The criteria of strongly exposed points in Orlicz spaces

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Abstract. In Orlicz spaces, the necessary and sufficient conditions of strongly exposed points are given.

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Let X be a Banach space, $S(X)$, $B(X)$ denote the unit sphere and unit ball of X, respectively. X^* denotes the dual of X. $x \in S(X)$ is called an exposed point of $B(X)$ provided there exists $f \in S(X^*)$, such that for all $y \in B(X) \setminus \{x\}$ $f(y) < f(x) = 1$. $x \in S(X)$ is called a strongly exposed point of $B(X)$ provided there exists $f \in S(X^*)$ such that for $x_n \in B(X)$, $f(x_n) \to f(x) = 1$ implies $\|x_n - x\| \to 0$. Obviously, a strongly exposed point is an exposed point.

The exposed points in Orlicz spaces have been discussed (see [1]). In this paper, we will give the criteria of strongly exposed points in Orlicz spaces.

For the sake of convenience, we still present the full proofs. The symbols used in this paper have the same meanings as [2]. $M(u)$, $N(v)$, denote a pair of complemented N-functions. $p(u)$, $q(v)$, denote their right-hand derivatives respectively. "$M \in \Delta_2$" ("$M \in \nabla_2$") means that $M(u)$ satisfies the Δ_2-condition (∇_2-condition) for large u. For the set of $\sum$-measurable functions over a finite nonatom measure space $(G, \sum, \mu)$.

$$\left\{x(t) : \exists\, c > 0 \ \text{ such that } \ R_M\left(\frac{x}{c}\right) = \int_G M\left(\frac{x(t)}{c}\right) dt < \infty\right\}$$

endowed with Luxemburg norm $\|x\|_{(M)} = \inf\{c > 0 : R_M(\frac{x}{c}) \le 1\}$ and Orlicz norm $\|x\|_M = \sup\{\int_G x(t)y(t)\,dt : R_N(y(t)) \le 1\} = \inf\{\frac{1}{k}(1 + R_M(kx)) : k > 0\}$, we denote them as $L_{(M)}$, L_M, respectively, and call them Orlicz spaces. In addition, for an element $x \in L_M$ (or $L_{(M)}$), we denote

$$K_M(x) = \left\{k > 0 : \|x\|_M = \frac{1}{k}\left(1 + R_M(kx)\right)\right\},$$

$$\xi_M(x) = \lim_{n\to\infty} \|x - x_n\|_M = \inf\left\{c > 0 : R_M\left(\frac{x}{c}\right) < \infty\right\} = \lim_{n\to\infty} \|x - x_n\|_{(M)},$$

where $x_n(t) = x(t)$ if $|x(t)| \le n$ and $x_n(t) = 0$ if $|x(t)| > n$.

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Theorem 1. *$x \in S(L_{(M)})$ is a strongly exposed point of $B(L_{(M)})$ if and only if*

(i) $M \in \Delta_2$;
(ii) $\mu\{t \in G : x(t) \in R \setminus S_M\} = 0$;
(iii) *denote $\{a_i\}$, $\{b_j\}$ as the sets of all those left extreme points and right extreme points of affine segments of $M(u)$, respectively, satisfying $p_-(a_i) = p(a_i)$, $p_-(b_j) = p(b_j)$, then*
$\mu\{t \in G : |x(t)| \in \{b_j\}\} = 0$, *or*
$\mu\{t \in G : |x(t)| \in \{a_i\}\} = 0$, *and there exists $\tau > 0$ such that* $\int_G N((1+\tau)p_-(x(t)))\,dt < \infty$.

Proof: Necessity. Since a strongly exposed point is a strongly extreme point, by Theorem in [2], (i) and (ii) are trivial. If (iii) were not true, we only need to discuss the following two cases:

(I) there exist affine segments of $M(u)$, $[a,c]$ and $[d,b]$, such that $p_-(a) = p(a)$, $p_-(b) = p(b)$ and $\mu G_a = \mu\{t \in G : |x(t)| = a\} > 0$, $\mu G_b = \mu\{t \in G : |x(t)| = b\} > 0$, without loss of generality, we assume $x(t) \geq 0$ for all $t \in G$. Take $E \subset G_a$, $F \subset G_b$ such that

$$M(a)\mu E + M(b)\mu F = M\left(\frac{1}{2}(a+c)\right)\mu E + M\left(\frac{1}{2}(d+b)\right)\mu F.$$

Put

$$x'(t) = x(t)\chi_{G\setminus E\setminus F} + \frac{1}{2}(a+c)\chi_E + \frac{1}{2}(d+b)\chi_F,$$

then $x \neq x'$, $R_M(x') = R_M(x) = 1$, so $\|x'(t)\|_{(M)} = 1$. Take a support functional $y(t)$ of $x(t)$, and $k \in K_N(y)$, then $p_-(x(t)) \leq ky(t) \leq p(x(t))$. Noticing

$$p_-(x(t)) = p_-(x'(t)) \leq ky(t) \leq p(x(t)) = p(x'(t)) \quad \text{whenever} \quad t \in G \setminus E \setminus F;$$

$$ky(t) = p(a) = p(\frac{1}{2}(a+c)) = p(x'(t)) \quad \text{whenever} \quad t \in E;$$

$$ky(t) = p(b) = p(\frac{1}{2}(d+b)) = p(x'(t)) \quad \text{whenever} \quad t \in F;$$

we have $\int_G x'(t)y(t)\,dt = \int_G x(t)y(t)\,dt = 1$, hence $x(t)$ is not a strongly exposed point of $B(L_{(M)})$.

