Criteria for weak compactness of vector-valued integration maps

S. Okada, W.J. Ricker

Abstract. Criteria are given for determining the weak compactness, or otherwise, of the integration map associated with a vector measure. For instance, the space of integrable functions of a weakly compact integration map is necessarily normable for the mean convergence topology. Results are presented which relate weak compactness of the integration map with the property of being a bicontinuous isomorphism onto its range. Finally, a detailed description is given of the compactness properties for the integration maps of a class of measures taking their values in ℓ^1, equipped with various weak topologies.

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Introduction

The importance of vector measures in modern analysis is well established. An important aspect of the theory is the integration map. Associated with each X-valued measure μ, with X a locally convex space (briefly, lcs), is its integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ given by $f \mapsto \int f\, d\mu$, for every $f \in \mathcal{L}^1(\mu)$. Here $\mathcal{L}^1(\mu)$ is the space of all $\mathbb{C}$-valued, μ-integrable functions; it is a lcs for the mean convergence topology (see Section 1). Many classical operators, such as the Fourier transform in $L^1(\mathbb{T})$, certain integral operators (e.g. Volterra), representations for Boolean algebras of projections (arising from normal operators) and so on, can be viewed as integration maps I_μ (or restrictions of such maps) for suitable measures μ and spaces X.

Properties of the operator I_μ, which is always linear and continuous, are closely related to the nature of the lcs $\mathcal{L}^1(\mu)$. For X a Banach space, the compactness properties of I_μ are investigated in detail in [5]. It turns out that such compactness results are not always a reliable guide as what to expect for X a lcs; the theory in such spaces (see [6]) is generally not attained from the Banach space case by simply replacing norms with seminorms. Genuinely new phenomena and difficulties occur.

Curiously though, all the examples exhibited in [6; § 3] of compact or weakly compact (briefly, w-compact) integration maps I_μ have the property that the lcs $\mathcal{L}^1(\mu)$ is normable, although μ itself takes its values in a non-normable lcs X.

One of the aims of this note is to show that this is not a coincidence, but a general phenomenon. In particular, it provides a criterion for deciding about w-compactness of I_μ; if $\mathcal{L}^1(\mu)$ is not normable, then I_μ cannot be w-compact. Here, w-compactness is meant in the sense of Grothendieck, that is, some neighbourhood of zero is mapped into a relatively w-compact set. We also exhibit other criteria which are either necessary or sufficient for compactness (resp. w-compactness) of I_μ. Several results are given which relate the w-compactness of I_μ with the property of I_μ being a bicontinuous isomorphism onto its range. For instance, if X is a Fréchet space and I_μ is w-compact, then I_μ cannot be a bicontinuous isomorphism onto its range. Examples are given of a class of measures μ in ℓ^1, considered not as a Banach space, but as a lcs equipped with one of the topologies $\sigma(\ell^1, c_0)$ or $\sigma(\ell^1, \ell^\infty)$, for which a complete description of the compactness properties of I_μ is possible.

1. Preliminaries

The continuous dual space of a locally convex Hausdorff space X (briefly, lcHs) is denoted by X'. The set of all continuous seminorms on X is denoted by $\mathcal{P}(X)$. The space X equipped with its weak topology $\sigma(X, X')$ is denoted by $X_{\sigma(X,X')}$. The space X' equipped with its weak-star topology $\sigma(X', X)$ is denoted by $X'_{\sigma(X',X)}$. We adopt the notation $\langle x', x\rangle = x'(x)$ for every $x \in X$ and $x' \in X'$. Given an X-valued set function m on a σ-algebra of sets and $x' \in X'$, let $\langle x', m\rangle$ denote the set function given by $\langle x', m\rangle(E) = \langle x',\ m(E)\rangle$ for every set E in the domain of m.

