Note on dense covers in the category of locales

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Abstract. In this note we are going to study dense covers in the category of locales. We shall show that any product of finitely regular locales with some dense covering property has this property as well.

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Introduction

This note is closely related to the paper [8] in which the preservation of some covering properties of locales w.r. to products is investigated. We shall now concentrate on the preservation of dense covering properties.

The paper is organized as follows. In Section 1 we introduce the new notions which are not contained in [8]. Section 2 contains the main result of the paper – the general dense cover theorem. The applications of the general dense cover theorem close Section 2. The Axiom of Choice is a prerequisite of our considerations.

The basic reference for the theory of locales is the classic book of Johnstone [3], for facts concerning topology in general we refer to [2], [6]. Our notation and terminology agree with the book [3] of Johnstone and with the papers [1], [7], [8].

First, let us clarify the terminology. For F, G sets, F a finite subset of G we write $F \subseteq\subseteq G$. If X is a topological space we put $\tau(X)$ for its frame of open subsets and $\Omega(X)$ for the corresponding locale. For a locale L we denote by $\tau(L)$ the frame we are dealing with. A nucleus $j : L \to L$ on a locale L is said to be

(i) *dense* if $j(a) = 0$ implies $a = 0$,
(ii) *codense* if $j(a) = 1$ implies $a = 1$.

Let L be a locale, $C, D \subseteq \tau(L)$. We shall say that D *finitely regular refines* C if, for each $d \in D$, there exists $F \subseteq\subseteq C$ such that $d \triangleleft \bigvee F$ (i.e. there is an element $e \in \tau(L)$ such that $e \wedge d = 0$, $e \vee \bigvee F = 1$). We shall write $D \triangleleft_f C$. We shall say that a locale L is *finitely regular* if each cover of L has a finitely regular refinement which is a cover as well.

Let L be a locale. *A dense cover* of L is a subset D of L such that $(\bigvee U)^* = 0$. The system of all dense covers of the locale L is denoted by $\mathcal{D}(L)$.

Let $\mathcal{B}, \mathcal{F}$ be functions with the domain $\mathcal{L}oc$ and the codomain $\mathcal{S}et$ as in [8] satisfying the following conditions:

(A0) $L \in \mathcal{L}oc$ implies $1 \in \mathcal{B}(L)$, $\mathcal{B}(L) \subseteq \tau(L)$ is closed under finite meets,
(A1) $L \in \mathcal{L}oc$, $t \in \mathcal{B}(L)$ implies $\{t\} \in \mathcal{F}(L)$, $\mathcal{F}(L) \subseteq 2^{\mathcal{B}(L)}$,
(A2) $L \in \mathcal{L}oc$, $t \in \mathcal{B}(L)$, $F \in \mathcal{F}(L)$ implies $\{t\} \wedge F \in \mathcal{F}(L)$,
(A3) $L \in \mathcal{L}oc$, $F \in \mathcal{F}(L)$ and for each $f \in \mathcal{F}(L)$ there is $F_f \in \mathcal{F}(L)$ such that $f = \bigvee F_f$ implies $\bigcup\{F_f : f \in F\} \in \mathcal{F}(L)$,
(A4) $L, K \in \mathcal{L}oc$, $F \in \mathcal{F}(L)$, $H \in \mathcal{F}(K)$ implies $\tau(\pi_L)(F) \wedge \tau(\pi_K)(H) \in \mathcal{F}(L\text{x}K)$.

If $F \in \mathcal{F}(L)$, we shall speak about a set of a type $\mathcal{F}$. Recall that the foregoing conditions **(A1)**, **(A2)** and **(A3)** in fact coincide with the conditions (i), (ii) and (iii) in [1]. It is sometimes convenient to suppose the following strengthening of the condition **(A1)**:

(A1') $L \in \mathcal{L}oc$, $C \subseteq\subseteq \mathcal{B}(L)$ implies $C \in \mathcal{F}(L)$, $\mathcal{F}(L) \subseteq 2^{\mathcal{B}(L)}$.

We shall say that a locale L is an *$\mathcal{FD}$-locale* if $C \in \mathcal{C}(L)$ implies there is $D \in \mathcal{D}(L) \cap \mathcal{F}(L)$ such that $D < C$. A morphism $f : L \to K$ of locales is said to be *$\mathcal{FD}$-good* if $\tau(f)(\mathcal{F}(K)) \subseteq \mathcal{F}(L)$. We shall say that a space X is an *$\mathcal{FD}$-space* if $\tau(X)$ is an $\mathcal{FD}$-locale.

