Sets of determination for parabolic functions on a half-space

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Abstract. We characterize all subsets M of $\mathbb{R}^n \times \mathbb{R}^+$ such that

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \sup_{X \in M} u(X)$$

for every bounded parabolic function u on $\mathbb{R}^n \times \mathbb{R}^+$. The closely related problem of representing functions as sums of Weierstrass kernels corresponding to points of M is also considered. The results provide a parabolic counterpart to results for classical harmonic functions in a ball, see References. As a by-product the question of representability of probability continuous distributions as sums of multiples of normal distributions is investigated.

Keywords: heat equation, parabolic function, Weierstrass kernel, set of determination, decomposition of $L_1(\mathbb{R}^n)$, normal distribution

Classification: 35K05, 35K15, 31B10, 60Exx

1. Preliminaries

In this paper the following notation is used: Small letters, such as x, y, will denote points in $\mathbb{R}^n$; capital letters, such as X, points in $\mathbb{R}^{n+1}$, and t denotes "time". (We will write $X = (x, t)$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.) The set $\mathbb{R}^n \times \{0\}$ is identified with $\mathbb{R}^n$, and, when there is no danger of confusion, the point $(y, 0) \in \mathbb{R}^n \times \{0\}$ is denoted by y. The Lebesgue measure in $\mathbb{R}^n$ will be denoted by λ.

Further notation is either standard or introduced when it is used.

Definition. *A real function u on an open set $G \subset \mathbb{R}^{n+1}$ having continuous partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i^2}$ for $i = 1, ..., n$, and satisfying the heat equation*

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

on G is called parabolic on G.

We will be interested in parabolic functions on $\mathbb{R}^n \times \mathbb{R}^+$.

The Weierstrass kernel for $\mathbb{R}^n \times \mathbb{R}^+$ with the pole in y in $\mathbb{R}^n$ is given by

$$p(X,y) = (4\pi t)^{-n/2} \exp(-\frac{\|x-y\|^2}{4t}),$$

where $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$.

A parabolic function which is the limit of an increasing sequence of bounded positive parabolic functions will be called quasi-bounded. The class of all functions u which can be expressed as a difference of two positive quasi-bounded parabolic functions will be denoted by $\mathcal{P}_1$.

Moreover, a parabolic function u will be called simple if there exists a λ-measurable subset A of $\mathbb{R}^n \times \{0\}$ such that $u(X) = \int_A p(X,y)\,dy$ for any $X \in \mathbb{R}^n \times \mathbb{R}^+$.

The class of all simple parabolic functions will be denoted by $\mathcal{P}_s$. Of course, $\mathcal{P}_s \subset \mathcal{P}_1$.

Theorem A. *A function u on $\mathbb{R}^n \times \mathbb{R}^+$ is a difference of two positive quasi-bounded parabolic functions if and only if there is a λ-measurable function f_u on $\mathbb{R}^n$ for which*

$$\int_{\mathbb{R}^n} \exp(-\frac{\|y\|^2}{4t})|f_u(y)|\,dy < \infty$$

for all $t > 0$ and

$$u(X) = \int_{\mathbb{R}^n} p(X,y) f_u(y)\,dy,\ X \in \mathbb{R}^n \times \mathbb{R}^+;$$

u is positive, if and only if f_u is positive λ-almost everywhere.

It is clear that the function f_u is uniquely determined λ-almost everywhere. In what follows, for any $u \in \mathcal{P}_1$, f_u will denote this function.

PROOF: See [5, p. 291]. □

Definition. *A point $Y = (y,0)$ is called a parabolic limit (resp. a 1-parabolic limit) of a sequence $\{X_k\}$, $X_k = (x_k, t_k)$, of points in $\mathbb{R}^n \times \mathbb{R}^+$, if $\{X_k\}$ converges to Y and $\liminf_{k\to\infty} t_k\|x_k - y\|^{-2} > 0$ (that is, all X_k belong to some paraboloid of revolution with vertex Y and opening upward)*
(resp. $\liminf_{k\to\infty} t_k\|x_k - y\|^{-2} \geq 1$).

Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$. A point $Y \in \mathbb{R}^n \times \{0\}$ is called a parabolic limit point (resp. a 1-parabolic limit point) of the set M if there exists a sequence $\{X_k\}$ such that every $X_k \in M$ and Y is a parabolic limit (resp. a 1-parabolic limit) of $\{X_k\}$.

A function f on $\mathbb{R}^n \times \mathbb{R}^+$ is said to have a parabolic limit q at Y if $\{f(X_k)\}$ converges to q whenever Y is a parabolic limit of $\{X_k\}$.

Theorem B (The Fatou limit theorem)**.** *Let u be a parabolic function in $\mathcal{P}_1$. Then for λ-almost all $y \in \mathbb{R}^n$ the function u has the parabolic limit $f_u(y)$ at $Y = (y, 0)$.*

Proof: See [5, p. 292]. $\square$

Lemma 1. *A parabolic function u on $\mathbb{R}^n \times \mathbb{R}^+$ is bounded if and only if there is a function $f_u \in L_\infty(\mathbb{R}^n)$ such that*

$$u(X) = \int_{\mathbb{R}^n} p(X, y) f_u(y)\, dy,\ \ X \in \mathbb{R}^n \times \mathbb{R}^+.$$

If it is the case, then $\sup\limits_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \operatorname*{ess\,sup}\limits_{y \in \mathbb{R}^n} f_u(y)$ *and* $\sup\limits_{X \in \mathbb{R}^n \times \mathbb{R}^+} |u(X)| = \|f_u\|_{L_\infty(\mathbb{R}^n)}$.

