On the Hölder continuity of solutions of nonlinear parabolic systems

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Abstract. Non-linear second order parabolic systems in the divergent form are considered. It is proved that under some restrictions on the modulus of ellipticity, all weak solutions are continuous.

Keywords: non-linear parabolic systems, weak solutions, regularity

Classification: 35D10, 35K55

We consider the parabolic system

$$\frac{\partial u}{\partial t} - \sum_{k=1}^{n} D_k A_k(x,t,u,Du) = 0 \tag{1}$$

of N equations ($N \geq 1$) in the domain $Q \subset \mathbf{R}^{n+1}$, ($n \geq 2$).

Here u and A_k are vector functions of dimension N, $D = (D_1, D_2, ..., D_n)$, $D_k = \frac{\partial}{\partial x_k}$. The coefficients $A_k(x,t,u,\xi)$ are assumed to be differentiable over all arguments except t.

Further, it is supposed that

$$\sum_{i,k}(\eta_k^i - \kappa \sum_{j,l} A_{kl}^{ij}(x,t,u,\xi)\eta_l^j)^2 \ \leq\ (1-\nu^2)|\eta|^2, \quad \nu > 0, \tag{2}$$

$$\begin{cases} \left|\frac{\partial A}{\partial u}\right| \leq c|\xi|^{\frac{2}{n+2}} + c|u|^{\frac{2}{n}} + f(x,t), \\ \left|\frac{\partial A}{\partial x}\right| \leq c|\xi| + c|u|^{1+\frac{2}{n}} + f^2(x,t), \end{cases} \tag{3}$$

where $A_{kl}^{ij} = \frac{\partial A_k^i}{\partial \xi_l^j}$ with $A = (A_1, A_2, ..., A_n)$, $\kappa > 0$ is a norming factor, $f \in L_a$ with $a > n+2$.

By c we denote any positive constant the value of which is of no importance.

Observe that the condition (2) follows from the pair of inequalities usually written in the form

$$\begin{cases} \sum_{i,j=1}^{N} \sum_{k,l=1}^{n} A_{kl}^{ij}(x,t,u,\xi)\eta_k^i\eta_l^j & \geq & \lambda|\eta|^2, \\ \sum_{i,k}(\sum_{j,l} A_{kl}^{ij}(x,t,u,\xi)\eta_l^j)^2 & \leq & \Lambda^2|\eta|^2, \end{cases} \tag{4}$$

where $\nu \geq \frac{\lambda}{\Lambda}$. Evidently, if (4) holds, we obtain easily that

$$\sum_{i,k}(\eta_k^i - \kappa \sum_{j,l} A_{kl}^{ij}\eta_l^j)^2 =$$
$$|\eta|^2 - 2\kappa \sum_{i,j,k,l} A_{kl}^{ij}\eta_l^j\eta_k^i + \kappa^2 \sum_{i,k}(\sum_{j,l} A_{kl}^{ij}\eta_l^j)^2 \quad \leq \quad (1 - 2\kappa\lambda + \kappa^2\Lambda^2)\ |\eta|^2;$$

setting $\kappa = \frac{\lambda}{\Lambda^2}$ we obtain (2) with $\nu = \frac{\lambda}{\Lambda}$.

Theorem. *In the condition* (2), *let*

$$\nu > 1 - \frac{2}{n}. \tag{5}$$

Then each generalized solution $u \in V_{2,loc}(Q)$ of the system (1) *belongs to $C_{loc}^{\alpha}(Q)$ for some $\alpha > 0$.*

(Here $V_{2,loc}(Q)$ is the space of measurable functions on Q with the expression

$$\sup_t \int |u(x,t)|^2\,dx + \int\int |Du(x,t)|^2\,dx\,dt$$

finite over each compact subcylinder contained in Q.)

It is a well known fact that weak solutions of nonlinear parabolic and elliptic systems can be discontinuous even if the coefficients are analytic. It is also known that this cannot take place in the case of one equation of the second order. In the course of the long-lasting effort to find conditions under which weak solutions of elliptic systems are regular it was Cordes' idea which appeared to be particularly fruitful. Cordes observed that the regularity properties of weak solutions can be substantially influenced by the modulus of ellipticity of the system. (For the further development see e.g. [1], [2].)

The success of Cordes' idea obtained in the elliptic case was not so far fully repeated in case of parabolic systems. The Hölder continuity is proved in the paper [3] by means of a modification of the technique of integral estimates with weights.

In my paper I prove this result under less restrictive assumption (5). The proof is based on Moser's iterative method.

Set

$$DuDv = \sum_{i,k} D_k u^i D_k v^i, \quad |Du|^2 = DuDu.$$

In the following lemma we give the Moser-type modification of natural energy space.

