proof of Corollary 3.1.1 (iv). □

**Remark 3.2.** In relation to the previous three corollaries it may be worth noting that the lcHs $\ell^1_{\sigma(\ell^1,c_0)}$ is a semireflexive, quasicomplete Montel space, that $\mathbb{C}^N$ is a complete, reflexive, Fréchet-Montel space, but that $\ell^1_{\sigma(\ell^1,\ell^\infty)}$ is neither semireflexive, Montel nor quasicomplete (it is sequentially complete). Of course, a continuous linear map from a lcHs into a Montel space is compact, if and only if, it is w-compact. This comment is relevant to Corollary 3.1.1 (ii) and Corollary 3.1.2 (ii). □
We can now exhibit an example showing that the converse of Lemma 1.2 fails (cf. Remark 1.3).

**Example 3.3.** Let \( S \) be the \( \sigma \)-algebra of Borel subsets of \([0, 1]\) and \( \lambda \) be Lebesgue measure on \( S \). Let \( g_1 = 1 \) and \( g_n = \chi_{E(n)} \), where \( E(n) = ((n + 1)^{-1}, n^{-1}] \) for each \( n = 2, 3, \ldots \). Since \( \{g_n\}_{n=1}^{\infty} \) is not unconditionally summable in \( L^\infty(\lambda) \), the integration map \( I_\nu : S \to \ell^1 \) (with \( \nu \) given by (3)) is not compact, [5; Proposition 3.6]. Let \( X = \ell^1_{\sigma(\ell^1, c_0)} \) and \( j : \ell^1 \to X \) be the identity map. It follows from Proposition 3.1 (i) that the measure \( \mu = j \circ \nu \) factors through \( \ell^1 \) via \( \nu \) and \( j \). Moreover, since \( j \) is a compact map and \( I_\mu = j \circ I_\nu \), it follows that \( I_\mu : \ell^1(\mu) \to X \) is compact.

We have already seen in the above example that the converse of Lemma 1.2 is not valid. However, for a particular setting, the converse does hold.

**Proposition 3.4.** Let \( Y \) be a lcHs and \( X = Y_{\sigma(Y,Y')} \). Let \( j : Y \to X \) be the identity map and \( \nu : S \to Y \) be a vector measure. Let \( \mu = j \circ \nu \). Suppose that the integration map \( I_\mu : \ell^1(\mu) \to X \) is w-compact. Then so is the integration map \( I_\nu : \ell^1(\nu) \to Y \).

**Proof:** By assumption, there is a neighbourhood \( V \) of 0 in \( \ell^1(\mu) \) whose image \( I_\mu(V) \) is relatively w-compact in \( X \). The set \( V \) is a neighbourhood of 0 also in \( \ell^1(\nu) \) because \( \ell^1(\mu) = \ell^1(\nu) \) as vector spaces and because the mean convergence topology on \( \ell^1(\nu) \) is stronger than that on \( \ell^1(\mu) \). Hence, \( I_\nu \) is w-compact because \( I_\nu(V) = I_\mu(V) \) is relatively w-compact in \( Y \).

The converse of Proposition 3.4 is not always valid. A counter-example will be given in the case when \( Y = \ell^2 \). It is interesting to know whether or not that is the case when \( Y = \ell^1 \).

**Example 3.5.** Let \( Y \) be the Hilbert space \( \ell^2 \) and \( X = \ell^2_{\sigma(\ell^2, \ell^2)} \). Let \( e_n, n \in \mathbb{N}, \) be the standard basis vectors in \( Y \) and \( \nu : 2^\mathbb{N} \to Y \) be the vector measure given by

\[
\nu(E) = \sum_{n \in E} n^{-1} e_n, \quad E \in 2^\mathbb{N}.
\]

Let \( j : Y \to X \) denote the identity map. Define a vector measure \( \mu : 2^\mathbb{N} \to X \) by \( \mu = j \circ \nu \). Then \( \ell^1(\mu) = \ell^1(\nu) \) (as vector spaces) and this space consists of precisely those functions \( f \) on \( \mathbb{N} \) such that \( \sum_{n=1}^{\infty} |f(n)/n| < \infty \).

Since \( Y \) is reflexive, the integration map \( I_\nu : \ell^1(\nu) \to Y \) is weakly compact. However, we shall show that the integration map \( I_\mu : \ell^1(\mu) \to X \) is not weakly compact. To this end, let \( Z \) denote the space \( \ell^2 \) equipped with the absolute weak topology \( |\sigma(\ell^2, \ell^2) \) (cf. [1; p. 166]). Namely, the topology on \( Z \) is generated by the seminorms \( q_\xi, \xi = (\xi_n)_{n=1}^{\infty} \in \ell^2, \) defined by

\[
q_\xi(x) = \sum_{n=1}^{\infty} |\xi_n x_n|, \quad x = (x_n)_{n=1}^{\infty} \in \ell^2.
\]
Then $|\sigma|(|\ell^2, \ell^2)$ is strictly weaker than the norm topology and strictly stronger than the weak topology. Let $k : Z \to X$ be the identity map and $\eta : 2^N \to Z$ be the vector measure satisfying $\mu = k \circ \eta$. Clearly $L^1(\eta) = L^1(\mu)$ as vector spaces (in fact, as lc spaces). A direct computation shows that the integration map $I_\eta$ is a bicontinuous isomorphism from $L^1(\eta)$ onto $Z$ and hence, $I_\eta$ is not $w$-compact by Remark 2.4 because $Z$ is not normable. Proposition 3.4 now implies that $I_\mu$ is not $w$-compact.

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