In the following we shall assume that $\mathcal{B}(L) = L$.

A subset F of a locale L is called *locally finite* if there is a cover C of L such that $F(c) = \{f \in F : c \wedge f \neq 0\} \subseteq\subseteq F$ for all $c \in C$. If $\mathcal{F}(L)$ is the system of locally finite subsets of L for every locale L, L is said to be *almost paracompact* if it is an $\mathcal{FD}$-locale.

Similarly, we call a subset F of a locale L *prime-finite* if, given any prime ($\wedge$-irreducible) element p of L, the inequality $f \not\leq p$ holds for at most a finite number of elements f of F. A subset G of a locale L is said to be *disjoint* if the meet of any two distinct elements of G is the bottom element of L. If a subset H of L is a union of countably many prime-finite (disjoint) subsets of L we shall speak about a *σ-prime-finite* (*σ-disjoint*) set.

If $\mathcal{F}(L)$ is the system of prime-finite subsets (σ-prime-finite, σ-disjoint subsets, countable subsets, subsets of a cardinality $\leq\mathbf{c}$) of L for every locale L, L is said to be *almost prime-metacompact* (*almost σ-prime-metacompact*, *almost screenable*, *almost Lindelöf*, *almost* **c***-compact*) if it is an $\mathcal{FD}$-locale. One easily checks that the conditions (A0)–(A4) are satisfied.

2. A general dense cover theorem

In the sequel, let $L_\gamma, \gamma \in \Gamma$, be a family of finitely regular $\mathcal{FD}$-locales, B' the set-theoretical product of the L_γ, $B = \{x' \in B' : \pi_\gamma(x') = 1$ for all but finitely many $\gamma \in \Gamma\}$, L is the categorical product of the L_γ, $s : \mathcal{P}(B') \to \mathcal{P}(L)$ a map which determines to each subset of B' its representation in L, $K \subseteq B$, $\bigvee s(K) = 1$, and let

$$\overline{K} = \{x \in B : x \in M' \subseteq\subseteq B, \bigvee s(M') = 1 \text{ implies there is a subset } {Q'}_x \subseteq B,$$
$$s({Q'}_x) \in \mathcal{D}(L), s({Q'}_x) < s(K \bigcup (M' - \{x\})), s({Q'}_x) \triangleleft_f s(K \bigcup (M' - \{x\})),$$
$$s({Q'}_x) \in \mathcal{F}(L)\}.$$

Lemma 2.1. *Let $S_\gamma, \gamma \in \Gamma$, be a system of covers of $L_\gamma, \gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. Then there are $T_\gamma, \gamma \in \Gamma$, $T_\gamma < S_\gamma$, $T_\gamma \triangleleft_f S_\gamma$, $T_\gamma \in \mathcal{D}(L) \cap \mathcal{F}(L)$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$.*

Proof: The proof follows the same arguments as the proof of Lemma 3.3 in [8]. □

Lemma 2.2. *Let $t \in B$, $s(t) \in \mathcal{B}(L)$, $A \subseteq\subseteq B$, $W \subseteq\subseteq \overline{K}$, $(K \cup A) \cap W = \emptyset, s(t) \triangleleft_f (A \cup W)$. Then there is a subset $T \subseteq B$ such that $s(T) < s(K \cup A)$, $s(T) \triangleleft_f s(K \cup A)$, $s(T) \in \mathcal{F}(L)$, $(\bigvee s(T))^{**} = (s(t))^{**}$.*

Proof: The proof will be done by induction according to $\operatorname{card} W$. If $\operatorname{card} W = 0$, it is enough to put $T = \{t\}$. Now, let $\operatorname{card} W = n > 0$ and the lemma holds for all $k < n$. Then there is an element $w \in W$ such that

$$\bigvee s(D(t) \cup A \cup (W - \{w\}) \cup \{w\}) = 1,$$

where $D(t) \subseteq\subseteq B$ satisfies $\bigvee s(D(t)) = s(t)^*$. Since $w \in W \subseteq\subseteq \overline{K}$, $s(D(t) \cup A \cup (W - \{w\}))$ is finite there is a subset $Q_w \subseteq B$ such that $s(Q_w) \in \mathcal{D}(L)$, $s(Q_w) < s(K \cup (D(t) \cup A \cup (W - \{w\})))$, $s(Q_w) \triangleleft_f s(K \cup (D(t) \cup A \cup (W - \{w\})))$, $s(Q_w) \in \mathcal{F}(L)$.