Proof: Since u is a bounded parabolic function, it belongs to $\mathcal{P}_1$. From Theorem A it follows that there is a λ-measurable function f_u such that

$$u(X) = \int_{\mathbb{R}^n} p(X, y) f_u(y)\, dy,\ \ X \in \mathbb{R}^n \times \mathbb{R}^+.$$

As for the constant function $u = c$ we have $f_u = c$ (λ-almost everywhere), by Theorem A it follows that $c - u \geq 0$ if and only if $c - f_u \geq 0$ λ-almost everywhere. Consequently, $\sup\limits_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \operatorname*{ess\,sup}\limits_{y \in \mathbb{R}^n} f_u(y)$. The rest is trivial. $\square$

The class of all bounded parabolic functions on $\mathbb{R}^n \times \mathbb{R}^+$ will be denoted by $\mathcal{P}_b$. Of course, $\mathcal{P}_s \subset \mathcal{P}_b \subset \mathcal{P}_1$.

Lemma 2. *Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$. Then the set of all parabolic limit points of M is λ-measurable.*

Proof: The point $Y = (y, 0)$ is a parabolic limit point of M if and only if

$$\limsup_{t_0 \to 0+} \sup_{\{x; (x, t_0) \in M\}} \frac{t_0}{\|x - y\|^2} > 0.$$

It follows from the definition of parabolic limits points. Remark that $\sup \emptyset = -\infty$, as usual.

Let M_0 be a countable dense subset of M. The point Y is a parabolic limit point of M if and only if Y is a parabolic limit point of M_0. That is why we can assume that M is a countable set.

For $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, define

$$\begin{aligned} g(t, x, y) &= \frac{t}{\|x - y\|^2}, && \text{if } x \neq y, \\ &= \infty, && \text{if } x = y. \end{aligned}$$

It is easy to see that the function g is a continuous function from $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$ into $(0, \infty]$. Fix x and t. The function $g(t, x, .)$ is measurable (even continuous) from $\mathbb{R}^n$ into $(0, \infty]$.

For fixed t, let us put

$$g_t(y) = \sup_{\{x;(x,t)\in M\}} \frac{t}{\|x-y\|^2}.$$

Let us recall that M is countable. Then g_t, being a supremum of a countable system of measurable functions, is measurable.

Because M is countable, only for countably many t the function g_t is strictly positive. Let us denote this set of t by T.

So, Y is a parabolic limit point of M if and only if

$$\limsup_{t\to 0, t\in T} g_t(y) > 0.$$

But then g, being a limes superior of a countable system of measurable functions, is measurable.

Thus the set of all parabolic limit points, which is $\{g > 0\}$, is measurable. $\square$

2. Sets of determination

Theorem 1. *Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$.*

If

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} u(X) = \sup_{X\in M} u(X)$$

for all bounded positive parabolic functions, then λ-almost every point $y \in \mathbb{R}^n \times \{0\}$ is a parabolic limit point of M.

Proof: Suppose it is not true.

The set of all parabolic limit points of M will be denoted by M_p. We know from the preceding lemma that the set M_p is measurable. Then its complement M_p' is also measurable and by our assumption $\lambda(M_p') > 0$.

For $k \in \mathbb{N}$ and $y \in \mathbb{R}^n \times \{0\}$, Γ_y^k will denote the set

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+;\ 1/k > t > \|x-y\|^2\}.$$

Then, for every $y \in M_p'$, there is $k_y \in \mathbb{N}$ such that $\Gamma_y^{k_y} \cap M$ is empty. Denote by D_k the set of $y \in M_p'$ for which $\Gamma_y^k \cap M$ is empty.

We will prove now that, for any k, the set D_k is a measurable subset of $\mathbb{R}^n \times \{0\}$. By definition

$$D_k = M_p' \cap \{y \in \mathbb{R}^n \times \{0\};\ M \cap \Gamma_y^k = \emptyset\}.$$

We know that M_p' is measurable. Thus it remains to prove that $\{y \in \mathbb{R}^n \times \{0\};\ M \cap \Gamma_y^k = \emptyset\}$ is measurable.

But $M \cap \Gamma_y^k = \emptyset$ if and only if for any $t \in (0, 1/k)$:

$$\sup_{\{x;\,(x,t)\in M\}} \frac{t}{\|x-y\|^2} \le 1.$$

Let M_0 be a countable dense subset of M. The point Y is a parabolic limit point of M if and only if Y is a parabolic limit point of M_0. That is why we can assume that M is a countable set. Let us fix t. Then

$$y \longmapsto \sup_{\{x;\,(x,t)\in M\}} \frac{t}{\|x-y\|^2}$$

is a measurable function and

$$\{y \in \mathbb{R}^n;\ \sup_{\{x;\,(x,t)\in M\}} \frac{t}{\|x-y\|^2} \le 1\}$$

is a measurable set.

This set can be different from $\mathbb{R}^n$ only for t such that there is $X = (x,t) \in M$. Let us denote this set by T. So, the intersection

$$\bigcap_{0<t<1/k} \{y \in \mathbb{R}^n;\ \sup_{\{x;\,(x,t)\in M\}} \frac{t}{\|x-y\|^2} \le 1\}$$

is equal to

$$\bigcap_{\substack{0<t<1/k\\ t\in T}} \{y \in \mathbb{R}^n;\ \sup_{\{x;\,(x,t)\in M\}} \frac{t}{\|x-y\|^2} \le 1\},$$

and, as an intersection of countable system of measurable sets, it is measurable. It means that D_k is measurable.

