

Free ℓ -groups and free products of ℓ -groups

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Abstract. In this paper we have given the construction of free ℓ -groups generated by a po-group and the construction of free products in any sub-product class \mathcal{U} of ℓ -groups. We have proved that the \mathcal{U} -free products satisfy the weak subalgebra property.

Keywords: lattice-ordered group (ℓ -group), free ℓ -group, free product of ℓ -groups, sub-product class of ℓ -groups

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1. Introduction

We use the standard terminologies and notations of [1], [2], [5]. The group operation of an ℓ -group is written by additive notation. A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G \mid x \geq 0\}$ is the positive semigroup of G . Let G and H be two po-groups. A map $\varphi : G \rightarrow H$ is called a po-group homomorphism, if φ is a group homomorphism and $x \geq y$ implies $\varphi(x) \geq \varphi(y)$ for any $x, y \in G$. A po-group homomorphism φ is called a po-group isomorphism, if φ is an injection and φ^{-1} is also a po-group homomorphism.

A partial ℓ -group G is a set with partial operations corresponding to the usual ℓ -group operations $\cdot, ^{-1}, |, \vee$ and \wedge such that whenever the operations are defined for elements of G , the the ℓ -group laws are satisfied. Suppose $[G, P]$ is a po-group. Then G has implicit partial operations \vee and \wedge as determined by the partial order. That is,

$$x \vee y = y \vee x = y \text{ if and only if } x \leq y$$

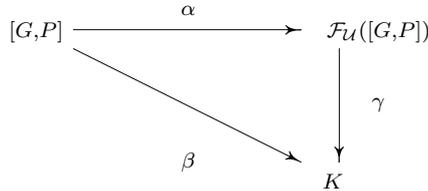
and

$$x \wedge y = y \wedge x = x \text{ if and only if } x \leq y.$$

Using these partial lattice operations together with the full group operations, G can be considered as a partial ℓ -group. Then we have the following definition as a special case of the \mathcal{U} -free algebra generated by a partial algebra.

Definition 1.1. Let \mathcal{U} be a class of ℓ -groups and $[G, P]$ be a po-group. The ℓ -group $\mathcal{F}_{\mathcal{U}}[G, P]$ is called the \mathcal{U} -free ℓ -group generated by $[G, P]$ (or \mathcal{U} -free ℓ -group over $[G, P]$) if the following conditions are satisfied:

- (1) $\mathcal{F}_{\mathcal{U}}([G, P]) \in \mathcal{U}$;
- (2) there exists a po-group isomorphism $\alpha : G \rightarrow \mathcal{F}_{\mathcal{U}}([G, P])$ such that $\alpha(G)$ generates $\mathcal{F}_{\mathcal{U}}([G, P])$ as an ℓ -group;
- (3) if $K \in \mathcal{U}$ and $\beta : G \rightarrow K$ is a po-group homomorphism, then there exists an ℓ -homomorphism $\gamma : \mathcal{F}_{\mathcal{U}}([G, P]) \rightarrow K$ such that $\gamma\alpha = \beta$.



Definition 1.2. Let \mathcal{U} be a class of ℓ -groups and $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ -groups in \mathcal{U} . The \mathcal{U} -free product of G_λ is an ℓ -group G , denoted by ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} G_\lambda$, together with a family of injective ℓ -homomorphisms $\alpha_\lambda : G_\lambda \rightarrow G$ (called coprojections) such that

- (1) ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} G_\lambda \in \mathcal{U}$;
- (2) $\bigcup_{\lambda \in \Lambda} \alpha_\lambda(G_\lambda)$ generates ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} G_\lambda$ as an ℓ -group;
- (3) if $K \in \mathcal{U}$ and $\{\beta_\lambda : G_\lambda \rightarrow K \mid \lambda \in \Lambda\}$ is a family of ℓ -homomorphisms, then there exists a (necessarily) unique ℓ -homomorphism $\gamma : G \rightarrow K$ satisfying $\beta_\lambda = \gamma\alpha_\lambda$ for all $\lambda \in \Lambda$.