Now, let us put $S = \{t\} \wedge Q_w$. Evidently, $s(S) < s(K \bigcup A \cup (W - \{w\}))$. Let us take arbitrary fixed $z \in S$. Then $z = t \wedge y$ for some $y \in Q_w$. Using the fact that $s(Q_w) \triangleleft_f s(K \cup (D(t) \cup A \cup (W - \{w\})))$, we have $s(y)^* \vee \bigvee s(D(t) \cup A \cup (W - \{w\}) \cup K') = 1$, for some finite $K' \subseteq\subseteq K$. This implies $s(z)^* \vee \bigvee (A \cup (W - \{w\}) \cup K') = 1$, i.e. $s(S) \triangleleft_f s(K \cup A \cup (W - \{w\}))$, $s(S) \in \mathcal{F}(L)$, $(\bigvee s(S))^{**} = s(t)^{**}$ by the condition (A2). Now, by the induction assumption applied to each $z \in S$, we have a collection of subsets $T_z \subseteq B$ such that $s(T_z) < s(K \cup A)$, $s(T_z) \triangleleft_f s(K \cup A)$, $(\bigvee s(T_z))^{**} = s(z)^{**}$, $s(T_z) \in \mathcal{F}(L)$ indexed by elements of S.

We put $T = \bigcup \{T_z : z \in S\}$. Then by the preceding and the condition (A3) we have that $T \subseteq B$ satisfies $s(T) < s(K \cup A)$, $s(T) \triangleleft_f s(K \cup A)$, $s(T) \in \mathcal{F}(L)$ and

$$(\bigvee s(T))^{**} = (\bigvee s(\bigcup \{T_z : z \in S\}))^{**} = (\bigvee \{s(z) : z \in S\})^{**} = s(t)^{**}.$$

□

Lemma 2.3. *$\overline{K} = B$.*

Proof: The proof is an immediate rewriting of the proof of the Proposition 3.5 in [8]. □

Theorem 2.4 (General dense cover theorem)**.** *Let $L_\gamma, \gamma \in \Gamma$, be a family of finitely regular $\mathcal{FD}$-locales. Then the product $L = \prod \{L_\gamma : \gamma \in \Gamma\}$ is a finitely regular $\mathcal{FD}$-locale L.*

Proof: Let $C \in \mathcal{C}(L)$. Then there is a basic refinement $s(K)$ of C. By Proposition 2.3 we have that $1 \in \overline{K} = B$. Now, let us put $M' = \{1\}$. Then there is a subset $Q \subseteq B, s(Q) = D$, $D \in \mathcal{D}(L) \cap \mathcal{F}(L)$, $D < s(K) < C$. □

Similarly as in [8], we obtain some interesting corollaries.

Corollary 2.5. *Let $X_\gamma, \gamma \in \Gamma$, be a family of finitely regular $\mathcal{FD}$-spaces, $f : \Omega(\prod X_\gamma) \to \prod \Omega(X_\gamma)$ be an induced sublocale morphism which is $\mathcal{FD}$-good. Then we have*

(i) *If f is a codense map then $\prod X_\gamma$ is a finitely regular $\mathcal{FD}$-space.*
(ii) *If Γ is countable and X_γ are Čech complete spaces, then $\prod X_\gamma$ is a finitely regular $\mathcal{FD}$-space.*
(iii) *If Γ is finite and X_γ are locally compact spaces for all but most one $\gamma \in \Gamma$ then $\prod X_\gamma$ is a finitely regular $\mathcal{FD}$-space.*

Proof: The proof copies the proof of the Corollary 3.7 in [8]. □

The following theorem is a straightforward application of the general dense theorem.

Theorem 2.6. *Let $L_\gamma, \gamma \in \Gamma$, be a family of finitely regular almost paracompact* (*almost σ-prime-metacompact, almost prime-metacompact, almost* **c**-*compact, almost Lindelöf, almost screenable*) *locales. Then the product $L = \prod\{L_\gamma : \gamma \in \Gamma\}$ is a finitely regular almost paracompact* (*almost σ-prime-metacompact, almost prime-metacompact, almost* **c**-*compact, almost Lindelöf, almost screenable*) *locale L.*

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