As $\bigcup_{k=1}^{\infty} D_k = M_p'$, the Lebesgue measure of at least one of the sets D_k is strictly positive. Let it be the set D_a where $a \in \mathbb{N}$. Then there is a bounded subset of D_a which has strictly positive Lebesgue measure. Denote this set by D.

For a point $X = (x,t)$ of $\mathbb{R}^n \times \mathbb{R}^+$ we define

$$A_X = \{(y,0) \in \mathbb{R}^n \times \{0\};\ \|x-y\|^2 < t\}.$$

(A_X is the ball in $\mathbb{R}^n \times \{0\}$ with the center $(x,0)$ and radius $t^{1/2}$.)
Denote $D' = (\mathbb{R}^n \times \{0\}) \setminus D$.

Let $X = (x,t) \in M \cap (\mathbb{R}^n \times (0,1/a))$ and let $y \in A_X$. Then $1/a > t > \|x-y\|^2$, so that $X \in \Gamma_y^a$. Consequently, $y \notin D_a$, and so $y \notin D$. We conclude that $A_X \subset D'$, whenever $X \in M \cap (\mathbb{R}^n \times (0,1/a))$.

For any measurable set $A \subset \mathbb{R}^n \times \{0\}$ we define

$$u_A(X) = \int_A p(X,y)\,dy,\ \ X \in \mathbb{R}^n \times \mathbb{R}^+.$$

It means $u_A \in \mathcal{P}_s$, thus $u_A \in \mathcal{P}_b$ and $-u_A \in \mathcal{P}_b$. By Lemma 1 we get u_A is positive, and if $\lambda(A) > 0$, then $\sup\limits_{\mathbb{R}^n \times \mathbb{R}^+} u_A(X) = 1$.

Denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$ by α_n. Then for any $X \in \mathbb{R}^n \times \mathbb{R}^+$:

$$\begin{aligned} u_{A_X}(X) &= \int_{A_X} p(X,y)\,dy = (4\pi t)^{-n/2} \int_{A_X} \exp(-\frac{\|x-y\|^2}{4t})\,dy \\ &\geq (4\pi t)^{-n/2} \int_{A_X} \exp(-\frac{t}{4t})\,dy = (4\pi t)^{-n/2} \lambda(A_X) \exp(-1/4) \\ &= (4\pi)^{-n/2} \alpha_n \exp(-1/4). \end{aligned}$$

We see that

$$c = \inf_{X \in \mathbb{R}^n \times \mathbb{R}^+} u_{A_X}(X) > 0.$$

The set D has a positive measure, hence the function u_D is positive, parabolic, bounded and

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u_D(X) = 1.$$

Since $u_D + u_{D'} = 1$, we get for every $N \subset \mathbb{R}^n \times \mathbb{R}^+$ the equality

$$\sup_{X \in N} u_D(X) = 1 - \inf_{X \in N} u_{D'}(X).$$

But A_X is a subset of D' for every $X \in M \cap (\mathbb{R}^n \times (0, 1/a))$.

It follows that

$$u_{D'}(X) \geq u_{A_X}(X) \geq c$$

for every $X \in M \cap (\mathbb{R}^n \times (0, 1/a))$.

Then

$$\sup_{X \in M \cap (\mathbb{R}^n \times (0,1/a))} u_D(X) \leq 1 - c.$$

Now $\sup\limits_{X \in M \cap (\mathbb{R}^n \times [1/a, \infty))} u_D(X)$ will be estimated. As D is bounded, there is

$d \in \mathbb{R}^+$ such that $D \subset [-d,d]^n$ and so we have for any $(x,t) \in \mathbb{R}^n \times [1/a, \infty)$

$$\begin{aligned} u_D(x,t) &= (4\pi t)^{-n/2} \int\limits_D \exp(-\frac{\|x-y\|^2}{4t})\,dy \\ &\leq (4\pi t)^{-n/2} \int\limits_{[-d,d]^n} \exp(-\frac{\|x-y\|^2}{4t})\,dy \leq (4\pi t)^{-n/2} \int\limits_{[-d,d]^n} \exp(-\frac{\|x-y\|^2}{4t})\,dy \\ &\leq (4\pi t)^{-n/2} \int\limits_{[-d,d]^n} \exp(-\frac{\|y\|^2}{4t})\,dy = \pi^{-n/2} \int\limits_{[-d/(2\sqrt{t}),d/(2\sqrt{t})]^n} \exp(-\|y\|^2)\,dy \\ &\leq \pi^{-n/2} \int\limits_{[-d\sqrt{a}/2,d\sqrt{a}/2]^n} \exp(-\|y\|^2)\,dy < 1. \end{aligned}$$

Thus

$$\sup_{X \in M \cap (\mathbb{R}^n \times [1/a,\infty))} u_D(X) < 1.$$

Consequently,

$$\sup_{X \in M} u_D(X) \leq \max\,(1-c, \sup_{X \in M \cap (\mathbb{R}^n \times [1/a,\infty))} u_D(X)) < 1,$$

contradicting our assumption. □

In fact, we proved a bit more than the assertion of Theorem 1. Namely we proved

Theorem 1'. *Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$.*
If

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \sup_{X \in M} u(X)$$

for all simple parabolic functions, then λ-almost every point $y \in \mathbb{R}^n \times \{0\}$ is a 1-parabolic limit point of M.