A family \mathcal{U} of ℓ -groups is called a sub-product class, if it is closed under taking (1) ℓ -groups and (2) direct products. All our sub-product classes of ℓ -groups are always assumed to contain along with a given ℓ -group all its ℓ -isomorphic copies. Clearly, all varieties of ℓ -groups are sub-product classes of ℓ -groups. Let \mathcal{L} , \mathcal{R} and \mathcal{A} be the varieties of all ℓ -groups, representable ℓ -groups and abelian ℓ -groups, respectively.

In this paper we will discuss the existence and constructions of free ℓ -groups generated by a po-group and free products in any sub-product classes of ℓ -groups. In what follows, \mathcal{U} is always denoted a sub-product class of ℓ -groups.

2. Construction for a \mathcal{U} -free ℓ -group generated by a po-group

In 1963 and 1965, E.C. Weinberg initially considered the \mathcal{A} -free ℓ -group generated by a po-group $[G, P]$. He has given a necessary and sufficient condition for existence and a simple description of $\mathcal{F}_{\mathcal{A}}([G, P])$ in [17], [18].

In 1970, P. Conrad generalized Weinberg’s result as follows.

Lemma 2.1 ([3]).

- (1) There exists an \mathcal{L} -free ℓ -group $\mathcal{F}_{\mathcal{L}}([G, P])$ generated by $[G, P]$, if and only if there exists a po-group isomorphism of $[G, P]$ into an ℓ -group, if and only if P is the intersection of right order on G .

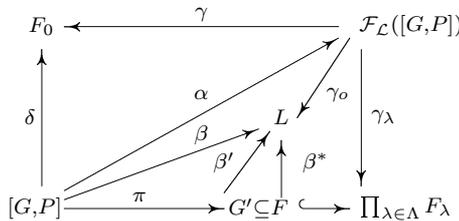
(2) Suppose that $P = \bigcap_{\lambda \in \Lambda} P_\lambda$ where $\{P_\lambda \mid \lambda \in \Lambda\}$ is the set of all right orders of G such that $P_\lambda \supseteq P$. If G_λ is G with one such right order, then denote by $A(G_\lambda)$ the ℓ -group of order preserving permutations of G_λ . Each $x \in G$ corresponds to an element ρ_x of $A(G_\lambda)$ defined by $\rho_x g = g + x$. Then $\mathcal{F}_{\mathcal{L}}([G, P])$ is the sublattice of $\prod_{\lambda \in \Lambda} A(G_\lambda)$ generated by the long constants $\langle g \rangle$ ($g \in G$).

By Grätzer existence theorem on a free algebra generated by a partial algebra (Theorem 28.2 of [6]) we have

Theorem 2.2. *There exists a \mathcal{U} -free ℓ -group $\mathcal{F}_{\mathcal{U}}([G, P])$ generated by a po-group $[G, P]$ if and only if $[G, P]$ is po-group isomorphic to an ℓ -group in \mathcal{U} .*

Lemma 2.3 (Lemma 11.3.1 of [5]). *Let L and L' be ℓ -groups and M be a subgroup of L which generates L as a lattice. Let $\varphi : M \rightarrow L'$ be a group homomorphism such that for each finite subset $\{x_{jk} \mid j \in J, k \in K\}$ of M , $\bigvee_{j \in J} \bigwedge_{k \in K} x_{jk} = 0$ implies $\bigvee_{j \in J} \bigwedge_{k \in K} \varphi(x_{jk}) = 0$. Then φ can be uniquely extended to an ℓ -homomorphism $\varphi' : L \rightarrow L'$.*

Let \mathcal{U} be a sub-product class of ℓ -groups. An ℓ -homomorphic image H of an ℓ -group G is said to be a \mathcal{U} -homomorphic image, if $H \in \mathcal{U}$. Suppose that a po-group $[G, P]$ is po-group isomorphic into an ℓ -group $F_0 \in \mathcal{U}$ with the po-group isomorphism δ . By Lemma 2.1 (1) there exists the \mathcal{L} -free ℓ -group $\mathcal{F}_{\mathcal{L}}([G, P])$ generated by $[G, P]$ with the po-group isomorphism $\alpha : [G, P] \rightarrow \mathcal{F}_{\mathcal{L}}([G, P])$. By Definition 1.1 there exists an ℓ -homomorphism $\gamma : \mathcal{F}_{\mathcal{L}}([G, P]) \rightarrow F_0$ such that $\gamma\alpha = \delta$. Let $D = \{F_\lambda \mid \lambda \in \Lambda\}$ be the set of all \mathcal{U} -homomorphic images of $\mathcal{F}_{\mathcal{L}}([G, P])$ with the ℓ -homomorphisms γ_λ ($\lambda \in \Lambda$). Thus $\gamma(\mathcal{F}_{\mathcal{L}}([G, P])) \in D$ and D is not empty.