Let $\mathcal{S}$ be a σ-algebra of subsets of a non-empty set Ω. Let $\mu : \mathcal{S} \to X$ be a vector measure, that is, a σ-additive set function. For every $x' \in X'$, the total variation measure of the scalar measure $\langle x', \mu\rangle$ is denoted by $|\langle x', \mu\rangle|$. Given $p \in \mathcal{P}(X)$, let $U_p^0 = \{x' \in X';\ |\langle x',\ x\rangle| \le 1,\ x \in p^{-1}([0,1])\}$. The *$p$-semivariation* of μ is the set function $p(\mu)$ given by

$$p(\mu)(E) = \sup\{|\langle x', \mu\rangle|(E);\ x' \in U_p^0\}, \qquad E \in \mathcal{S}.$$

A scalar-valued, $\mathcal{S}$-measurable function f on Ω is called *μ-integrable* if it is $\langle x', \mu\rangle$-integrable, for every $x' \in X'$, and if there is a unique function $f\mu : \mathcal{S} \to X$ satisfying

$$\langle x',\ (f\mu)(E)\rangle = \int_E f\, d\langle x', \mu\rangle, \qquad x' \in X',\ E \in \mathcal{S}.$$

In this case, $f\mu$ is also σ-additive by the Orlicz-Pettis lemma, and will be called the *indefinite integral* of f with respect to μ. We also use the classical notation

$$\int_E f\, d\mu = (f\mu)(E), \qquad E \in \mathcal{S}.$$

The vector space of all μ-integrable functions on Ω is denoted by $\mathcal{L}^1(\mu)$. An element of $\mathcal{L}^1(\mu)$ is called *μ-null* if its indefinite integral is the zero measure.

The subspace of $\mathcal{L}^1(\mu)$ consisting of all μ-null functions is denoted by $\mathcal{N}(\mu)$. For every $p \in \mathcal{P}(X)$, the seminorm $f \mapsto p(f\mu)(\Omega)$, for $f \in \mathcal{L}^1(\mu)$, is also denoted by $p(\mu)$. The space $\mathcal{L}^1(\mu)$ is equipped with the lc-topology defined by the family of seminorms $p(\mu)$, $p \in \mathcal{P}(X)$. This topology is called the *mean convergence topology.* The lcHs associated with $\mathcal{L}^1(\mu)$ is the quotient space $\mathcal{L}^1(\mu)/\mathcal{N}(\mu)$.

The integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is defined by

$$I_\mu(f) = (f\mu)(\Omega) = \int_\Omega f\, d\mu, \qquad f \in \mathcal{L}^1(\mu).$$

It is clear that I_μ is linear and continuous.

Definition 1.1. The measure $\mu : \mathcal{S} \to X$ is said to *factor* through a lcHs Y if there exist a vector measure $\nu : \mathcal{S} \to Y$ and a continuous linear map $j : Y \to X$ such that

(i) $\mathcal{L}^1(\mu) = \mathcal{L}^1(\nu)$ as lcs,
(ii) $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ as sets, and
(iii) $I_\mu = j \circ I_\nu$.

In this case we say that μ *factors through* Y *(via* ν *and* j); see [6; § 1].

Lemma 1.2. *Let j be a continuous linear map from a lcHs Y into a lcHs X and $\nu : \mathcal{S} \to Y$ be a vector measure. Let $\mu = j \circ \nu$. Suppose that μ factors through Y via ν and j. Then, if the integration map $I_\nu : \mathcal{L}^1(\nu) \to Y$ is w-compact (resp. compact, nuclear) so is the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$.*

Proof: The statements for compact and w-compact maps are clear. For the case concerning nuclear maps see [8; Proposition 47.1]. □

Remark 1.3. It is shown in Section 3 (see Example 3.3) that the converse of Lemma 1.2 is not always valid. □

Lemma 1.4. *Let Y be a lcHs and $\nu : \mathcal{S} \to Y$ be a vector measure. Let $X = Y_{\sigma(Y,Y')}$ and $j : Y \to X$ be the identity map. Suppose that the measure $\mu = j \circ \nu$ factors through Y via ν and j. Then the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is compact (= w-compact), if and only if, the integration map $I_\nu : \mathcal{L}^1(\nu) \to Y$ is w-compact.*

Proof: Follows from the fact that a subset A of Y is w-compact, if and only if, $j(A)$ is compact in X. □

We conclude this section with a technical lemma needed later.