Lemma. *Let $u = (u^1, u^2, ..., u^N)$ be a function which is differentiable almost everywhere. Put $v = u|u|^s$ with $s \in (-1, \infty)$.*

Then for $\mu(s) = 1 - (\frac{s}{2+s})^2$ we have

$$DuDv \geq \mu^{\frac{1}{2}}(s)|Du||Dv|\,.$$

Proof: Denote

$$I = |u|^s|Du|^2\,, \quad J = |u|^{s-2}\sum_k(\sum_i u^i D_k u^i)^2.$$

Then

$$DuDv = I + sJ\,, \quad |u|^{-s}|Dv|^2 = I + (2s + s^2)J\,.$$

As $s > -1$ and $I \geq J$ (which follows from Hölder's inequality), we obtain that $DuDv \geq 0$.

The assertion of the lemma now follows from the relation

$$(DuDv)^2 - \mu(s)|Du|^2|Dv|^2 = s^2(\frac{I}{2+s} - J)^2 \geq 0.$$

$\square$

Remark. In the proof of Theorem we use this result only with $s \in (0, \infty)$. Its other part ($s \in (-1, 0]$) is helpful when we consider the removable singularities of solutions.

Proof of the Theorem: Using finite difference operators in a standard way leads us to the fact that

$$Du \in V_{2,loc}(Q), \quad \frac{\partial u}{\partial t} \in L_{2,loc}(Q).$$

(In what follows, all the spaces are local in Q.) Further, the local variant of Gehring's lemma (see [4], appendix) yields that D^2u and $\frac{\partial u}{\partial t}$ belong to L_p for some $p > 2$.

Let now $z_o = (x_o, t_o)$ be an arbitrary point in Q, $\varrho \leq \frac{1}{3}\,\mathbf{dist}(z_o, \partial Q)$. Define the cut-off function

$$\varphi(z) = \psi(\frac{|x - x_o|}{\varrho} - 1)\psi(\frac{t_o - t}{\varrho})$$

where $z = (x, t)$ and ψ is a smooth decreasing function on $\mathbf{R}$ with $\psi(\tau) = 1$ for $\tau < 0$ and $\psi(\tau) = 0$ for $\tau > 1$.

Set $v_m = u_m|u_m|^s$ for $s > 0$ where $u_m = D_m u$, $(m \in (1, 2, ..., n))$. Rewrite the system (1) in the form

$$\kappa\frac{\partial u^i}{\partial t} - \triangle u^i = \sum_k D_k(\kappa A_k^i - D_k u^i),$$

where κ is the norming factor from the condition (2). Multiplying the system by $D_m(v_m\varphi^2)$ we integrate it over the domain

$$Q_1 = \{z = (x,t);\ |x - x_o| < 2\varrho \text{ and } |t - t_o| < \varrho\}.$$

As $u \in W_p^{2,1}$ with $p > 2$ the integral is finite for sufficiently small positive s. Integrating by parts with respect to the variable x we obtain

$$(6) \quad \int_{Q_1} \Big(\frac{\kappa}{2+s}\frac{\partial}{\partial t}|u_m|^{2+s} + Du_m Dv_m\Big)\varphi^2\, dz =$$
$$\int_{Q_1} \sum_{i,k}\Big\{\Big[\Big(\kappa\sum_{j,l} A_{kl}^{ij} D_l u_m^j - D_k u_m^i\Big) + \sum_j \frac{\partial A_k^i}{\partial u^j} u_m^j + \frac{\partial A_k^i}{\partial x_m}\Big]\varphi^2 D_k v_m^i +$$
$$\kappa D_m A_k^i v_m^i D_k\varphi^2\Big\}\, dz$$

The first member in the right-hand side of the formula (6) can be estimated by means of Hölder's inequality and of the assumption (2) as

$$\Big|\sum_{i,k}\Big(\kappa\sum_{j,l} A_{kl}^{ij} D_l u_m^j - D_k u_m^i\Big) D_k v_m^i\Big| \le (1-\nu^2)^{\frac{1}{2}} |Du_m||Dv_m|.$$

Using now the Lemma, we obtain from (6) the estimate

$$(7) \quad \max_{|t-t_o|<\varrho} \frac{\kappa}{2+s}\int_{\Omega_t} |u_m|^{2+s}\varphi^2 dx +$$
$$\big(\mu(s)^{\frac{1}{2}} - (1-\nu^2)^{\frac{1}{2}}\big)\Big) \int_{Q_1} |Du_m|^2 |u_m|^s \varphi^2\, dz \le$$
$$c\int_{Q_1} \Big[\Big(\Big|\frac{\partial A}{\partial u}\Big||u_m| + \Big|\frac{\partial A}{\partial x}\Big|\Big)|Dv_m|\varphi^2 + (|D_m A||v_m| + |u_m|^{2+s})\varphi\Big]\, dz,$$

where

$$\Omega_t = \{(x,t);\ |x - x_o| < 2\varrho\}.$$

In the case that $\frac{s}{2+s} < \nu$, we obtain the same kind of estimate as (7) for the left-hand side

$$\max_{|t-t_o|<\varrho} \int_{\Omega_t} |u_m|^{2+s}\varphi^2\, dx + \int_{Q_1} |Du_m|^2|u_m|^s\varphi^2\, dz.$$

(The eventual increase of the constant c does not play any important role.)

