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A metrizable completely regular ordered space

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Abstract. We construct a completely regular ordered space $(X, \mathcal{T}, \leq)$ such that X is an I-space, the topology $\mathcal{T}$ of X is metrizable and the bitopological space $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise regular, but not pairwise completely regular. (Here $\mathcal{T}^\sharp$ denotes the upper topology and $\mathcal{T}^\flat$ the lower topology of X.)

Keywords: completely regular ordered, strictly completely regular ordered, pairwise completely regular, pairwise regular, I-space

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1. Introduction

It is the aim of this note to answer affirmatively the following question posed in [2] (see also Problem 11 of [5]):

Problem. Does there exist a completely regular ordered space $(X, \mathcal{T}, \leq)$ such that X is an I-space, the topology $\mathcal{T}$ of X is normal and the bitopological space $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise regular, but not pairwise completely regular (compare [2, Proposition 1])?

For a discussion of the origin and importance of that question we refer the reader to [2], [5], [6]. Before presenting the construction let us introduce the necessary terminology and recall some facts.

A topological space X endowed with a partial order $\leq$ is called a *topological ordered space.* A mapping $f : X \to Y$ between two topological ordered spaces X and Y is said to be *increasing* (*decreasing*) if $f(x) \leq f(y)$ ($f(x) \geq f(y)$) whenever $x, y \in X$ and $x \leq y$.

Given a topological ordered space X a subset A of X is called an *upper set* of X if $x \leq y$ and $x \in A$ imply that $y \in A$. Similarly, we say that a subset A of X is a *lower set of* X if $y \leq x$ and $x \in A$ imply that $y \in A$. For any subset E of X, $i(E)$ ($d(E)$) will denote the intersection of all upper (lower) sets of X containing E. A topological ordered space X is called an *I-space* [7] if $d(G)$ and $i(G)$ are open whenever G is an open subset of X. In the following we shall consider the bispace

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$(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ associated with a given topological ordered space $(X, \mathcal{T}, \leq)$ where $\mathcal{T}^\sharp$ denotes the collection of $\mathcal{T}$-open upper sets of X and $\mathcal{T}^\flat$ denotes the collection of $\mathcal{T}$-open lower sets of X.

We recall that a topological ordered space X is said to be *completely regular ordered* if there exists a quasi-uniformity $\mathcal{U}$ on X that determines X (i.e. $\mathcal{T}(\mathcal{U}^*)$ is the topology of X and $\cap\mathcal{U}$ is the partial order of X). Here $\mathcal{U}^*$ denotes the coarsest uniformity finer than $\mathcal{U}$ on X. Clearly each completely regular ordered space has a completely regular topology and a closed partial order. (Let us mention that in [8] it is shown that these two conditions are not sufficient for a topological ordered space to be completely regular ordered.) On the other hand, it will follow from the examples presented in this paper that even under strong conditions the bispace associated with a completely regular ordered space need not be pairwise completely regular.

A completely regular ordered space X is called *strictly completely regular ordered* if given a closed lower (resp. upper) set A and a point $x \in X \setminus A$, there exists a continuous increasing function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(x) = 1$ (resp. $f(A) = 1$ and $f(x) = 0$). In [2, Proposition 1] it is noted that a completely regular ordered space $(X, \mathcal{T}, \leq)$ is strictly completely regular ordered if and only if the bispace $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise completely regular.

2. Results

We first describe a relatively simple example of a metrizable completely regular ordered I-space that is not strictly completely regular ordered, since its associated bispace is not pairwise regular. Example 1 nicely illustrates the basic idea used in our main example (Example 2).

Example 1. *A countable, completely regular ordered I-space $(X, \mathcal{T}, \leq)$ determined by a quasi-uniformity having a countable base such that $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is not pairwise regular.*

Let
$$X = \{\infty\} \cup ([\omega \cup (\omega \times \omega)] \times \{1\}) \cup ([\omega \cup (\omega \times \omega)] \times \{0\})$$
and let Δ_X be the diagonal of X. Define a partial order $\leq$ on X as follows: Set $\leq = \Delta_X \cup <$ where $< = \{((k, m, 0), (k, m, 1)) : k, m \in \omega\}$.

Fix $n \in \omega$. Set

$$\begin{aligned}\mathcal{G}_n = \{&\{\infty\} \cup ([(\omega \setminus n) \cup ((\omega \setminus n) \times (\omega \setminus n))] \times \{1\})\} \cup \\ &\{\{p\} \times (\omega \setminus n) \times \{1\} : p \in n\} \cup \\ &\{\{(p, 1)\} \cup [(\omega \setminus n) \times \{p\} \times \{1\}] : p \in n\} \cup \\ &\{\{(k, m, 1)\} : k, m \in n\} \cup \\ &\{\{(p, 0)\} \cup [\{p\} \times (\omega \setminus n) \times 2] : p \in n\} \cup \\ &\{\{(k, m)\} \times 2 : k, m \in n\}.\end{aligned}$$

Furthermore let $T_n = \bigcap_{G\in\mathcal{G}_n}([(X \setminus G) \times X] \cup [G \times G])$.

