A property of B_2-groups

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Abstract. It is shown, under ZFC, that a B_2-group has the interesting property of being $\aleph_0$-prebalanced in every torsion-free abelian group in which it is a pure subgroup. As a consequence, we obtain alternate proofs of some well-known theorems on B_2-groups.

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Introduction

All groups considered here, unless otherwise stated, are additively written torsion-free abelian groups. For unexplained terminology and notations, we refer to Fuchs [F-1]. A torsion-free abelian group G of infinite rank is called a B_2-group if, for some ordinal τ, G is the union of a continuous well-ordered ascending chain of pure subgroups,

$$(*) \qquad 0 = G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset \ldots\ldots, \quad (\alpha < \tau)\ldots\ldots$$

such that, for each $\alpha < \tau$, $G_{\alpha+1} = G_\alpha + B_\alpha$, where B_α is a finite rank pure subgroup of a completely decomposable group. Such groups B_α are also called Butler groups. Recently Fuchs [F-2] made striking advances in the study of B_2-groups by employing the concept of $\aleph_0$-prebalancedness introduced in [BF]. In this note we prove that a B_2-group has the interesting property of being $\aleph_0$-prebalanced in every torsion-free group in which it is a pure subgroup. A noteworthy corollary is that a B_2-group A is a pure subgroup of index $\leq \aleph_1$ in a B_1-group G, then G itself becomes a B_2-group. Taking $A = 0$ leads to a well-known theorem ([DHR]) that a B_1-group of cardinality $\leq \aleph_1$ is a B_2-group. An adaptation of our methods also leads to a direct and simple proof of a theorem of Hill and Megibben ([HM]) that completely decomposable groups are absolutely separative.

Preliminaries

A torsion-free group G is called a B_1-group if $\mathrm{Bext}^1(G,T) = 0$ for all torsion groups T. (Here Bext^1 denotes the subfunctor of Ext^1 consisting of all the balanced extensions.) The chain of subgroups $(*)$ defined above for a B_2-group G is called a B_2-filtration of G. Let A be a pure subgroup of a torsion-free group G. A is called decent (prebalanced) in G if whenever L/A is a finite rank (rank one) pure subgroup of G/A, then $L = A + B$, for some finite rank Butler group B.

A is a TEP subgroup of G if, for any torsion group T, every homomorphism from A to T extends to a homomorphism from G to T. A is said to be $\aleph_0$-prebalanced ([BF]) in G if, for each $g \in G \setminus A$ there is a countable subset $\{a_1, a_2, \dots\} \subset A$ such that for each $a \in A$, there is an $n < \omega$ with $t(g+a) \leq \sup\{t(g+a_1), \dots t(g+a_n)\}$ where $t(x)$ denotes the type of x. In the last definition, if A satisfies the stronger condition that $\chi(g+a) \leq \chi(g+a_i)$ for some $i < \omega$, then A is said to be separative (or in the terminology of [HM], separable) in G, where, as usual, $\chi(x)$ denotes the characteristic of x. An $\aleph_0$-prebalanced chain for a group G is a continuous well-ordered ascending chain of $\aleph_0$-prebalanced subgroups

$$0 = G_0 \subset G_1 \subset \dots \subset G_\alpha \subset \dots G_\tau = G \quad \text{(for some ordinal } \tau)$$

where all the factors $G_{\alpha+1}/G_\alpha$ are of rank one. A key result of Fuchs ([F-2, Corollary 2.4]) is that if G has an $\aleph_0$-prebalanced chain, then G is of the form $G = C/K$, where C is completely decomposable and K is a balanced B_2-subgroup. Another useful idea that we need from [BF] is the balanced-projective resolution of a group G relative to a pure subgroup A. To form this, consider all the rank-1 pure subgroups J_α in $G \setminus A$ and let C be the direct sum of all these J_α's. Then the map $C \to B$ induced by the inclusion of the J_α in G together with the inclusion of A in G induces a balanced exact sequence

$$0 \longrightarrow K \longrightarrow A \oplus C \longrightarrow G \longrightarrow 0$$

which is called the balanced-projective resolution of G relative to A. An important result of Bican-Fuchs ([BF, Theorem 3.2]) that we shall be using asserts that if G/A is countable, then A is $\aleph_0$-prebalanced in G exactly when K is a B_2-group. We shall also need a result from [R] that if A is a TEP subgroup of B and if both A and B are B_2-groups, then so is B/A. The reader is referred to [BF], [F-2] and [R] for background details.

The results

We shall begin with the following simple lemma.

Lemma 1. *Let A and S be subgroups of a torsion-free group G. If $A \cap S$ is pure and decent in A, then S is pure and decent in $A + S$.*

Proof: We first show that given any finite subset X of $A + S$, there is a finite rank Butler subgroup B such that $B+S$ is pure in $A+S$ and contains X. Without loss of generality, we may assume that $X \subset A$. By the decency of $A \cap S$, there is a finite rank Butler subgroup B of A such that $B + (A \cap S)$ is pure in A and contains X. It is then readily seen that both $B + S$ and S are pure in $A + S$. From this the decency of S follows. □

Bican and Fuchs [BF] showed, under $V = L$, that every B_1-group is "absolutely $\aleph_0$-prebalanced", that is, it is an $\aleph_0$-prebalanced subgroup of every group in which it is a pure subgroup. The next theorem says that this holds for any B_2-group and we prove this under ZFC.