Now we apply Young's inequality to the right-hand side of (7). Taking account of the fact that from the condition (3) and from $Du \in V_2$ it follows $\frac{\partial A}{\partial u} \in L_b$ with $b = \min\{\frac{(n+2)^2}{n}, a\}$ we get

$$(8) \quad \max_{|t-t_o|<\varrho} \int_{\Omega_t} |u_m|^{2+s}\varphi^2\, dx + \int_{Q_1} |Du_m|^2|u_m|^s\varphi^2\, dz$$
$$\le c\int_{Q_1} \Big[(1+|Du|)^q + |u|^{2+4/n}|Du|^s + \Big|\frac{\partial A}{\partial u}\Big|^b + f^a\Big]\, dz,$$

where $q = \max\{b\frac{2+s}{b-2},\ \frac{as}{a-4}\}$. By the restrictions $a < \frac{(n+2)^2}{n}$ and $s < a-4$, we obtain the value of q as $q = a\frac{2+s}{a-2}$.

As for u and Du, the condition $Du \in V_2$ yields that $Du \in L_{2\frac{n+2}{n}}$ and $u \in L_{2\frac{n+2}{n-2}}$. (In the case of $n = 2$ the solution u is integrable with any positive power.)

Take $s_1 \in (0, \frac{2\nu}{1-\nu})$ so small that all the considerations that we have made until now are correct for each s, $s \le s_1$. As z_o was an arbitrary point of Q, the estimate (8) yields

$$|Du|^{1+s_1/2} \in V_2.$$

The imbedding theorem for the space V_2 gives then

$$Du \in L_{(2+s_1)\frac{n+2}{n}}, \quad u \in L_{(2+s_1)\frac{n+2}{n-2}}$$

So all the reasonings leading to the estimate (8) can be repeated for $s \le s_2$, where $s_2 \in (0, \frac{2\nu}{1-\nu})$ satisfies the inequalities

$$\begin{aligned} s_2 &\le (2+s_1)\frac{n+2}{n}(1-\frac{2}{p}), \\ 2+s_2 &\le (2+s_1)\frac{n+2}{n}\frac{a-2}{a}, \\ s_2 &\le (s_1+\frac{4}{n})\frac{n+2}{n}. \end{aligned}$$

As $p > 2$ and $a > n+2$, the value of s_2 can be chosen in a way that $s_2 > s_1$. From (8) we have $|Du|^{1+s_2/2} \in V_2$.

Analogously, we can determine s_3 starting from s_2 and so on. It is evident now that after a finite number of steps we obtain that $|Du|^{1+s/2} \in V_2$ for any $s < s^* = \min\{\frac{2\nu}{1-\nu}, a-4\}$.

It follows now from (5) and from $a > n+2$ that $s^* > n-2$. Thus for each t the function $Du(x,t)$ is integrable with respect to x to power bigger than n and so (by the imbedding theorem) u is locally Hölder continuous with respect to x in Q.

As $\frac{\partial u}{\partial t} \in L_2$, it follows in the standard way that the Hölder continuity with respect to t with the exponent $\alpha = \frac{\beta}{n+2\beta}$. Namely, we have that

$$\begin{aligned} &|u(x,t) - u(x,\tau)| \\ &\qquad \le |u(x,t) - u(y,t)| + |u(x,\tau) - u(y,\tau)| + |u(y,t) - u(y,\tau)| \\ &\qquad\qquad \le c|x-y|^\beta + |u(y,t) - u(y,\tau)| \end{aligned}$$

where y is an arbitrary point of the ball $B(x,\varrho) = \{y;\ |y-x| < \varrho\}$.

Fubini's theorem now gives that

$$\int |\frac{\partial u}{\partial t}(y,t)|^2\,dt < \infty$$

for almost all y, and for such y the function $u(y,t)$ is Hölder continuous with respect to t with the exponent $\frac{1}{2}$:

$$|u(y,t)-u(y,\tau)| \le c|t-\tau|^{\frac{1}{2}}(\int_\tau^t |\frac{\partial u}{\partial t}(y,h)|^2\,dh)^{\frac{1}{2}}.$$

Evidently,

$$\min_{y\in B(x,\varrho)} \int_\tau^t |\frac{\partial u}{\partial t}(y,h)|^2\,dh \le |B(x,\varrho)|^{-1}\int_{B(x,\varrho)}\int_\tau^t |\frac{\partial u}{\partial t}|^2\,dh\,dy \le c\varrho^{-n}.$$

Taking $\varrho = |t-\tau|^{\frac{1}{n+2\beta}}$, we obtain finally $|u(x,t)-u(x,\tau)| \le c|t-\tau|^\alpha$ with $\alpha = \frac{\beta}{n+2\beta}$. □

Acknowledgement. This work was written during my stay in the Department of Mathematical Analysis of Charles University, Prague, and was supported by the grant n. 201/93/2171 of Grant Agency of Czech Republic.

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(*Received* February 22, 1994)