Then $\{T_n : n \in \omega\}$ is a countable transitive subbase for a quasi-uniformity $\mathcal{U}$ on X.

It is readily checked that for any $x \in X$, $\bigcap_{n\in\omega} T_n(x) = i(x)$. Therefore $\cap\mathcal{U} = \leq$. We consider the completely regular ordered space $(X, \mathcal{T}(\mathcal{U}^*), \cap\mathcal{U})$. Note that $(X, \mathcal{T}(\mathcal{U}^*))$ is a countable metrizable space. By considering the sets $(T_n \cap T_n^{-1})(x)$ for $n \in \omega$ and $x \in X$, we see that the topology $\mathcal{T}(\mathcal{U}^*)$ on X can be described as follows: The points (k, m, t) where $k, m \in \omega$ and $t \in 2$ are isolated. A neighborhood base of the point ∞ is given by

$$\{\{\infty\} \cup ([(\omega \setminus n) \cup ((\omega \setminus n) \times (\omega \setminus n))] \times \{1\}) : n \in \omega\}.$$

Similarly,

$$\{\{(p, 0)\} \cup [\{p\} \times (\omega \setminus n) \times \{0\}] : n \in \omega\}$$

is a neighborhood base at $(p, 0)$ where $p \in \omega$ and

$$\{\{(p, 1)\} \cup [(\omega \setminus n) \times \{p\} \times \{1\}] : n \in \omega\}$$

is a neighborhood base at $(p, 1)$ where $p \in \omega$.

Obviously, the space $(X, \mathcal{T}(\mathcal{U}^*), \cap\mathcal{U})$ is an I-space, since for each $G \in \mathcal{T}(\mathcal{U}^*)$, $d(G) \setminus G$ and $i(G) \setminus G$ consist of isolated points by the definition of $\leq$.

Set $F = \omega \times \{0\}$. Then F is a closed upper set and $\infty \notin F$. Let G_1 be an open lower set containing ∞ and let G_2 be an open upper set containing F. There is an $n \in \omega$ such that $(\omega \setminus n) \times (\omega \setminus n) \times \{0\} \subseteq G_1$, because G_1 is an open lower set containing ∞. Moreover $(n, k, 0)$ belongs to G_2 for all but finitely many $k \in \omega$, since $(n, 0) \in G_2$ and G_2 is open. We conclude that $G_1 \cap G_2 \neq \emptyset$. Thus $(X, (\mathcal{T}(\mathcal{U}^*))^\sharp, (\mathcal{T}(\mathcal{U}^*))^\flat)$ is not pairwise regular. (Note that this example answers Problem 11 of [3] positively.)

In connection with Example 1 the following observation may be noteworthy.

Remark. *Let $(X, \mathcal{T}, \leq)$ be a completely regular ordered second countable I-space such that $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise regular. Then $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise completely regular.*

Proof: Let $\mathcal{B}$ be a countable base for $\mathcal{T}$. Then $\{i(B) : B \in \mathcal{B}\}$ is a countable base for $\mathcal{T}^\sharp$, because $(X, \mathcal{T}, \leq)$ is an I-space. Similarly, $\{d(B) : B \in \mathcal{B}\}$ is a countable base for $\mathcal{T}^\flat$. Since $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise regular and both topologies $\mathcal{T}^\sharp$ and $\mathcal{T}^\flat$ are second countable, by a result of J.C. Kelly [1, Theorem 2.8] the bispace $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is quasi-pseudo-metrizable, in particular it is pairwise normal. It follows that $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise completely regular [4].

We are now ready to discuss our main example.