Theorem 2. *Every B_2-group is absolutely $\aleph_0$-prebalanced.*

Proof: Let A be a B_2-group with and axiom-3 family $\mathbb{C}$ of pure decent subgroups so chosen that for each $Y \in \mathbb{C}$, A/Y is again a B_2-group (see [AH] for the construction of $\mathbb{C}$). Suppose A is a pure subgroup of a torsion-free group B with B/A countable. Then $B = A + S$, where S is a countable pure subgroup. Moreover, by the usual back and-forth argument, we could assume that $A \cap S = Y \in \mathbb{C}$. By Lemma 1, S is decent and pure in B. Moreover, $B/S \cong A/Y$ is a B_2-group. Since S is decent and countable, the pre-image of a B_2-filtration of B/S gives rise to an $\aleph_0$-prebalanced chain in B. In order to show that A is $\aleph_0$-prebalanced in B, consider a relative balanced-projective resolution (as explained in the Preliminaries)

$$0 \longrightarrow K \longrightarrow A \oplus X \longrightarrow B \to 0$$

where X is completely decomposable. Let

$$0 \longrightarrow M \longrightarrow X' \longrightarrow A \longrightarrow 0$$

be a balanced-projective resolution of A with X' completely decomposable. Then the obvious epimorphism $X' \oplus X \to A \oplus X$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & X' \oplus X & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & A \oplus X & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here all the rows and columns are balanced exact. Since B has an $\aleph_0$-prebalanced chain, Corollary 2.4 of [F-2] implies that L is a B_2-group. Since $A \oplus X$ is a B_2-group, the middle column is TEP exact and, moreover, by [F-2] and [R], M is a B_2-group. Clearly the first column is now TEP exact and Theorem 3 of [R] then yields that K is also a B_2-group. An appeal to Theorem 3.2 of [BF] (alluded to in the Preliminaries) leads to the conclusion that A is $\aleph_0$-prebalanced in B.

$\square$

Corollary 3. *Suppose A is a B_2-group which is a pure subgroup of a torsion-free group B with B/A having cardinality $\leq \aleph_1$. Then*

(a) *B has an $\aleph_0$-prebalanced chain and* $\mathrm{Bext}^2(B,T)=0$ *for all torsion groups T.*
(b) *If B is a B_1-group, then B is also a B_2-group.*

Proof: (a) Now B is a union of a smooth ascending chain of pure subgroups

$$A = A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots\ ,\ \alpha < \omega_1, \ldots\ldots \tag{1}$$

where, for each α, $A_{\alpha+1}/A_\alpha$ is countable. Since a countable extension of an absolutely $\aleph_0$-prebalanced subgroup is again absolutely $\aleph_0$-prebalanced, the chain (1) gives rise to a $\aleph_0$-prebalanced chain for B. By Corollary 2.3 of [F-2], $\mathrm{Bext}^2(B,T)=0$.

(b) Follows from the fact (Theorem 4.1 of [F-2]) that a B_1-group with an $\aleph_0$-prebalanced chain is a B_2-group. □

In Corollary 3 (b) if we take $A=0$, then we obtain the following

Corollary 4 ([DHR]). *A B_1-group of cardinality $\leq \aleph_1$ is a B_2-group.*

Corollary 5. *If A is a pure B_2-subgroup of a finitely Butler group B with B/A countable, then B itself is a B_2-group.*

Proof: Since B is finitely Butler, the countable subgroup S in the first part of the proof of Theorem 2 is Butler and decent in B with B/S a B_2-group. Clearly B is then a B_2-group. □

Note: The group ΠZ, the direct product of $\aleph_0$ copies of the group Z of integers, shows that Corollary 5 is false if B/A is uncountable.

If A is a completely decomposable group, then the subgroup S in the proof of Theorem 2 can actually be a direct summand, as the following lemma shows.

Lemma 6. *Suppose A is a completely decomposable group and is a pure subgroup of a torsion-free group B with B/A countable. Then $B = A' \oplus S$, where $A' \subset A$ and S is countable.*

Proof: Now $B = A + X$, where X is a suitable countable pure subgroup of B. Then we can write $A = A' \oplus Y$, where Y is countable and $X \cap A \subset Y$. If $S = Y + X$, then clearly $B = A' + S$. Moreover, $A' \cap S = A' \cap A \cap S \subset A' \cap Y = 0$, so that $B = A' \oplus S$. □

As an application we get a direct and simpler proof of theorem of Hill and Megibben ([HM]) that completely decomposable are absolutely separative.

Theorem 7 ([HM]). *A completely decomposable group A is separative in every torsion-free group containing A as a pure subgroup.*

Proof: Let A be a pure subgroup of a torsion-free group G. Let $g \in G \setminus A$. If $B = \langle A, g\rangle^*$, the pure subgroup generated by A and g, then by Lemma 6 $B = A' \oplus S$, $A = A' \oplus C$, where S is countable and $C = A \cap S$. Write $g = a' + s$, where $a' \in A'$ and $s \in S$. Clearly, $H = \{-a' + c : c \in C\}$ is a countable subset of A. We claim that for any $a \in A$, there is an $h \in H$ such that $\chi(g+a) \le \chi(g+h)$. Indeed if $a = x + y$, with $x \in A'$ and $y \in C$, then we have $\chi(g + a) = \chi(a' + s + x + y) = \chi((a' + x) + (s + y)) \le \chi(s + y) = \chi(g + h)$, where $h = -a' + y \in L$. Thus A is separative in G. □

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