For each $\lambda \in \Lambda$, $\gamma_\lambda\alpha$ is a po-group homomorphism of $[G, P]$ into F_λ . Then the direct product $\prod_{\lambda \in \Lambda} F_\lambda$ is an ℓ -group in \mathcal{U} . Let π be the natural map of G onto the subgroup G' of long constants of $\prod_{\lambda \in \Lambda} F_\lambda$. That is,

$$\pi(g) = (\dots, \gamma_\lambda\alpha(g), \dots)$$

for $g \in G$. Since $\gamma\alpha = \delta$ is a po-group isomorphism, π is a po-group isomorphism. Let F be the sublattice of $\prod_{\lambda \in \Lambda} F_\lambda$ generated by G' . Then F is an ℓ -subgroup of $\prod_{\lambda \in \Lambda}$, and so $F \in \mathcal{U}$.

Theorem 2.4. *Suppose that a po-group $[G, P]$ is po-group isomorphic into an ℓ -group in a sub-product class of ℓ -groups. Then the \mathcal{U} -free ℓ -group $\mathcal{F}_{\mathcal{U}}([G, P])$ generated by $[G, P]$ is the sublattice F of the direct product $\prod_{\lambda \in \Lambda} F_{\lambda}$ generated by the po-group isomorphic image G' of G where $\{F_{\lambda} \mid \lambda \in \Lambda\}$ are all ℓ -homomorphic images of the \mathcal{L} -free ℓ -group $\mathcal{F}_{\mathcal{L}}([G, P])$ generated by $[G, P]$.*

PROOF: We have already known that $F \in \mathcal{U}$ and $[G, P]$ is po-group isomorphic into F . Suppose that β is a po-group homomorphism of $[G, P]$ into an ℓ -group $L \in \mathcal{U}$. Then there exists an ℓ -homomorphism $\gamma_o : \mathcal{F}_{\mathcal{L}}([G, P]) \rightarrow L$ such that $\gamma_o \alpha = \beta$. So $\gamma'(\mathcal{F}_{\mathcal{L}}([G, P])) \in D$. For $g' = \pi(g) \in G'$ ($g \in G$), put

$$\beta'(g') = \beta(g).$$

Then β' is a group homomorphism of G' into L and $\beta'\pi = \beta$. By Lemma 2.3 we only need to show that for each finite subset $\{g_{jk} \mid j \in J, k \in K\} \subseteq G$, $\bigvee_{j \in J} \bigwedge_{k \in K} \beta'\pi(g_{jk}) \neq 0$ implies $\bigvee_{j \in J} \bigwedge_{k \in K} \pi(g_{jk}) \neq 0$. In fact,

$$\bigvee_{j \in J} \bigwedge_{k \in K} \gamma_o \alpha(g_{jk}) = \bigvee_{j \in J} \bigwedge_{k \in K} \beta(g_{jk}) = \bigvee_{j \in J} \bigwedge_{k \in K} \beta'\pi(g_{jk}) \neq 0.$$

Hence

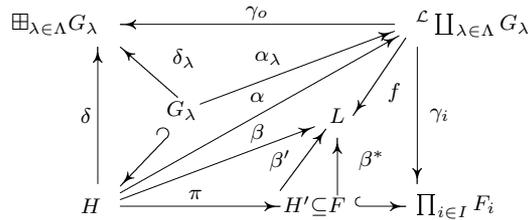
$$\begin{aligned} \bigvee_{j \in J} \bigwedge_{k \in K} \pi(g_{jk}) &= \bigvee_{j \in J} \bigwedge_{k \in K} (\dots, \gamma_o \alpha(g_{jk}), \dots) \\ &= (\dots, \bigvee_{j \in J} \bigwedge_{k \in K} \gamma_o \alpha(g_{jk}), \dots) \neq 0. \end{aligned}$$