Example 2. *We construct a metrizable completely regular ordered I-space $(X,\mathcal{T},\leq)$ such that $(X,\mathcal{T}^\sharp,\mathcal{T}^\flat)$ is pairwise regular, but not pairwise completely regular.*

Let

$$X=\{\infty\}\cup[(\omega^\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times\omega\times\{1\}]\cup[(\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times\omega\times\{0\}]$$

and let Δ_X be the diagonal of X. Define a partial order $\leq$ on X in the following way. Let $\leq=\Delta_X\cup<$ where

$$\begin{aligned}<=\{((k,m,f,1,l,0),(k,m,f,0,l,1)):k,m,l\in\omega,f\in\omega^\omega\}\cup\\ \{((k,m,f,1,l,1),(k,m,f,0,l+1,0)):k,m,l\in\omega,f\in\omega^\omega\}.\end{aligned}$$

Fix $n\in\omega$ and $f\in\omega^\omega$. Set

$$\begin{aligned}\mathcal{G}_{n,f}=\{&X\setminus([(\omega^\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times l\times\{1\}]\cup\\ &[(\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times l\times\{0\}]):l\in n\}\cup\\ \{&X\setminus([(\omega^\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times l\times\{1\}]\cup\\ &[(\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times(l+1)\times\{0\}]):l\in n\}\cup\\ \{&i((k,m,e,s,l,r)):k,m,l\in n;e\in\omega^\omega;s,r\in 2\}\cup\\ \{&\{(k,l,0)\}\cup i[\{k\}\times(\omega\setminus n)\times\omega^\omega\times 2\times\{l\}\times\{0\}]:k,l\in n\}\cup\\ \{&\{(f,l,1)\}\cup i[(\mathrm{graph}\,(f)\setminus\mathrm{graph}\,(f|n))\times\{f\}\times 2\times\{l\}\times\{1\}]:l\in n\}.\end{aligned}$$

In the same way as in Example 1 we define a transitive relation $T_{n,f}$ on X with the help of the collection $\mathcal{G}_{n,f}$ of X and let $\mathcal{U}$ be the quasi-uniformity on X generated by the subbase consisting of the relations $T_{n,f}$ where $n\in\omega$ and $f\in\omega^\omega$. Note that $\cap\mathcal{U}=\leq$.

The promised example is the completely regular ordered space $(X,\mathcal{T}(\mathcal{U}^*),\cap\mathcal{U})$. Considering the sets $(T_{n,f}\cap T_{n,f}^{-1})(x)$ where $n\in\omega$, $f\in\omega^\omega$ and $x\in X$ we can describe the topology $\mathcal{T}(\mathcal{U}^*)$ on X as follows:

The set

$$\begin{aligned}\{\{\infty\}\cup[(\omega^\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times(\omega\setminus n)\times\{1\}]\cup\\ [(\omega\cup(\omega\times\omega\times\omega^\omega\times 2))\times(\omega\setminus n)\times\{0\}]:n\in\omega\}\end{aligned}$$

is a neighborhood base of open sets at ∞.

The points (k,m,f,s,l,r) are isolated for $k,m,l\in\omega$, $f\in\omega^\omega$ and $s,r\in 2$.

The set

$$\{\{(k,l,0)\}\cup[\{k\}\times(\omega\setminus n)\times\omega^\omega\times 2\times\{l\}\times\{0\}]:n\in\omega\}$$

is a neighborhood base of open sets at the points $(k,l,0)$ where $k,l\in\omega$.

Finally, the set

$$\{\{(f,l,1)\} \cup [(\operatorname{graph}(f) \setminus \operatorname{graph}(f|n)) \times \{f\} \times 2 \times \{l\} \times \{1\}] : n \in \omega\}$$

is a neighborhood base of open sets at the points $(f,l,1)$ where $f \in \omega^\omega$ and $l \in \omega$.

Clearly the space $(X, \mathcal{T}(\mathcal{U}^*), \cap\mathcal{U})$ is an I-space by the argument given in Example 1. It is straightforward to check that X is a (regular Hausdorff) space having a σ-locally finite base. Thus X is metrizable.

Next we show that the bispace $(X, (\mathcal{T}(\mathcal{U}^*))^\sharp, (\mathcal{T}(\mathcal{U}^*))^\flat)$ is pairwise regular:

Let F be a proper closed lower set and let $x \in X \setminus F$. There is a basic open neighborhood G_x of x (of the form just described above) such that $G_x \cap F = \emptyset$. Observe that $i(G_x) \cap F = \emptyset$. It is easy to check that $i(G_x)$ is $\mathcal{T}(\mathcal{U}^*)$-closed. Then $x \in i(G_x)$, $F \subseteq X \setminus i(G_x)$, $i(G_x)$ is an open upper set and $X \setminus i(G_x)$ is an open lower set. Thus the proof of this case is complete.

Suppose now that F is a proper closed upper set and $x \in X \setminus F$.

If $x \neq \infty$, an argument completely analogous to the one just presented applies.