Theorem 2. *Let M be a subset of $\mathbb{R}^n \times \mathbb{R}^+$ and let λ-almost every point $(y,0) \in \mathbb{R}^n \times \{0\}$ be a parabolic limit point of M. Then*

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} |u(X)| = \sup_{X \in M} |u(X)|$$

for every bounded parabolic function u on $\mathbb{R}^n \times \mathbb{R}^+$.

PROOF: Let u be a bounded parabolic function on $\mathbb{R}^n \times \mathbb{R}^+$. By Lemma 1, there exists $f \in L_\infty(\mathbb{R}^n)$ such that

$$u(X) = \int\limits_{\mathbb{R}^n} p(X,y) f(y)\,dy,\ X \in \mathbb{R}^n \times \mathbb{R}^+$$

and

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} |u(X)| = \|f\|_{L_\infty}.$$

It is clear that

$$\sup_{X\in M} |u(X)| \le \sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} |u(X)| = \|f\|_{L_\infty}.$$

By the hypothesis, λ-almost every point of $\mathbb{R}^n \times \{0\}$ is a parabolic limit point of M. From the Fatou limit theorem, for λ-almost every $y \in \mathbb{R}^n$,

$$\lim u(X_k) = f(y),$$

whenever y is a parabolic limit of $\{X_k\}$. Thus

$$\sup_{X\in M} |u(X)| \ge \|f\|_{L_\infty},$$

so that

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} |u(X)| = \sup_{X\in M} |u(X)|.$$

$\square$

3. A decomposition theorem for $L_1(\mathbb{R}^n)$

Let us introduce the following notation: The closure of $M \subset \mathbb{R}^n \times \mathbb{R}^+$ in the set $\mathbb{R}^n \times \mathbb{R}^+$ (which is the set $\overline{M} \cap (\mathbb{R}^n \times \mathbb{R}^+)$) will be denoted by $\widetilde{M}$.

The support of a measure ν on $\mathbb{R}^n \times \mathbb{R}^+$ with respect to $\mathbb{R}^n \times \mathbb{R}^+$ (it means the complement in $\mathbb{R}^n \times \mathbb{R}^+$ of the largest open set $G \subset \mathbb{R}^n \times \mathbb{R}^+$ such that $\nu(G) = 0$) will be denoted by $s(\nu)$.

Theorem 3. *Let M be a subset of $\mathbb{R}^n \times \mathbb{R}^+$ and ν be a σ-finite measure on $\mathbb{R}^n \times \mathbb{R}^+$ such that $s(\nu) = \widetilde{M}$. Let*

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} |u(X)| = \sup_{X\in M} |u(X)|$$

for every bounded parabolic function u on $\mathbb{R}^n \times \mathbb{R}^+$.

Then, for any f in $L_1(\mathbb{R}^n)$, there exists Φ in $L_1(\nu)$ such that

$$f = \int_{\mathbb{R}^n\times\mathbb{R}^+} \Phi(X) p(X, .)\, d\nu(X) \tag{1}$$

λ-almost everywhere and

$$\|f\|_{L_1(\mathbb{R}^n)} = \inf\, \{\|\Phi\|_{L_1(\nu)};\ (1)\ \textit{holds for some}\ \Phi \in L_1(\nu)\}.$$

Further, there exists a sequence $\{X_k\}$, $X_k \in M$ *and* $\{\lambda_k\} \in l_1$ *such that*

$$f = \sum_{k=1}^{\infty} \lambda_k p(X_k, .) \tag{2}$$

λ*-almost everywhere and*

$$\|f\|_{L_1(\mathbb{R}^n)} = \inf \{\sum |\lambda_k|;\ (2) \text{ holds for some } \{X_k\} \text{ in } M\}.$$

The second part is an easy consequence of the first one. Just take a dense countable subset of M, denote it $\{X_k\}$, and take ν the counting measure on it.

We will need the following version of the closed range theorem (see [7, p. 97]). *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces,* T *a bounded linear mapping of* $\mathcal{X}$ *into* $\mathcal{Y}$*. If there exists a constant* $c > 0$ *such that* $\|T^* y^*\| \geq c\|y^*\|$ *for all* $y^* \in \mathcal{Y}^*$ *then* $T\mathcal{X} = \mathcal{Y}$. In our situation, $\mathcal{X} = L_1(\nu)$, $\mathcal{Y} = L_1(\mathbb{R}^n)$ and for $\Phi \in L_1(\nu)$ we define

$$T_\nu \Phi = \int\limits_{\mathbb{R}^n \times \mathbb{R}^+} p(X, .)\Phi(X)\, d\nu(X).$$

Lemma 3. *The mapping* T_ν *is a bounded linear mapping of* $L_1(\nu)$ *into* $L_1(\mathbb{R}^n)$*,* $\|T_\nu\| = 1$*;* T_ν^* *is the parabolic extension mapping* $L_\infty(\mathbb{R}^n)$ *into* $L_\infty(\nu)$*.*