Therefore β' can be uniquely extended to an ℓ -homomorphism $\beta^* : F \rightarrow L$. □

3. Construction of \mathcal{U} -free products

Let \mathcal{U} be a sub-product class of ℓ -groups and $\{G_{\lambda} \mid \lambda \in \Lambda\}$ be a family of ℓ -groups in \mathcal{U} . By Corollary 2 of Theorem 2 of [6] the \mathcal{U} -free product $\mathcal{U} \prod_{\lambda \in \Lambda} G_{\lambda}$ always exists. Specifically, there exists an \mathcal{L} -free product $\mathcal{L} \prod_{\lambda \in \Lambda} G_{\lambda}$ with the coprojection α_{λ} . In [7]–[14] J. Martinez, W. Powell and C. Tsinakis have given several descriptions and some properties for the free products in the varieties \mathcal{R} and \mathcal{A} . W.C. Holland and E. Scrimger have given a description for \mathcal{L} -free product. Let H be the group free product of $\{G_{\lambda} \mid \lambda \in \Lambda\}$. Let $P = \{h \in H \mid h \text{ be a sum of conjugates in } H \text{ of elements of } \bigcup_{\lambda \in \Lambda} G_{\lambda}^+\}$ and $P' = \{Q \mid Q \text{ is the positive cone of a right order on } H \text{ with } P \subseteq Q\}$. Then $[H, P']$ is a po-group and its \mathcal{L} -free extension $\mathcal{F}_{\mathcal{L}}([H, P'])$ by the ℓ -ideal generated by $\{g^- \wedge g^+ \mid g \in \bigcup_{\lambda \in \Lambda} G_{\lambda}\}$ (Theorem 3.7 of [4]). There exists a group homomorphism $\alpha : H \rightarrow \mathcal{L} \prod_{\lambda \in \Lambda} G_{\lambda}$ which extends every α_{λ} ($\lambda \in \Lambda$).

It is clear that the cardinal sum $\boxplus G_{\lambda}$ is an ℓ -group in \mathcal{U} and every G_{λ} ($\lambda \in \Lambda$) can naturally be embedded into $\boxplus_{\lambda \in \Lambda} G_{\lambda}$ as an ℓ -group with embedding δ_{λ} . Hence there exists a group homomorphism $\delta : H \rightarrow \boxplus_{\lambda \in \Lambda} G_{\lambda}$ which extends each δ_{λ}



($\lambda \in \Lambda$) and there exists an ℓ -homomorphism $\gamma_o : \mathcal{L} \prod_{\lambda \in \Lambda} G_\lambda \rightarrow \boxplus_{\lambda \in \Lambda} G_\lambda$ such that $\gamma_o \alpha_\lambda = \delta_\lambda$ for each $\lambda \in \Lambda$. Let $D = \{F_i \mid i \in I\}$ be the set of all \mathcal{U} -homomorphic images of $\mathcal{L} \prod_{\lambda \in \Lambda} G_\lambda$ with the ℓ -homomorphisms γ_i ($i \in I$). Thus, $\boxplus_{\lambda \in \Lambda} G_\lambda \in D$ and D is not empty. For each $\lambda \in \Lambda$ and each $i \in I$, $\gamma_i \alpha_\lambda$ is an ℓ -homomorphism of G_λ into F_i . The direct product $\prod_{i \in I} F_i$ is an ℓ -group in \mathcal{U} . For each $\lambda \in \Lambda$, let π_λ be the natural ℓ -homomorphism of G_λ onto the ℓ -subgroup G'_λ of $\prod_{i \in I} F_i$. That is,

$$\pi_\lambda(g_\lambda) = (\dots, \gamma_i \alpha_\lambda(g_\lambda), \dots)$$

for $g_\lambda \in G_\lambda$. Let H' be the subgroup of $\prod_{i \in I} F_i$ generated by $\bigcup_{\lambda \in \Lambda} G'_\lambda$. Let π be the group homomorphism of H onto H' which extends every π_λ ($\lambda \in \Lambda$). That is,