Lemma 1.5. *Let Z be a Banach space and Z' be the dual Banach space. Let $j : Z' \to Z'_{\sigma(Z',Z)}$ be the identity map. A continuous linear map T from a Banach space W into Z' is nuclear, if and only if, $j \circ T : W \to Z'_{\sigma(Z',Z)}$ is nuclear.*

Proof: Since $Z'_{\sigma(Z',Z)}$ is quasicomplete, it follows from [8; Corollary 1, p. 482] that there exist a bounded sequence $\{w'_n\}_{n=1}^\infty$ in W', a bounded sequence $\{z'_n\}_{n=1}^\infty$

in $Z'_{\sigma(Z',Z)}$ and an absolutely convergent series of scalars $\Sigma_{n=1}^{\infty} a_n$ such that

$$(j \circ T)w = \sum_{n=1}^{\infty} a_n \langle w'_n, w \rangle z'_n, \qquad w \in W.$$

Since $\sum_{n=1}^{\infty} |a_n|.|\langle w'_n, w \rangle|.\|z'_n\|$ is finite, we have

$$Tw = \sum_{n=1}^{\infty} a_n \langle w'_n, w \rangle j^{-1}(z'_n), \qquad w \in W.$$

Again by [8; Corollary 1, p. 482], T is nuclear.

The converse statement is clear. □

2. w-Compactness criteria

In this section we present some general criteria which are sufficient to guarantee compactness and/or w-compactness of integration maps.

A lcs Z is called *seminormable* if its topology is the same as that determined by a single seminorm. If Z is Hausdorff then, of course, the single seminorm is a norm and we use the term *normable*. If, in addition, Z is sequentially complete, then it must be complete for this norm, that is, Z is a Banach space.

Proposition 2.1. *Let X be a lcHs and $\mu : \mathcal{S} \to X$ be a vector measure. Then the following two statements are equivalent.*

(i) *There is a neighbourhood V of 0 in $\mathcal{L}^1(\mu)$ such that its image $I_\mu(V)$ is a bounded subset of X.*

(ii) *The lcs $\mathcal{L}^1(\mu)$ is seminormable* (*i.e. the quotient space $\mathcal{L}^1(\mu)/\mathcal{N}(\mu)$ is normable*).

If X is sequentially complete, then either of (i) *or* (ii) *is equivalent to the following statement.*

(iii) *The lcs $\mathcal{L}^1(\mu)$ is a complete seminormed space* (*i.e. the quotient space $\mathcal{L}^1(\mu)/\mathcal{N}(\mu)$ is a Banach space*).

Proof: The implications (iii) ⇒ (ii) ⇒ (i) are clear. So, suppose that (i) holds. Take a seminorm $p \in \mathcal{P}(X)$ satisfying

$$\{g \in \mathcal{L}^1(\mu);\ p(\mu)(g) \le 1\} \subseteq V. \tag{1}$$

Denote the left-hand-side of (1) by V_p. Let $q \in \mathcal{P}(X)$ be arbitrary. The boundedness of $I_\mu(V_p)$ implies that

$$I_\mu(V_p) \subseteq C_q\{x \in X;\ q(x) \le 1\},$$

for some positive constant C_q. Let $g \in \mathcal{L}^1(\mu)$. If $p(\mu)(g) \neq 0$, then it follows easily that

$$q(I_\mu g) \le C_q p(\mu)(g). \tag{2}$$

If $p(\mu)(g) = 0$, then $\alpha g \in V_p \subseteq V$ and so $\alpha I_\mu g \in I_\mu(V_p)$, for all scalars α. Since $I_\mu(V_p)$ is bounded, this forces $I_\mu g = 0$ and so again (2) holds. Accordingly, (2) holds for every $g \in \mathcal{L}^1(\mu)$. It then follows from [2; Ch.I, Proposition 1.11] that

$$q(\mu)(g) \leq 4 \sup_{E\in\mathcal{S}} q(\int_E g\, d\mu) \leq 4C_q p(\mu)(g), \qquad g \in \mathcal{L}^1(\mu).$$

This shows that the mean convergence topology on $\mathcal{L}^1(\mu)$ can be defined by the single seminorm $p(\mu)$. In other words, (ii) holds. A further consequence is that there is a finite measure $\lambda : \mathcal{S} \to [0,\infty)$ with respect to which the set functions $q(\mu)$, for $q \in \mathcal{P}(X)$, are absolutely continuous; that is, $q(\mu)(E) \to 0$ for all $q \in \mathcal{P}(X)$ as $\lambda(E) \to 0$, $E \in \mathcal{S}$. This follows from the fact that there is a finite measure on $\mathcal{S}$ with respect to which the set function $p(\mu)$ is absolutely continuous; see [4; Ch.II, Theorem 1.1], for example. It follows that the scalar measures $\langle x', \mu\rangle$, for $x' \in X'$, are absolutely continuous with respect to λ.