Proof: Using the Fubini theorem we arrive at

$$\begin{aligned}
\|T_\nu \Phi\|_{L_1(\mathbb{R}^n)} &= \int\limits_{\mathbb{R}^n} |T_\nu \Phi|\, d\lambda = \int\limits_{\mathbb{R}^n} |\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} p(X, y)\Phi(X)\, d\nu(X)|\, d\lambda(y) \\
&\leq \int\limits_{\mathbb{R}^n} (\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} p(X, y)|\Phi(X)|\, d\nu(X))\, d\lambda(y) \\
&= \int\limits_{\mathbb{R}^n \times \mathbb{R}^+} (\int\limits_{\mathbb{R}^n} p(X, y)|\Phi(X)|\, d\lambda(y))\, d\nu(X) \\
&= \int\limits_{\mathbb{R}^n \times \mathbb{R}^+} |\Phi(X)|(\int\limits_{\mathbb{R}^n} p(X, y)\, d\lambda(y))\, d\nu(X) \\
&= \int\limits_{\mathbb{R}^n \times \mathbb{R}^+} |\Phi(X)|\, d\nu(X) = \|\Phi\|_{L_1(\nu)}.
\end{aligned}$$

So, the first part of Lemma is proved.

Let $g \in L_\infty(\mathbb{R}^n)$ and $\Phi \in L_1(\nu)$. Using again the Fubini theorem we have

$$[\Phi, T_\nu^* g] = [T_\nu \Phi, g] = \int\limits_{\mathbb{R}^n} g T_\nu \Phi\, d\lambda = \int\limits_{\mathbb{R}^n} g(y)(\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} \Phi(X) p(X, y)\, d\nu(X))\, d\lambda(y) =$$

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} \Phi(X)(\int\limits_{\mathbb{R}^n} g(y)p(X, y)\, d\lambda(y))\, d\nu(X) = [\Phi, \int\limits_{\mathbb{R}^n} p(X, y) g(y)\, d\lambda(y)].$$

$\square$

Proof of Theorem 3. We shall prove the existence of a constant $c > 0$ such that $\|T_\nu^* g\|_{L_\infty(\nu)} \geq c\|g\|_{L_\infty(\mathbb{R}^n)}$ for all $g \in L_\infty(\mathbb{R}^n)$ and the first part of the theorem will be proved.

The function $T_\nu^* g$ is bounded and parabolic on $\mathbb{R}^n \times \mathbb{R}^+$. Then, by hypothesis, Lemma 3 and Lemma 1,

$$\sup_{X \in M} |(T_\nu^* g)(X)| = \sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} |(T_\nu^* g)(X)| = \|g\|_{L_\infty(\mathbb{R}^n)}.$$

Since $T_\nu^* g$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^+$ and $s(\nu) = \widetilde{M}$,

$$\|T_\nu^* g\|_{L_\infty(\nu)} = \sup_{X \in M} |(T_\nu^* g)(X)|.$$

Consequently,

$$\|T_\nu^* g\|_{L_\infty(\nu)} = \|g\|_{L_\infty(\mathbb{R}^n)}.$$

So, we can take $c = 1$. The first part of Theorem 3 is proved.

To prove the other part, define the space

$$\mathcal{Z} = L_1(\nu) / \ker T_\nu.$$

For $z \in \mathcal{Z}$ and $\Phi \in z$ put $\mathcal{S}z = T_\nu \Phi$.

Then $\mathcal{S}$ is an invertible bounded linear mapping of $\mathcal{Z}$ into $L_1(\mathbb{R}^n)$ and so its adjoint $\mathcal{S}^*$ is an invertible bounded linear mapping of $L_\infty(\mathbb{R}^n)$ into $\mathcal{Z}^*$ (see [7, p. 94]).

Let $z \in \mathcal{Z}$, $\Phi \in z$ and $g \in L_\infty(\mathbb{R}^n)$. Then we have

$$(\mathcal{S}^* g)(z) = [Sz, g] = [T_\nu \Phi, g] = [\Phi, T_\nu^* g].$$

If $\varepsilon > 0$, there exists $\Phi_0 \in L_1(\nu)$ with $\|\Phi_0\|_{L_1(\nu)} = 1$ and

$$|[\Phi_0, T_\nu^* g]| > \|T_\nu^* g\|_{L_\infty(\nu)} - \varepsilon.$$

Let z_0 denote the coset of Φ_0 in $\mathcal{Z}$. Then

$$|(\mathcal{S}^* g)(z_0)| > \|T_\nu^* g\|_{L_\infty(\nu)} - \varepsilon,$$

and

$$\|z_0\|_{\mathcal{Z}} \leq \|\Phi_0\|_{L_1(\nu)} = 1.$$

Therefore, the norm of the functional $\mathcal{S}^* g$ satisfies

$$\|\mathcal{S}^* g\|_{\mathcal{Z}^*} > \|T_\nu^* g\|_{L_\infty(\nu)} - \varepsilon = \|g\|_{L_\infty(\mathbb{R}^n)} - \varepsilon.$$

Since ε was arbitrary, we proved that

$$\|\mathcal{S}^* g\|_{\mathcal{Z}^*} \geq \|g\|_{L_\infty(\mathbb{R}^n)}$$

for any $g \in L_\infty(\mathbb{R}^n)$, and so, using the fact that the norm of any operator is the same as the norm of its adjoint (see [7, p. 93]) and the obvious fact that $(\mathcal{S}^*)^{-1} = (\mathcal{S}^{-1})^*$, we have

$$\|\mathcal{S}^{-1}\| = \|(\mathcal{S}^*)^{-1}\| \leq 1.$$

Fix $f \in L_1(\mathbb{R}^n)$ and put $z = \mathcal{S}^{-1} f$. Then

$$\|z\|_{\mathcal{Z}} \leq \|f\|_{L_1(\mathbb{R}^n)},$$

that is

$$\inf \{\|\Phi\|_{L_1(\nu)}; T_\nu \Phi = f\} \leq \|f\|_{L_1(\mathbb{R}^n)}.$$

By Lemma 3 we have

$$\|f\|_{L_1(\mathbb{R}^n)} = \|T_\nu \Phi\|_{L_1(\mathbb{R}^n)} \leq \|T_\nu\|.\|\Phi\|_{L_1(\nu)} = \|\Phi\|_{L_1(\nu)}.$$