$$\pi(h) = (\dots, \gamma_i \alpha(h), \dots)$$

for $h \in H$. Since $\boxplus_{\lambda \in \Lambda} G_\lambda \in D$ and every δ_λ ($\lambda \in \Lambda$) is an ℓ -isomorphism, π_λ is an ℓ -isomorphism for each $\lambda \in \Lambda$. Let F be the sublattice of $\prod_{i \in I} F_i$ generated by H' . For each $h \in H$, put $h' = \pi(h)$. Since $\prod_{i \in I} F_i$ is a distributive lattice,

$$F = \left\{ \bigvee_{j \in J} \bigwedge_{k \in K} h'_{jk} \mid h_{jk} \in H, \quad J \text{ and } K \text{ finite} \right\}.$$

Then we have the following construction theorem for ${}^{\mathcal{U}} \prod_{\lambda \in \Lambda} G_\lambda$.

Theorem 3.1. *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of ℓ -groups in a sub-product class of ℓ -groups. Then the \mathcal{U} -free product ${}^{\mathcal{U}} \prod_{\lambda \in \Lambda} G_\lambda$ is the sublattice F of the direct product $\prod_{i \in I} F_i$ generated by the group homomorphic image H' of the group free product H of G_λ , where $\{F_i \mid i \in I\}$ are all \mathcal{U} -homomorphic images of the \mathcal{L} -free product $\mathcal{L} \prod_{\lambda \in \Lambda} G_\lambda$.*

PROOF: We have seen that $F \in \mathcal{U}$ and each G_λ ($\lambda \in \Lambda$) can be embedded into F as an ℓ -group. Suppose that $L \in \mathcal{U}$ and $\{\beta_\lambda : G_\lambda \rightarrow L \mid \lambda \in \Lambda\}$ is a family of ℓ -homomorphisms. We must show that there exists a unique ℓ -homomorphism $\beta^* : F \rightarrow L$ such that $\beta^* \pi_\lambda = \beta_\lambda$. By the universal property of group free product, there exists a group homomorphism $\beta : H \rightarrow L$ which extends every β_λ ($\lambda \in \Lambda$). For any $h' = \pi(h) \in H'$, put

$$\beta'(h') = \beta(h).$$

By the universal property of an \mathcal{L} -free product, there exists a unique ℓ -homomorphism $f : \mathcal{L} \coprod_{\lambda \in \Lambda} G_\lambda \rightarrow L$ such that $\beta_\lambda = f\alpha_\lambda$ for each $\lambda \in \Lambda$. Then

$$f\alpha = \beta'\pi = \beta.$$

By Lemma 2.3 we only need to show that for each finite subset $\{h_{jk} \mid j \in J, k \in K\} \subseteq H$, $\bigvee_{j \in J} \bigwedge_{k \in K} \beta'\pi(h_{jk}) \neq 0$ implies $\bigvee_{j \in J} \bigwedge_{k \in K} \pi(h_{jk}) \neq 0$. In fact,

$$\bigvee_{j \in J} \bigwedge_{k \in K} f\alpha(h_{jk}) = \bigvee_{j \in J} \bigwedge_{k \in K} \beta'\pi(h_{jk}) \neq 0.$$

Because $f(\mathcal{L} \coprod_{\lambda \in \Lambda} G_\lambda) \in D$, $\bigvee_{j \in J} \bigwedge_{k \in K} \gamma_i\alpha(h_{jk}) \neq 0$ for some $i \in I$. So

$$\begin{aligned} \bigvee_{j \in J} \bigwedge_{k \in K} \pi(h_{jk}) &= \bigvee_{j \in J} \bigwedge_{k \in K} (\dots, \gamma_i\alpha(h_{jk}), \dots) \\ &= (\dots, \bigvee_{j \in J} \bigwedge_{k \in K} \gamma_i\alpha(h_{jk}), \dots) \neq 0. \end{aligned}$$

Therefore β' can be uniquely extended to an ℓ -homomorphism $\beta^* : F \rightarrow L$. \square

By using the similar proof as the one for Theorem 3.1 we can get the following result.

