Checking positive definiteness or stability of symmetric interval matrices is NP-hard

Jiří Rohn*

Abstract. It is proved that checking positive definiteness, stability or nonsingularity of all [symmetric] matrices contained in a symmetric interval matrix is NP-hard.

Keywords: positive definiteness, stability, nonsingularity, NP-hardness

Classification: 15A48, 15A18, 68Q25

As is well known, a square (not necessarily symmetric) matrix A is called positive definite if $x^T Ax > 0$ for each $x \neq 0$, stable if $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of A, and Schur stable if $\varrho(A) < 1$. We prove here that checking these properties is NP-hard (see [1]) for a symmetric interval matrix $A^I = [\underline{A}, \overline{A}] := \{A; \underline{A} \leq A \leq \overline{A}\}$. By definition, A^I is called symmetric if both $\underline{A}$ and $\overline{A}$ are symmetric; hence, a symmetric A^I may contain nonsymmetric matrices. If A^I is symmetric and $A \in A^I$, then $\frac{1}{2}(A + A^T) \in A^I$. Let $\lambda_{\min}(A)$ denote the minimal eigenvalue of a symmetric matrix A. We have these results:

Theorem. *For a symmetric interval matrix A^I with rational bounds, each of the following problems is NP-hard:*

(i) *check whether each $A \in A^I$ is positive definite,*
(ii) *check whether each symmetric $A \in A^I$ is positive definite,*
(iii) *check whether each $A \in A^I$ is stable,*
(iv) *check whether each symmetric $A \in A^I$ is stable,*
(v) *check whether each $A \in A^I$ is nonsingular,*
(vi) *check whether each symmetric $A \in A^I$ is nonsingular,*
(vii) *check whether each symmetric $A \in A^I$ is Schur stable,*
(viii) *given rational numbers a, b, $a < b$, check whether $\lambda_{\min}(A) \in (a, b)$ for each symmetric $A \in A^I$.*

Proof: Let us call a *symmetric* real $n \times n$ matrix $A = (a_{ij})$ an MC-matrix if $a_{ii} = n$ and $a_{ij} \in \{0, -1\}$ for $i \neq j$ $(i, j = 1, \ldots, n)$. Then for each $x \neq 0$ we have $x^T Ax \geq n\|x\|_2^2 - \sum_{i \neq j} |x_i x_j| = (n + 1)\|x\|_2^2 - \|x\|_1^2 \geq \|x\|_2^2 > 0$, hence A is

*This work was supported by the Charles University Grant Agency under grant GAUK 237.

positive definite (and so is A^{-1}). For an MC-matrix A and a positive integer L, let us form three symmetric interval matrices

$$A^I = \left[A^{-1} - \frac{1}{L}ee^T, A^{-1} + \frac{1}{L}ee^T\right],$$

$$A_0^I = \left[-A^{-1} - \frac{1}{L}ee^T, -A^{-1} + \frac{1}{L}ee^T\right]$$

and

$$A_1^I = \left[I + \frac{1}{m}(-A^{-1} - \frac{1}{L}ee^T), I + \frac{1}{m}(-A^{-1} + \frac{1}{L}ee^T)\right],$$

where $e = (1, 1, \ldots, 1)^T$, I is the unit matrix and $m = \|A^{-1}\|_\infty + \frac{n}{L} + 1$. Hence, $A_0^I = \{-A; A \in A^I\}$, $A_1^I = \{I + \frac{1}{m}A; A \in A_0^I\}$ and $\varrho(A') \le \|A'\|_\infty < m$ for each $A' \in A^I$. We shall prove that the following assertions are mutually equivalent:

0) $z^T A z \ge L$ for some $z \in \{-1, 1\}^n$,
1) A^I contains a matrix which is not positive definite,
2) A^I contains a symmetric matrix which is not positive definite,
3) A_0^I contains an unstable matrix,
4) A_0^I contains a symmetric unstable matrix,
5) A^I contains a singular matrix,
6) A^I contains a symmetric singular matrix,
7) A_1^I contains a symmetric matrix which is not Schur stable,
8) $\lambda_{\min}(A') \notin (0, m)$ for some symmetric $A' \in A^I$.