Assume now that X is sequentially complete. Statement (iii) then follows from [4; Ch. IV, Theorem 7.3] and [7; Proposition 2.1]. □

Since w-compact sets are bounded, an immediate consequence is that the examples of w-compact integration maps I_μ exhibited in [6], namely Examples 3.1 and 3.2 and Proposition 3.8, necessarily have normable spaces $\mathcal{L}^1(\mu)$. Proposition 2.1 can also be used to check that an integration map is not w-compact. For instance, the lcHs X in Example 1.7 of [6] is quasicomplete and it was shown for the vector measure $\mu : \mathcal{S} \to X$ given there, that $\mathcal{L}^1(\mu)$ is not normable. So, by Proposition 2.1, the associated integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is not w-compact.

We now consider the connection between w-compactness of I_μ and the property of I_μ being a bicontinuous isomorphism onto its range.

Let T be a continuous linear map from a lcHs U into a lcHs W. We say that T *factors through a lcHs* Z if there exist continuous linear maps $R : U \to Z$ and $S : Z \to W$ such that $T = S \circ R$.

Remark 2.2. (i) Let X be a non-reflexive Pták space. Let T be a bijective, continuous linear map from X onto a lcHs Y. Then T does not factor through any reflexive Banach space, [6; Lemma 3.5]. We note that every Fréchet lcs is a Pták space and hence, in particular, Banach spaces are Pták spaces.

(ii) If a vector measure $\mu : \mathcal{S} \to X$ factors through a lcHs Y (cf. Definition 1.1), then the associated integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ also factors through Y. □

For clarity of presentation, in the remainder of this paper the space $\mathcal{L}^1(\mu)$ of all μ-integrable functions, for a given vector measure μ, will be identified with its associated Hausdorff space $\mathcal{L}^1(\mu)/\mathcal{N}(\mu)$.

Proposition 2.3. *Let X be a Fréchet lcs and $\mu : \mathcal{S} \to X$ be a vector measure such that $\mathcal{L}^1(\mu)$ is a non-reflexive Fréchet space.*

(i) *If the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is an injective, continuous linear map with closed range, then I_μ cannot be w-compact.*

(ii) *If $I_\mu : \mathcal{L}^1(\mu) \to X$ is w-compact, then I_μ cannot be a bicontinuous isomorphism onto its range.*

Proof: (i) Suppose I_μ were w-compact, where we consider μ and I_μ as taking their values in the Fréchet lcs $\underset{\sim}{X} = I_\mu(\mathcal{L}^1(\mu))$. By Remark 2.5 of [6], applied to $\underset{\sim}{X}$, the measure μ would factor through a reflexive Banach space Y and hence, the integration map $I_\mu : \mathcal{L}^1(\mu) \to \underset{\sim}{X}$ would factor through Y, by Remark 2.2 (ii). This contradicts Remark 2.2 (i).

(ii) If I_μ were a bicontinuous isomorphism onto its range $Z = I_\mu(\mathcal{L}^1(\mu))$, then Z would be a Fréchet lcs and so, by part (i), $I_\mu : \mathcal{L}^1(\mu) \to Z$ could not be w-compact. This contradicts the hypothesis.

□

We note that Proposition 2.3 (ii) implies immediately that the w-compact integration map I_μ of Example 3.2 in [6] cannot be a bicontinuous isomorphism onto its range.

Remark 2.4. A slight variation of Proposition 2.3 (i) is as follows:
Let $\mu : \mathcal{S} \to X$ be a vector measure with values in a non-normable lcHs X such that its integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is a bicontinuous isomorphism of $\mathcal{L}^1(\mu)$ onto X. Then I_μ cannot be w-compact.
For, otherwise X would have a bounded neighbourhood of 0, which would force X to be normable. □

Example 2.5. Let $\mathbb{N}$ denote the natural numbers. Let $X = \mathbb{C}^{\mathbb{N}}$, equipped with the seminorms given by

$$q_n : x \mapsto \max_{1 \le r \le n} |x_r|, \qquad x = (x_j)_{j=1}^\infty \in X,$$

for each $n = 1, 2, \dots$. Then X is a separable, reflexive Fréchet space. Let $\mathcal{S} = 2^{\mathbb{N}}$ and $\mu(E) = \chi_E$, for each $E \in \mathcal{S}$. Then I_μ is a bicontinuous isomorphism of $\mathcal{L}^1(\mu)$ onto X (see Remark 2.7 below) and hence, I_μ is not w-compact (by Remark 2.4). This is despite the fact that X is reflexive; for reflexive Banach spaces X this cannot happen as I_μ is always w-compact in such spaces. □

We now exhibit a class of measures μ for which the criterion given by Remark 2.4 is especially effective; Example 2.5 is a particular case of such a measure μ.