Theorem 3.2. *Suppose that \mathcal{U} is a sub-product class of ℓ -groups which is contained in \mathcal{A} and $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in \mathcal{U} . Then the \mathcal{U} -free product $\mathcal{U} \coprod_{\lambda \in \Lambda} G_\lambda$ is the sublattice of $\prod_{i \in I} F_i$ generated by the group homomorphic image H' of the group free product H of G_λ , where $\{F_i \mid i \in I\}$ are all ℓ -homomorphic images of the \mathcal{A} -free product $\mathcal{A} \coprod_{\lambda \in \Lambda} G_\lambda$.*

4. The weak subalgebra property

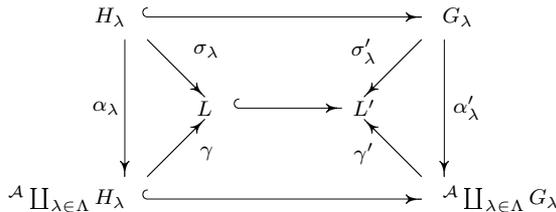
Let \mathcal{U} be a sub-product class of ℓ -groups. \mathcal{U} -free products are said to have the subalgebra property if for any family $\{G_\lambda \mid \lambda \in \Lambda\}$ in \mathcal{U} with ℓ -subgroups $H_\lambda \in G_\lambda$, $\mathcal{U} \coprod_{\lambda \in \Lambda} H_\lambda$ is simply the ℓ -subgroup of $\mathcal{U} \coprod_{\lambda \in \Lambda} G_\lambda$ generated by $\bigcup_{\lambda \in \Lambda} H_\lambda$. It is well known that \mathcal{A} -free products satisfy the subalgebra property (Theorem 3.2 of [11]). \mathcal{U} -free products are said to have the weak subalgebra property if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in \mathcal{U} with ℓ -subgroups $H_\lambda \subseteq G_\lambda$ and any family of ℓ -homomorphisms $\sigma_\lambda : H_\lambda \rightarrow L \in \mathcal{U}$ can be extended to a family of ℓ -homomorphisms $\sigma'_\lambda : G_\lambda \rightarrow L' \in \mathcal{U}$ such that L is an ℓ -subgroup of L' and $\sigma'_\lambda|_{H_\lambda} = \sigma_\lambda$, then $\mathcal{U} \coprod_{\lambda \in \Lambda} H_\lambda$ is the ℓ -subgroup of $\mathcal{U} \coprod_{\lambda \in \Lambda} G_\lambda$ generated by $\bigcup_{\lambda \in \Lambda} H_\lambda$.

Theorem 4.1. *Suppose that \mathcal{U} is a sub-product class of ℓ -groups which is contained in \mathcal{A} . Then \mathcal{U} -free products satisfy the weak subalgebra property.*

PROOF: Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in \mathcal{U} with ℓ -subgroups $H_\lambda \subseteq G_\lambda$ and any family of ℓ -homomorphisms $\sigma_\lambda : H_\lambda \rightarrow L \in \mathcal{U}$ can be extended to a family

of ℓ -homomorphisms $\sigma'_\lambda : G_\lambda \rightarrow L' \in \mathcal{U}$ such that L is an ℓ -subgroup of L' and $\sigma'_\lambda|_{H_\lambda} = \sigma_\lambda$. We see that ${}^A \prod_{\lambda \in \Lambda} H_\lambda$ is the ℓ -subgroup of ${}^A \prod_{\lambda \in \Lambda} G_\lambda$ generated by $\bigcup_{\lambda \in \Lambda} H_\lambda$.

(1) First we show that any ℓ -homomorphism $\gamma : {}^A \prod_{\lambda \in \Lambda} H_\lambda \rightarrow L \in \mathcal{U}$ can be extended to an ℓ -homomorphism $\gamma' : {}^A \prod_{\lambda \in \Lambda} G_\lambda \rightarrow L' \in \mathcal{U}$ such that L is an ℓ -subgroup of L' and $\gamma'|_{{}^A \prod_{\lambda \in \Lambda} H_\lambda} = \gamma$. In fact, any ℓ -homomorphism $\gamma : {}^A \prod_{\lambda \in \Lambda} H_\lambda \rightarrow L \in \mathcal{U}$ induces a family of ℓ -homomorphisms $\sigma_\lambda : H_\lambda \rightarrow L \in \mathcal{U}$ such that $\gamma \alpha_\lambda = \sigma_\lambda$ for each $\lambda \in \Lambda$ where α_λ is the inclusion map. Thus σ_λ can