So, the opposite inequality holds as well. □

Theorem 4. *Let ν be a σ-finite measure on $\mathbb{R}^n \times \mathbb{R}^+$ and $s(\nu) = \widetilde{M}$. Assume that for every function $f \in L_1(\mathbb{R}^n)$ there exists Φ in $L_1(\mathbb{R}^n)$ such that*

$$f = \int_{\mathbb{R}^n \times \mathbb{R}^+} \Phi(X) p(X, .)\, d\nu(X) \tag{1}$$

λ-almost everywhere and

$$\|f\|_{L_1(\mathbb{R}^n)} = \inf \{\|\Phi\|_{L_1(\nu)};\ (1) \textit{ holds for some } \Phi \textit{ in } L_1(\nu)\}.$$

Then

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \sup_{X \in M} u(X)$$

for any quasi-bounded positive parabolic function u on $\mathbb{R}^n \times \mathbb{R}^+$.

Proof: Put $c = \sup_{X \in M} u(X)$. If $c = \infty$, $\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \infty$. So, suppose that $c < \infty$.

Let $\varepsilon > 0$. If we fix $X_0 \in \mathbb{R}^n \times \mathbb{R}^+$, then $p(X_0, .) \in L_1(\mathbb{R}^n)$ and $\|p(X_0, .)\|_{L_1(\mathbb{R}^n)} = 1$. By assumptions there is a function $\Phi \in L_1(\nu)$ such that

$$p(X_0, .) = \int_{\mathbb{R}^n \times \mathbb{R}^+} \Phi(X) p(X, .)\, d\nu(X) \leq \int_{\mathbb{R}^n \times \mathbb{R}^+} |\Phi(X)| p(X, .)\, d\nu(X)$$

and

$$\|\Phi\|_{L_1(\nu)} < 1 + \varepsilon.$$

As u is a quasi-bounded positive parabolic function, $u \in \mathcal{P}$ and we can integrate the first inequality with respect to $f_u \, d\lambda$. Using the Fubini theorem and the fact that $u \leq c$ on $s(\nu)$, we have

$$u(X_0) = \int\limits_{\mathbb{R}^n} p(X_0, y) f_u(y)\, dy \leq \int\limits_{\mathbb{R}^n} (\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} |\Phi(X)| p(X,y) d\nu(X)) f_u(y)\, dy =$$

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} |\Phi(X)| (\int\limits_{\mathbb{R}^n} p(X,y) f_u(y)\, dy)\, d\nu(X) = \int\limits_{\mathbb{R}^n \times \mathbb{R}^+} |\Phi(X)| u(X)\, d\nu(X) \leq$$

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} c.|\Phi(X)|\, d\nu(X) = c\|\Phi\|_{L_1(\nu)} \leq c(1+\varepsilon).$$

Since X_0 and ε were arbitrary, we have $\sup\limits_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = c.$ $\square$

Of course, the following special form of Theorem 4 holds:

Theorem 4'. *Let M be a subset of $\mathbb{R}^n \times \mathbb{R}^+$. Assume that for every function $f \in L_1(\mathbb{R}^n)$ there exist $\{\lambda_k\}_{k=1}^{\infty}$ in l_1 and a sequence $\{X_k\}_{k=1}^{\infty}$ of points in M such that*

$$f = \sum_{k=1}^{\infty} \lambda_k p(X_k, .) \tag{2}$$

$\lambda-$almost everywhere and

$$\|f\|_{L_1(\mathbb{R}^n)} = \inf \{ \sum_{k=1}^{\infty} |\lambda_k|;\ (2) \text{ holds for some } \{X_k\} \text{ in } M\}.$$

Then

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \sup_{X \in M} u(X)$$

for any quasi-bounded positive parabolic function u.

4. The main results

Theorem 5. *Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$. Then the following statements are equivalent:*

(i)

$$\sup_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \sup_{X \in M} u(X)$$

for all simple parabolic functions u;

(ii)

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} u(X)=\sup_{X\in M} u(X)$$

for all bounded positive parabolic functions u*;*

(iii)

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} u(X)=\sup_{X\in M} u(X)$$

for all bounded parabolic functions u*;*

(iv)

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} u(X)=\sup_{X\in M} u(X)$$

for all quasi-bounded positive parabolic functions u*;*

(v)

$$\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} |u(X)|=\sup_{X\in M} |u(X)|$$

for all bounded parabolic functions u*;*

(vi) *for* λ*-almost every point* $Y\in\mathbb{R}^n\times\{0\}$ *there is a sequence of points of* M *for which* Y *is a parabolic limit;*

(vii) *for* λ*-almost every point* $Y\in\mathbb{R}^n\times\{0\}$ *there is a sequence of points of* M *for which* Y *is a* 1*-parabolic limit;*