If $x = \infty$, then there is an $n \in \omega$ such that F is contained in the open set

$$G := [(\omega^\omega \cup (\omega \times \omega \times \omega^\omega \times 2)) \times n \times \{1\}] \cup [(\omega \cup (\omega \times \omega \times \omega^\omega \times 2)) \times n \times \{0\}].$$

Thus for the basic open neighborhood

$$\begin{aligned} G_\infty := X \setminus ([(\omega^\omega \cup (\omega \times \omega \times \omega^\omega \times 2)) \times (n+1) \times \{1\}] \cup \\ [(\omega \cup (\omega \times \omega \times \omega^\omega \times 2)) \times (n+1) \times \{0\}]) \end{aligned}$$

of ∞ it follows that $d(G_\infty) \cap i(G) = \emptyset$. We have shown that $(X, (\mathcal{T}(\mathcal{U}^*))^\sharp, (\mathcal{T}(\mathcal{U}^*))^\flat)$ is pairwise regular.

It remains to prove that $(X, (\mathcal{T}(\mathcal{U}^*))^\sharp, (\mathcal{T}(\mathcal{U}^*))^\flat)$ is not pairwise completely regular. To this end we are going to verify that $(X, \mathcal{T}(\mathcal{U}^*), \cap\mathcal{U})$ is not strictly completely regular ordered.

For each $n \in \omega$ set $F_n = \omega \times \{(n,0)\}$. Then F_0 is a closed upper set. Suppose that there exists a continuous increasing function $h : X \to [0,1]$ such that $h(\infty) = 0$ and $h(F_0) = 1$. Note that in order to obtain a contradiction, it suffices to show that for any $n \in \omega$,

$$\limsup_{k\to\infty} h(k,n,0) = 1 \ \ (*),$$

since if we choose $f_n \in F_n$ whenever $n \in \omega$ such that $1 = \lim_{n\to\infty} h(f_n)$, then $1 = \lim_{n\to\infty} h(f_n) = h(\infty) = 0$, because $\lim_{n\to\infty} f_n = \infty$ and h is continuous.

Clearly assertion $(*)$ is true for $n = 0$. Suppose that condition $(*)$ holds for $n \in \omega$, but that for some $\epsilon > 0$, $\limsup_{k\to\infty} h(k, n+1, 0) < 1 - 2\epsilon$. By our assumption on n we have that $A := \{l \in \omega : h(l,n,0) > 1 - \epsilon\}$ is infinite. By continuity of h for any $l \in A$ there is $s_l \in \omega$ such that $s \geq s_l$ implies that $h(l,s,g,1,n,0) > 1 - \epsilon$ whenever $g \in \omega^\omega$.

Suppose that $g' \in \omega^\omega$ such that $g'(l) \geq s_l$ for infinitely many $l \in A$. Because $(l, g'(l), g', 0, n, 1) > (l, g'(l), g', 1, n, 0)$ whenever $l \in \omega$, $\lim_{l\to\infty}(l, g'(l), g', 0, n, 1) = (g', n, 1)$ and h is both continuous and increasing, we conclude that $h(g', n, 1) \geq 1 - \epsilon$.

On the other hand, since $\limsup_{k\to\infty} h(k, n+1, 0) < 1 - 2\epsilon$, the set $D = \{l \in A : h(l, n+1, 0) < 1 - 2\epsilon\}$ has infinitely many elements. By continuity of h we can choose for each $l \in D$ an element $g_1(l) \in \omega$ with $g_1(l) \geq s_l$ such that for any $p \in \omega^\omega$, $h(l, g_1(l), p, 0, n+1, 0) < 1 - 2\epsilon$. Set $g_1(k) = 0$ if $k \in \omega \setminus D$. Then $h(l, g_1(l), g_1, 0, n+1, 0) < 1 - 2\epsilon$ whenever $l \in D$. For each $l \in \omega$, $(l, g_1(l), g_1, 0, n+1, 0) > (l, g_1(l), g_1, 1, n, 1)$. By monotonicity of h it follows that $1 - 2\epsilon > h(l, g_1(l), g_1, 1, n, 1)$ whenever $l \in D$. Because $\lim_{l\to\infty} h(l, g_1(l), g_1, 1, n, 1) = h(g_1, n, 1)$ by continuity of h, we see that $1 - 2\epsilon \geq h(g_1, n, 1)$ — a contradiction, since $g_1(l) \geq s_l$ for all $l \in D$. We conclude that for any $n \in \omega$, $\limsup_{k\to\infty} h(k, n, 0) = 1$ and that the bispace $(X, (\mathcal{T}(\mathcal{U}^*))^\sharp, (\mathcal{T}(\mathcal{U}^*))^\flat)$ is not pairwise completely regular.

Problem. Is there an ordered I-space $(X, \mathcal{T}, \leq)$ determined by a quasi-uniformity with a countable base such that $(X, \mathcal{T}^\sharp, \mathcal{T}^\flat)$ is pairwise regular, but not pairwise completely regular?

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