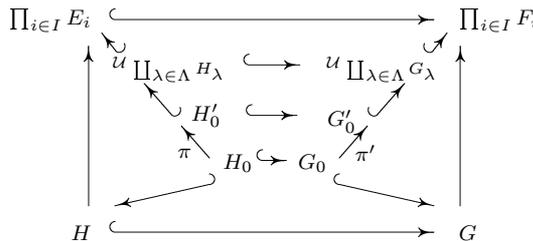


be extended to a family of ℓ -homomorphisms $\sigma'_\lambda : G_\lambda \rightarrow L' \in \mathcal{U}$ such that L is an ℓ -subgroup of L' and $\sigma'_\lambda|_{H_\lambda} = \sigma_\lambda$. By the universal property there exists an ℓ -homomorphism $\gamma' : {}^A \prod_{\lambda \in \Lambda} G_\lambda \rightarrow L'$ such that $\gamma' \alpha'_\lambda = \sigma'_\lambda$ for each $\lambda \in \Lambda$ where α'_λ is the inclusion map. Hence

$$\sigma_\lambda = \sigma'_\lambda|_{H_\lambda} = (\gamma' \alpha'_\lambda)|_{H_\lambda} = \gamma'|_{H_\lambda}$$

for each $\lambda \in \Lambda$. By the uniqueness $\gamma'|_{{}^A \prod_{\lambda \in \Lambda} H_\lambda} = \gamma$.

(2) Now we show that ${}^U \prod_{\lambda \in \Lambda} H_\lambda$ is the ℓ -subgroup of ${}^U \prod_{\lambda \in \Lambda} G_\lambda$ generated by the $\bigcup_{\lambda \in \Lambda} H_\lambda$. Let $G_0 = \bigoplus_{\lambda \in \Lambda} G_\lambda$, $H_0 = \bigoplus_{\lambda \in \Lambda} H_\lambda$, $G = {}^A \prod_{\lambda \in \Lambda} G_\lambda$ and $H = {}^A \prod_{\lambda \in \Lambda} H_\lambda$. Then G_0 and H_0 are subgroups of G and H , respectively, and H_0 is a subgroup of G_0 , H is a subgroup of G . Let $D = \{F_i \mid i \in I\}$ be the set of all \mathcal{U} -homomorphic images of G with the ℓ -homomorphisms γ'_i ($i \in I$). For each



$i \in I$, $\gamma'_i|_H(H)$ is a \mathcal{U} -homomorphic image of H . Conversely, if E is a \mathcal{U} -homomorphic image of H with the ℓ -homomorphism γ . It follows from (1) that γ can be extended to an ℓ -homomorphism $\gamma' : G \rightarrow F \in \mathcal{U}$ such that E is an

ℓ -subgroup of F and $\gamma'|_H = \gamma$. Hence the set of all \mathcal{U} -homomorphic images of H is $C = \{E_i \mid i \in I\}$ where E_i is an ℓ -subgroup of F_i and $\gamma'_i|_H(H) = E_i$ for each $i \in I$. By Theorem 3.2 we see that \mathcal{U} -free product ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} G_\lambda$ is the sublattice of the direct product $\prod_{i \in I} F_i$ generated by the group homomorphic image G'_0 of G_0 with the group homomorphism π' , and the \mathcal{U} -free product ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} H_\lambda$ is the sublattice of the direct product $\prod_{i \in I} E_i$ generated by the group homomorphic image H'_0 of H_0 with the group homomorphism π . $\pi'|_{G_\lambda}$ and $\pi|_{H_\lambda}$ are all ℓ -homomorphisms for $\lambda \in \Lambda$. Hence ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} G_\lambda$ is the ℓ -subgroup of $\prod_{i \in I} F_i$ generated by $\bigcup_{\lambda \in \Lambda} G'_\lambda$ where $G'_\lambda = \pi'(G_\lambda)$ and ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} H_\lambda$ is the ℓ -subgroup of $\prod_{i \in I} E_i$ generated by $\bigcup_{\lambda \in \Lambda} H'_\lambda$ where $H'_\lambda = \pi(H_\lambda)$. From the above we see that $\prod_{i \in I} E_i$ is an ℓ -subgroup of $\prod_{i \in I} F_i$ and