Let X be a lcHs and $L(X)$ be the space of all continuous linear operators of X into X. With respect to the topology of pointwise convergence in X (i.e. the strong operator topology), $L(X)$ is also a lcHs; it is denoted by $L_s(X)$. For the definition of a *spectral measure* $P : \mathcal{S} \to L_s(X)$ we refer to [3]. These are generalizations of the resolution of the identity for normal operators in Hilbert space. A spectral measure P is called *equicontinuous* if its range $P(\mathcal{S})$ is an equicontinuous subset of $L(X)$. Given $x \in X$, the *cyclic space* $P(\mathcal{S})[x]$ generated by x with respect to P is defined to be the closed linear span of the set $\{P(E)x;\ E \in \mathcal{S}\}$. For each $x \in X$, let $Px : \mathcal{S} \to X$ denote the X-valued measure $E \mapsto P(E)x$, $E \in \mathcal{S}$.

Proposition 2.6. *Let X be a quasicomplete lcHs such that $L_s(X)$ is sequentially complete and $P : \mathcal{S} \to L_s(X)$ be an equicontinuous spectral measure with range $P(\mathcal{S})$ a closed subset of $L_s(X)$.*

(i) *For each $x \in X$, the integration map $I_{Px} : \mathcal{L}^1(Px) \to X$ is a bicontinuous isomorphism of $\mathcal{L}^1(Px)$ onto the cyclic space $P(\mathcal{S})[x]$.*

(ii) *If the cyclic space $P(\mathcal{S})[x]$ is non-normable, then the integration map I_{Px} is not w-compact.*

Proof: Part (i) is just [3; Proposition 2.1], while part (ii) follows from (i) and Remark 2.4. □

We note that the condition of the range $P(\mathcal{S})$ being closed in $L_s(X)$ is automatically satisfied in separable Fréchet spaces, [3], [7].

Remark 2.7. The claim made in Example 2.5 that the integration map I_μ given there is a bicontinuous isomorphism onto $X = \mathbb{C}^{\mathbb{N}}$ follows from Proposition 2.6. For, in the notation of Example 2.5, given a subset E of $\mathbb{N}$ define the projection $P(E)$ by $P(E)x = \chi_E x$ (coordinatewise multiplication), for each $x \in X$. Since X is barrelled, the spectral measure is necessarily equicontinuous. Moreover, as X is a separable Fréchet space, $P(\mathcal{S})$ is a closed subset of $L_s(X)$. In addition, the element $\mathbb{1} \in X$ (consisting of 1 in every co-ordinate) is a cyclic vector for P, that is, $P(\mathcal{S})[\mathbb{1}] = X$. Since $\mu = P\mathbb{1}$, we can apply Proposition 2.6. □

3. Examples

In this section we exhibit some examples of measures in lc-spaces which arise from Banach spaces with their weak or weak-star topologies. For the particular Banach space ℓ^1 quite detailed information is available. The dual operator to a continuous linear operator T between lc-spaces is denoted by T'.

Proposition 3.1. *Let j be an injective, continuous linear map from the Banach space ℓ^1 into a lcHs X such that $(j')^{-1}(\{f_1\}) \neq \phi$, where $f_1 = (1, 0, 0, \dots)$ is considered as an element of ℓ^∞. Let $\lambda : \mathcal{S} \to [0, \infty)$ be a finite measure. Let $g_1 = \mathbb{1}$ be the function constantly equal to 1 and $g_n \in \mathcal{L}^\infty(\lambda)$, $n = 2, 3, \dots$, satisfy*

$$\sum_{n=1}^{\infty} |\langle g_n,\ f\rangle| < \infty, \qquad f \in \mathcal{L}^1(\lambda).$$

Let $\mathbf{e}_n$, $n \in \mathbb{N}$, be the standard basis vectors of ℓ^1 and $\nu : \mathcal{S} \to \ell^1$ be the vector measure given by

$$\nu(E) = \sum_{n=1}^{\infty} \langle g_n,\ \chi_E\rangle \mathbf{e}_n, \qquad E \in \mathcal{S}. \tag{3}$$

Finally, let $\mu = j \circ \nu$. Then the following statements hold.