(viii) *if* ν *is a* σ*-finite Borel measure with* $s(\nu)=\widetilde{M}$*, then for every* $f\in L_1(\mathbb{R}^n)$ *there exists* $\Phi\in L_1(\nu)$ *such that*

$$f=\int\limits_{\mathbb{R}^n\times\mathbb{R}^+}\Phi(X)p(X,.)\,d\nu(X) \tag{1}$$

λ*-almost everywhere and*

$$\|f\|_{L_1(\mathbb{R}^n)}=\inf\,\{\|\Phi\|_{L_1(\nu)};\ (1)\ \textit{holds for some}\ \Phi\in L_1(\nu)\};$$

(ix) *for every* $f\in L_1(\mathbb{R}^n)$*, there is a sequence* $\{X_k\}$*,* $X_k\in M$ *and* $\{\lambda_k\}\in l_1$ *such that*

$$f=\sum_{k=1}^{\infty}\lambda_k p(X_k,.) \tag{2}$$

λ*-almost everywhere and*

$$\|f\|_{L_1(\mathbb{R}^n)}=\inf\,\{\textstyle\sum|\lambda_k|;\ (2)\ \textit{holds for some}\ \{X_k\}\ \textit{in}\ M\}.$$

PROOF: (i) $\Rightarrow$ (vii) by Theorem 1'; (vii)$\Rightarrow$ (vi) is clear; (vi) $\Rightarrow$ (v) by Theorem 2; (v) $\Rightarrow$ (viii) by Theorem 3; (viii) $\Rightarrow$ (ix) is easy; (ix) $\Rightarrow$ (iv) by Theorem 4'; (iv) $\Rightarrow$ (ii) is clear; (ii) $\Rightarrow$ (iii) is easy; (iii) $\Rightarrow$ (i) is clear. $\square$

Theorem 6. *Let ν be a σ-finite Borel measure on $\mathbb{R}^n \times \mathbb{R}^+$. Then for every $f \in L_1(\mathbb{R}^n)$ there exists $\Phi \in L_1(\nu)$ such that*

$$f = \int_{\mathbb{R}^n \times \mathbb{R}^+} \Phi(X) p(X, .) \, d\nu(X) \tag{1}$$

and

$$\|f\|_{L_1(\mathbb{R}^n)} = \inf \{\|\Phi\|_{L_1(\nu)};\ (1) \text{ holds for some } \Phi \in L_1(\nu)\},$$

if and only if λ-almost every point $Y \in \mathbb{R}^n \times \{0\}$ is a parabolic limit point of $s(\nu)$.

Proof: The result is obtained by combining Theorem 4, Theorem 1, Theorem 2 and Theorem 3. □

5. Application

Definition. *Normal distribution $N(\mu, \sigma^2)$ on $\mathbb{R}$ with parameters μ and σ^2 is a continuous probability distribution on $\mathbb{R}$ with a density*

$$\phi(y; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(y-\mu)^2}{2\sigma^2});$$

normal distribution $N_n(\mu, \Sigma)$ on $\mathbb{R}^n$ with parameters μ and Σ, where $\mu \in \mathbb{R}^n$ and $\Sigma = (\sigma_{k,j})$ is a positive definite matrix $n \times n$, is a continuous probability distribution with density

$$\phi(y; \mu, \Sigma) = \phi(y_1, y_2, ..y_n; \mu, \Sigma) =$$

$$(2\pi)^{-n/2} |\Sigma|^{-n/2} \exp(-\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (y_j - \mu_j)(y_k - \mu_k)\sigma^{j,k}),$$

where $|\Sigma|$ is the determinant of Σ and $(\sigma^{j,k}) = \Sigma^{-1}$.

If Σ is a σ^2-multiple of the unit matrix U we have $|\Sigma| = \sigma^{2n}$, $\Sigma^{-1} = \sigma^{-2}U$ and

$$\phi(y; \mu, \Sigma) = \phi(y; \mu, \sigma^2) = (2\pi)^{-n/2} \sigma^{-n} \exp(-\frac{\|y-\mu\|^2}{2\sigma^2}).$$

Such normal distribution will be denoted by $N_n(\mu, \sigma^2)$ and the class of all such normal distributions will be denoted by $\mathcal{S}_U$. Let us remark that, if $n = 1$, then $\mathcal{S}_U$ is a class of all normal distributions.

(We recall that a continuous probability distribution on $\mathbb{R}^n$ is a probability measure on $\mathbb{R}^n$ which is absolutely continuous with respect to λ.)

Definition. *A set N is called non-tangentially dense in $\mathbb{R}^n \times \mathbb{R}^+$ if for λ-almost every point $(y, 0)$ of $\mathbb{R}^n \times \{0\}$ there is a sequence $\{(\mu_k, \sigma_k)\}$ of points of N such that $(y, 0)$ is a limit of this sequence and $\liminf_{k\to\infty} \sigma_k^{-1} \|\mu_k - y\| > 0$. (It means that there is a cone $C_y \subset \mathbb{R}^n \times \mathbb{R}^+$ with vertex in $(y, 0)$ and open upward such that M is a limit point of $M \cap C_y$.)*

Theorem 7. *Let $\mathcal{S} \subset \mathcal{S}_U$. Then the set $N = \{(\mu, \sigma) \in \mathbb{R}^n \times \mathbb{R}^+;\ N_n(\mu, \sigma^2) \in \mathcal{S}\}$ is non-tangentially dense if and only if for any continuous probability distribution P there is a sequence $\{N_n(\mu_k, \sigma_k^2)\}$ of normal distributions on $\mathbb{R}^n$ of $\mathcal{S}$ and $\{\lambda_k\} \in l_1$ such that P can be expressed in the form*

$$P = \sum_{k=1}^{\infty} \lambda_k N_n(\mu_k, \sigma_k^2), \tag{3}$$

and

$$\inf \{\sum_{k=1}^{\infty} |\lambda_k|;\ (3) \text{ holds for some } N_n(\mu_k, \sigma_k^2) \text{ in } \mathcal{S}\} = 1. \tag{4}$$

Proof: A function f is a density of some continuous distribution if and only if $f \geq 0$, $f \in L_1(\mathbb{R}^n)$ and $\|f\|_{L_1(\mathbb{R}^n)} = 1$.