We prove 0) $\Rightarrow$ 6) $\Rightarrow$ 2) $\Rightarrow$ 8) $\Rightarrow$ 2) $\Rightarrow$ 4) $\Rightarrow$ 7) $\Rightarrow$ 4) $\Rightarrow$ 3) $\Rightarrow$ 1) $\Rightarrow$ 5) $\Rightarrow$ 0). 0) $\Rightarrow$ 6): If $z^T A z \ge L$ for some $z \in \{-1, 1\}^n$, then the matrix $A' = A^{-1} - (z^T A z)^{-1} z z^T$ is symmetric, belongs to A^I and satisfies $A' A z = 0$, hence it is singular. 6) $\Rightarrow$ 2) is obvious. 2) $\Leftrightarrow$ 8): For a symmetric $A' \in A^I$, since $\varrho(A') < m$, we have that A' is not positive definite if and only if $\lambda_{\min}(A') \notin (0, m)$. 2) $\Rightarrow$ 4): If a symmetric $A' \in A^I$ is not positive definite, then $\lambda_{\max}(-A') = -\lambda_{\min}(A') \ge 0$, hence $-A'$ is unstable and $-A' \in A_0^I$. 4) $\Leftrightarrow$ 7): For each symmetric $A' \in A_0^I$, since $\varrho(A') < m$, we have that A' is unstable if and only if $I + \frac{1}{m}A' \in A_1^I$ is not Schur stable. 4) $\Rightarrow$ 3) is obvious. 3) $\Rightarrow$ 1): If $\tilde{A} \in A_0^I$ is unstable, then by Bendixson theorem $0 \le \operatorname{Re}\lambda \le \lambda_{\max}(\frac{1}{2}(\tilde{A} + \tilde{A}^T))$, hence for $A' = -\frac{1}{2}(\tilde{A} + \tilde{A}^T)$ we have $A' \in A^I$ and $\lambda_{\min}(A') \le 0$, so that A' is not positive definite. 1) $\Rightarrow$ 5): Let $\tilde{A} \in A^I$ be not positive definite. Put $t_0 = \sup\left\{t \in [0, 1]\,; A^{-1} + t(\frac{1}{2}(\tilde{A} + \tilde{A}^T) - A^{-1}) \text{ is positive definite}\right\}$. Then the matrix $A' = A^{-1} + t_0(\frac{1}{2}(\tilde{A} + \tilde{A}^T) - A^{-1})$ is symmetric, belongs to A^I (due to its convexity) and is positive semidefinite, but not positive definite, hence $\lambda_{\min}(A') = 0$, so that A' is singular. 5) $\Rightarrow$ 0): Let $A'x = 0$ for some $A' \in A^I$, $x \ne 0$. Define $z \in \{-1, 1\}^n$ by $z_j = 1$ if $x_j \ge 0$ and $z_j = -1$ otherwise ($j = 1, \ldots, n$). Then $e^T|x| = z^T x = z^T A(A^{-1} - A')x \le |z^T A| \frac{1}{L} ee^T |x|$, which implies $L \le |z^T A| e = z^T A z$ (since A is diagonally dominant). This proves that the assertions 0) to 8) are equivalent. Now, in [3, Theorem 2.6] it is proved that the decision problem

Instance. An MC-matrix A and a positive integer L.

Question. Is $z^T Az \geq L$ for some $z \in \{-1, 1\}^n$?

is NP-complete. In view of the above equivalences, this problem can be polynomially reduced to each of the problems (i)–(viii), hence all of them are NP-hard. □

Comments. The result (v) was proved in [3, Theorem 2.8]; here it was included for completeness. Cf. also Nemirovskii's results in [2]. Characterizations of positive definiteness, stability and Schur stability of symmetric interval matrices are given in [4].

References

[1] Garey M.E., Johnson D.S., *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, 1979.

[2] Nemirovskii A., *Several NP-hard problems arising in robust stability analysis*, Math. of Control, Signals, and Systems **6** (1993), 99–105.

[3] Poljak S. and Rohn J., *Checking robust nonsingularity is NP-hard*, Math. of Control, Signals, and Systems **6** (1993), 1–9.

[4] Rohn J., *Positive definiteness and stability of interval matrices*, SIAM J. Matrix Anal. Appl. **15** (1994), 175–184.

Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Prague 1, Czech Republic

(*Received* March 23, 1994)