(i) *The measure $\mu : \mathcal{S} \to X$ factors through ℓ^1 via ν and j.*

(ii) *If* $\{g_n\}_{n=1}^{\infty}$ *is unconditionally summable in* $\mathcal{L}^\infty(\lambda)$*, then the integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ *is compact.*
(iii) *If* $\{g_n\}_{n=1}^{\infty}$ *is absolutely summable in* $\mathcal{L}^\infty(\lambda)$*, then the integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ *is nuclear.*

Proof: (i) The continuity of j implies that $\mathcal{L}^1(\nu) \subset \mathcal{L}^1(\mu)$. Choose a vector $x' \in X'$ such that $j'(x') = f_1$, in which case

$$\langle x', \mu\rangle = \langle x',\ j \circ \nu\rangle = \langle j'(x'),\ \nu\rangle = \langle f_1,\ \nu\rangle = \lambda, \tag{4}$$

and so $\mathcal{L}^1(\mu) \subset \mathcal{L}^1(\lambda) = \mathcal{L}^1(\nu)$. Accordingly, $\mathcal{L}^1(\mu) = \mathcal{L}^1(\nu) = \mathcal{L}^1(\lambda)$ as vector spaces. By (4) we conclude that $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\lambda)$ are isomorphic. The identity $I_\mu = j \circ I_\nu$ is a consequence of the fact that the $\mathcal{S}$-simple functions are dense in both $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\nu)$. The equality $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ follows from the injectivity of j. Hence, (i) holds.

Statements (ii) and (iii) follow from part (i), Lemma 1.2 and [5; Proposition 3.6].

□

Special choices of the space X in Proposition 3.1 give a way of producing integration maps with specific properties.

Corollary 3.1.1. *Let* $X = \ell^1_{\sigma(\ell^1, c_0)}$ *and* $j : \ell^1 \to X$ *be the identity map. Let the measure* λ*, the sequence* $\{g_n\}_{n=1}^{\infty}$ *in* $\mathcal{L}^\infty(\lambda)$ *and the vector measure* ν *be as in Proposition* 3.1*. Let* $\mu = j \circ \nu$*.*

(i) *The measure* $\mu : \mathcal{S} \to X$ *factors through the Banach space* ℓ^1 *via* ν *and* j*.*
(ii) *The integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ *is compact* (= *w-compact*)*.*
(iii) I_μ *is nuclear, if and only if,* I_ν *is nuclear.*
(iv) *If the Banach space* $\mathcal{L}^1(\lambda)$ *is infinite-dimensional, then the integration map* I_μ *is not a bicontinuous isomorphism onto its range.*

Proof: (i) Let $f_1 \in \ell^\infty$ be as in Proposition 3.1. Since $j'(f_1) = f_1$, Proposition 3.1 (i) implies (i).

(ii) Since $I_\mu = j \circ I_\nu$ with j compact, it follows that I_ν is compact.

(iii) See Lemma 1.5.

(iv) By the proof of Proposition 3.1, the spaces $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\lambda)$ are isomorphic Banach spaces; in particular, $\mathcal{L}^1(\mu)$ is non-reflexive. Statement (iv) follows from (ii). □

Corollary 3.1.2. *Let* $X = \ell^1_{\sigma(\ell^1, \ell^\infty)}$ *and* $j : \ell^1 \to X$ *be the identity map. Let the measure* λ*, the sequence* $\{g_n\}_{n=1}^{\infty}$ *in* $\mathcal{L}^\infty(\lambda)$ *and the measure* ν *be as in Proposition* 3.1*. Let* $\mu = j \circ \nu$*.*

(i) *The measure* $\mu : \mathcal{S} \to X$ *factors through the Banach space* ℓ^1 *via* ν *and* j*.*

(ii) *The integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ *is compact* (= w-*compact*), *if and only if, the integration map* $I_\nu : \mathcal{L}^1(\nu) \to \ell^1$ *is compact.*
(iii) *The integration map* I_μ *is nuclear, if and only if, the integration map* I_ν *is nuclear.*
(iv) *If the Banach space* $\mathcal{L}^1(\lambda)$ *is infinite-dimensional, then the integration map* I_μ *is not a bicontinuous isomorphism onto its range.*

Proof: Part (i) follows as in the proof of Corollary 3.1.1 (i). Part (ii) is a consequence of part (i) and Lemma 1.4.