Let us introduce substitution $t = \sigma^2/2$, $\mu = x$. Then $\phi(y, \mu, \sigma^2) = p((x,t), y)$. Denote $M = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+; (\mu, \sigma) \in N\}$.

The set N is non-tangentially dense if and only if for almost every $(y, 0)$ there is a sequence $(\mu_k, \sigma_k) \in N$, the sequence converges to $(y, 0)$ and $\liminf_{k\to\infty} \sigma_k \|\mu_k - y\|^{-1} > 0$, it means $\liminf_{k\to\infty} \sigma_k^2/2\|\mu_k - y\|^{-2} > 0$. Using the above substitution we have that it is equivalent to the existence of the sequence (x_k, t_k) of elements of M which converges to $(y, 0)$ and $\liminf_{k\to\infty} t_k \|x_k - y\|^{-2} > 0$. It means N is non-tangentially dense if and only if λ-almost every point of $\mathbb{R}^n \times \{0\}$ is a parabolic limit point of M.

Then, by Theorem 5, for any density f there is a sequence (x_k, t_k) of elements of M (it means $(\mu_k, \sigma_k) \in N$) and $\{\lambda_k\} \in l_1$

$$f = \sum_{k=1}^{\infty} \lambda_k p(X_k, .) \tag{5}$$

λ-almost everywhere and

$$1 = \|f\|_{L_1(\mathbb{R}^n)} = \inf \{\sum_{k=1}^{\infty} |\lambda_k|;\ (5) \text{ holds for some } \{x_k, t_k\} \text{ in } M\} \tag{6}$$

and so

$$f = \sum_{k=1}^{\infty} \phi(., \mu_k, \sigma_k^2) \tag{7}$$

$$1 = \inf \{\sum_{k=1}^{\infty} |\lambda_k|;\ (7) \text{ holds for some } N_n(\mu_k, \sigma_k^2) \text{ in } \mathcal{S}\}. \tag{8}$$

Integrating (7) with respect to λ over any λ-measurable set A and using the Lebesgue Convergence Theorem (can be used thanks to (8)) we have (3) and (4).

Let us suppose that any continuous distribution can be expressed in (3) and that (4) is true. Let P be a continuous distribution and f its density. Then (3) can be written in the form for any λ-measurable set A

$$\int_A f\,d\lambda = \sum_{k=1}^{\infty} \lambda_k \int_A \phi(.,\mu_k,\sigma_k^2)\,d\lambda.$$

Then, using the Lebesgue Convergence Theorem again (thanks to (4)), we have

$$\int_A f\,d\lambda = \int_A \sum_{k=1}^{\infty} \lambda_k \phi(.,\mu_k,\sigma_k^2)\,d\lambda.$$

Choosing A be a ball with the center y and radius r, dividing the equality by the λ-measure of the ball and letting $r \to 0$, we have (7) and (8) and thus (5) and (6).

Let us remark that for any $f \in L_1(\mathbb{R}^n)$ there are real numbers a_1, a_2 and functions f_1, f_2 which are densities of some distributions such that $f = a_1\, f_1 - a_2\, f_2$ and $\|f\|_{L_1(\mathbb{R}^n)} = a_1\, \|f_1\|_{L_1(\mathbb{R}^n)} + a_2\, \|f_2\|_{L_1(\mathbb{R}^n)}$. (Really, put $a_1 = \|f^+\|_{L_1(\mathbb{R}^n)}$ and $a_2 = \|f^-\|_{L_1(\mathbb{R}^n)}$. If $a_1 \neq 0$, put $f_1 = \frac{1}{a_1} f^+$, else $f_1 = \pi^{-n/2}\exp(-\|.\|^2)$, and if $a_2 \neq 0$, then put $f_2 = \frac{1}{a_2} f^-$, else $f_2 = \pi^{-n/2}\exp(-\|.\|^2)$.)

Then (5) and (6) are true for any $f \in L_1(\mathbb{R}^n)$ (of course, without 1) and, by Theorem 5, λ-almost every point of $\mathbb{R}^n \times \{0\}$ is a parabolic limit point of M, and then, as we know, N is non-tangentially dense. $\square$

6. Remark

Similar problems as in Sections 1–4 have been recently investigated for classical harmonic functions on a ball in [2], [3], [4], [6] and for more general domains in [1]. The proofs in Section 2 were inspired by [6]; those in Section 3 by [4]. Sufficiency of the non-tangential density of the set N in Theorem 7 can be obtained as a special case of Theorem 2.9 in [8]. (From this, using substitution $\sigma = \sqrt{2t}$, $x = \mu$, Theorem 5 (vi) $\Rightarrow$ (viii) follows.)

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