$$\pi'|_{H_0} = \pi.$$

Therefore ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} H_\lambda$ is the ℓ -subgroup of $\prod_{i \in I} F_i$ generated by $\bigcup_{\lambda \in \Lambda} H'_\lambda$, and so ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} H_\lambda$ is also the ℓ -subgroup of ${}^{\mathcal{U}}\coprod_{\lambda \in \Lambda} G_\lambda$ generated by $\bigcup_{\lambda \in \Lambda} H'_\lambda$. \square

5. An example

Theorem 2.4 and Theorem 2.1 are applicable to all varieties of ℓ -groups. But here we give an example of a class of ℓ -groups which is not a variety.

An ℓ -group G is said to be weak Hamiltonian if each closed convex ℓ -subgroup of G is normal. Let WH be the set of all weak Hamiltonian ℓ -groups. It is easy to show that WH is a sub-product class of ℓ -groups (see [16]) and

$$\mathcal{A} \subseteq \text{WH} \subseteq \mathcal{R} \subseteq \mathcal{L}.$$

So we have the construction theorem for the WH-free product.

Theorem 5.1. *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in WH. Then ${}^{\text{WH}}\coprod_{\lambda \in \Lambda} G_\lambda$ is the sublattice of $\prod_{i \in I} F_i$ generated by the group homomorphic image H' of the group free product H of G_λ , where $\{F_i \mid i \in I\}$ are all weak Hamiltonian ℓ -homomorphic images of ${}^{\mathcal{L}}\coprod_{\lambda \in \Lambda} G_\lambda$.*

REFERENCES

- [1] Anderson M., Feil T., *Lattice-Ordered Groups (An Introduction)*, D. Reidel Publishing Company, 1988.
- [2] Conrad P., *Lattice-Ordered Groups*, Tulane Lecture Notes, Tulane University, 1970.
- [3] ———, *Free lattice-ordered groups*, J. of Algebra **16** (1970), 191–203.
- [4] Holland W.C., Scringer E., *Free products of lattice-ordered groups*, Algebra Universalis **2** (1972), 247–254.
- [5] Glass A.M.W., Holland W.C., *Lattice-Ordered Groups(Advances and Techniques)*, Kluwer Academic Publishers, 1989.
- [6] Grätzer G., *Universal Algebra*, 2nd ed., Springer-Verlag, New York, 1979.
- [7] Martinez J., *Free products in varieties of lattice-ordered groups*, Czech. Math. J. **22** (1972), 535–553.

- [8] ———, *Free products of abelian ℓ -groups*, Czech. Math. J. **23** (1973), 349–361.
- [9] Martinez J. (ed.), *Ordered Algebraic Structure*, Kluwer Academic Publishers, 1989, pp. 11–49.
- [10] Powell W.B., Tsinakis C., *Free products of abelian ℓ -groups are cardinally indecomposable*, Proc. A.M.S. **86** (1982), 385–390.
- [11] ———, *Free products in the class of abelian ℓ -groups*, Pacific J. Math. **104** (1983), 429–442.
- [12] ———, *The distributive lattice free products as a sublattice of the abelian ℓ -group free product*, J. Austral. Math. Soc. **34** (1993), 92–100.
- [13] ———, *Free products of lattice-ordered groups*, Algebra Universalis **18** (1984), 178–198.
- [14] ———, *Disjointness conditions for free products of ℓ -groups*, Arkiv. der Math. **46** (1986), 491–498.
- [15] ———, *Disjoint sets in free products of representable ℓ -groups*, Proc. A.M.S. **104** (1988), 1014–1020.
- [16] Dao-Rong Ton, *Free weak Hamiltonian ℓ -groups*, Northeast Mathematics **10** (2) (1994), 235–240.
- [17] Weinberg E.C., *Free lattice-ordered abelian groups*, Math. Ann. **151** (1963), 187–199.
- [18] ———, *Free lattice-ordered abelian groups II*, Math. Ann. **159** (1965), 217–222.

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