(iii) Let $Z = \ell^1_{\sigma(\ell^1, c_0)}$ and $k : X \to Z$ be the identity map. Then the measure $k \circ \mu : \mathcal{S} \to Z$ factors through X via μ and k so that $I_{k\circ\mu} = k \circ I_\mu$. By part (i), we have $j \circ I_\nu = I_\mu$, and hence, $I_{k\circ\mu} = k \circ I_\mu = (k \circ j) \circ I_\nu$. Therefore, if I_μ is nuclear, then so is $I_{k\circ\mu}$ and hence, I_ν is nuclear by Corollary 3.1.1 (iii). The converse implication is clear.

(iv) If I_μ were a bicontinuous isomorphism then, on the infinite-dimensional linear subspace $I_\nu(\mathcal{L}^1(\lambda)) = j^{-1}(I_\mu(\mathcal{L}^1(\mu)))$ of ℓ^1, the norm topology and the weak topology would coincide, which is a contradiction. □

Corollary 3.1.3. *Let* X *be the Fréchet space* $\mathbb{C}^{\mathbb{N}}$ *and* $j : \ell^1 \to X$ *be the natural injection. Let the measure* λ*, the sequence* $\{g_n\}_{n=1}^\infty$ *in* $\mathcal{L}^\infty(\lambda)$ *and the measure* ν *be as in Proposition* 3.1. *Let* $\mu = j \circ \nu$.
(i) *The measure* $\mu : \mathcal{S} \to X$ *factors through the Banach space* ℓ^1 *via* ν *and* j.
(ii) *The integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ *is compact* (= w-*compact*).
(iii) *The integration map* I_μ *is nuclear, if and only if, the integration map* I_ν *is nuclear.*
(iv) *If the Banach space* $\mathcal{L}^1(\lambda)$ *is infinite-dimensional, then the integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ *is not an isomorphism onto its range.*

Proof: (i) The arguments in the proof of Corollary 3.1.1 (i) apply.

(ii) Since X is a Montel space, the map j is compact. Hence, $I_\mu = j \circ I_\nu$ is compact and thus, also w-compact.

(iii) Since $\mathcal{L}^1(\lambda) = \mathcal{L}^1(\mu)$ is barrelled and X is complete, statement (iii) can be proved as in Corollary 3.1.1 (iii) by using the analogue of Lemma 1.5 with $Z = \mathbb{C}^{\mathbb{N}}$; again apply [8; Corollary 1, p. 482].

(iv) Use the same argument as in the proof of Corollary 3.1.1 (iv). □

Remark 3.2. In relation to the previous three corollaries it may be worth noting that the lcHs $\ell^1_{\sigma(\ell^1, c_0)}$ is a semireflexive, quasicomplete Montel space, that $\mathbb{C}^{\mathbb{N}}$ is a complete, reflexive, Fréchet-Montel space, but that $\ell^1_{\sigma(\ell^1, \ell^\infty)}$ is neither semireflexive, Montel nor quasicomplete (it is sequentially complete). Of course, a continuous linear map from a lcHs into a Montel space is compact, if and only if, it is w-compact. This comment is relevant to Corollary 3.1.1 (ii) and Corollary 3.1.2 (ii). □

We can now exhibit an example showing that the converse of Lemma 1.2 fails (cf. Remark 1.3).

Example 3.3. Let $\mathcal{S}$ be the σ-algebra of Borel subsets of $[0,1]$ and λ be Lebesgue measure on $\mathcal{S}$. Let $g_1 = \mathbb{1}$ and $g_n = \chi_{E(n)}$, where $E(n) = ((n+1)^{-1},\ n^{-1}]$ for each $n = 2, 3, \dots$. Since $\{g_n\}_{n=1}^{\infty}$ is not unconditionally summable in $\mathcal{L}^{\infty}(\lambda)$, the integration map $I_\nu : \mathcal{S} \to \ell^1$ (with ν given by (3)) is not compact, [5; Proposition 3.6]. Let $X = \ell^1_{\sigma(\ell^1, c_0)}$ and $j : \ell^1 \to X$ be the identity map. It follows from Proposition 3.1 (i) that the measure $\mu = j \circ \nu$ factors through ℓ^1 via ν and j. Moreover, since j is a compact map and $I_\mu = j \circ I_\nu$, it follows that $I_\mu : \mathcal{L}^1(\mu) \to X$ is compact. $\square$

We have already seen in the above example that the converse of Lemma 1.2 is not valid. However, for a particular setting, the converse does hold.

Proposition 3.4. *Let Y be a lcHs and $X = Y_{\sigma(Y,Y')}$. Let $j : Y \to X$ be the identity map and $\nu : \mathcal{S} \to Y$ be a vector measure. Let $\mu = j \circ \nu$. Suppose that the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is w-compact. Then so is the integration map $I_\nu : \mathcal{L}^1(\nu) \to Y$.*

Proof: By assumption, there is a neighbourhood V of 0 in $\mathcal{L}^1(\mu)$ whose image $I_\mu(V)$ is relatively w-compact in X. The set V is a neighbourhood of 0 also in $\mathcal{L}^1(\nu)$ because $\mathcal{L}^1(\mu) = \mathcal{L}^1(\nu)$ as vector spaces and because the mean convergence topology on $\mathcal{L}^1(\nu)$ is stronger than that on $\mathcal{L}^1(\mu)$. Hence, I_ν is w-compact because $I_\nu(V) = I_\mu(V)$ is relatively w-compact in Y. $\square$

The converse of Proposition 3.4 is not always valid. A counter-example will be given in the case when $Y = \ell^2$. It is interesting to know whether or not that is the case when $Y = \ell^1$.

Example 3.5. Let Y be the Hilbert space ℓ^2 and $X = \ell^2_{\sigma(\ell^2,\ell^2)}$. Let $\mathbf{e}_n$, $n \in \mathbb{N}$, be the standard basis vectors in Y and $\nu : 2^{\mathbb{N}} \to Y$ be the vector measure given by

$$\nu(E) = \sum_{n \in E} n^{-1}\mathbf{e}_n, \qquad E \in 2^{\mathbb{N}}.$$

Let $j : Y \to X$ denote the identity map. Define a vector measure $\mu : 2^{\mathbb{N}} \to X$ by $\mu = j \circ \nu$. Then $\mathcal{L}^1(\mu) = \mathcal{L}^1(\nu)$ (as vector spaces) and this space consists of precisely those functions f on $\mathbb{N}$ such that $\sum_{n=1}^{\infty} |f(n)/n|^2 < \infty$.

Since Y is reflexive, the integration map $I_\nu : \mathcal{L}^1(\nu) \to Y$ is weakly compact. However, we shall show that the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is not weakly compact. To this end, let Z denote the space ℓ^2 equipped with the absolute weak topology $|\sigma|(\ell^2,\ell^2)$ (cf. [1; p. 166]). Namely, the topology on Z is generated by the seminorms q_ξ, $\xi = (\xi_n)_{n=1}^{\infty} \in \ell^2$, defined by

$$q_\xi(x) = \sum_{n=1}^{\infty} |\xi_n x_n|, \quad x = (x_n)_{n=1}^{\infty} \in \ell^2.$$

Then $|\sigma|(\ell^2,\ \ell^2)$ is strictly weaker than the norm topology and strictly stronger than the weak topology. Let $k : Z \to X$ be the identity map and $\eta : 2^{\mathbb{N}} \to Z$ be the vector measure satisfying $\mu = k \circ \eta$. Clearly $\mathcal{L}^1(\eta) = \mathcal{L}^1(\mu)$ as vector spaces (in fact, as lc spaces). A direct computation shows that the integration map I_η is a bicontinuous isomorphism from $\mathcal{L}^1(\eta)$ onto Z and hence, I_η is not w-compact by Remark 2.4 because Z is not normable. Proposition 3.4 now implies that I_μ is not w-compact. □

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Mathematics Department, University of Tasmania, Hobart 7001, Australia

School of Mathematics, University of New South Wales, Sydney 2052, Australia

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