(II) there exists an affine segment $[a,b]$ satisfying $p_-(b) = p(b)$, $\mu G_b = \mu\{t \in G : x(t) = b\} > 0$, and for any $\varepsilon > 0$, $\int_G N((1+\varepsilon)p_-(x(t)))\,dt = \infty$.

Take $y \in S(L_N)$ satisfying $\int_G x(t)y(t)\,dt = 1$ and take $k \in K_N(y)$, then $p_-(x(t)) \leq ky(t) \leq p(x(t))$. Clearly, for any $\varepsilon > 0$, $\int_G N((1+\varepsilon)ky(t))\,dt = \infty$, so $\xi_N(ky) = 1$, hence $\|(ky(t))\chi_{G\setminus G_n}\|_N \to 1$ $(n \to \infty)$, where $G_n = \{t \in G : ky(t) \leq n\}$. By Hahn-Banach theorem, there exist $\{u_m\}_{n=1}^{\infty} \subset S(L_{(M)})$ such that $u_n(t) = u_n(t)\chi_{G\setminus G_n}$, and $\int_{G\setminus G_n} u_n(t)ky(t)\,dt \to 1$ $(n \to \infty)$. Obviously, for n large enough, $G_b \subset G_n$, $c = (M(b) - M(a))\mu G_b < 1$. Put

$$x_n(t) = x(t)\chi_{G_n\setminus G_b} + a\chi_{G_b} + cu_n(t),$$

then $R_M(x_n) \le R_M(x\chi_{G_n\setminus G_b}) + M(a)\mu G_b + cR_M(u_n) = R_M(x\chi_{G_n}) \le R_M(x) = 1$, hence we have $\|x_n\|_{(M)} \le 1$, and $\|x_n - x\|_{(M)} \ge (b-a)\|\chi_{G_b}\|_{(M)} > 0$. On the other hand,

$$\begin{aligned}
&\int_G x_n(t)ky(t)\,dt = \int_{G_n\setminus G_b} x(t)ky(t)\,dt + ap(b)\mu G_b + c\int_{G\setminus G_n} u_n(t)ky(t)\,dt \\
&= \int_{G_n\setminus G_b} (M(x(t)) + N(ky(t)))\,dt \\
&+ (M(a) + N(p(b)))\mu G_b + (M(b) - M(a))\mu G_b(1 + o(\frac{1}{n})) \\
&= \int_{G_n} M(x(t))\,dt + \int_{G_n} (ky(t))\,dt + o(\frac{1}{n}) \longrightarrow R_M(x) + R_N(ky) \\
&= 1 + R_N(ky) = k
\end{aligned}$$

i.e. $\int_G x_n(t)y(t)\,dt \to 1$. So $x(t)$ is not a strongly exposed point of $B(L_{(M)})$. Combining (I), (II), we obtain that (iii) is also necessary.

Sufficiency. Still assume $x(t) \ge 0$ for all $t \in G$, only discuss the following two cases.

(I) $\mu\{t \in G : x(t) \in \{b_j\}\} = 0$.

Denote $E_i = \{t \in G : x(t) = a_i\}$, denote the set of all discontinuous points of $p(u)$ as $\{r_n\}$ (either r_n is an extreme point of affine segments or not), denote $e_n = \{t \in G : x(t) = r_n\}$. For every n, take $\varepsilon_n > 0$ such that $p_-(r_n) + \varepsilon_n < p(r_n)$, and put $G_0 = G \setminus (\bigcup_i E_i) \setminus \bigcup_n e_n)$,

$$W(t) = p_-(x(t))\chi_{G\setminus\bigcup_n e_n} + \sum_n (p_-(r_n) + \varepsilon_n)\chi_{e_n}, \quad y(t) = \frac{W(t)}{\|W(t)\|_N}.$$

From

$$\begin{aligned}
1 &\ge \int_G x(t)y(t)\,dt = \frac{1}{\|W(t)\|_N}\Big(\int_{G\setminus\bigcup_n e_n} x(t)p_-(x(t))\,dt + \sum_n r_n(p_-(r_n) + \varepsilon_n)\mu e_n\Big) \\
&= \frac{1}{\|W(t)\|_N}\Big(\int_{G\setminus\bigcup_n e_n} (M(x(t)) + N(p_-(x(t))))\,dt \\
&\quad + \sum_n (M(r_n) + N(p_-(r_n) + \varepsilon_n))\mu e_n\Big) \\
&= \frac{1}{\|W(t)\|_N}\Big(\int_G M(x(t))\,dt + \int_G N(W(t))\,dt\Big) \ge \frac{1 + R_N(W(t))}{\|W(t)\|_N} \ge \|y(t)\|_N = 1
\end{aligned}$$

we get $\int_G x(t)y(t)\,dt = 1$, $k = \|W(t)\|_N \in K_N(y(t))$. For $\{x_n(t)\}_{n=1}^{\infty} \subset S(L_{(M)})$, $\int_G x_n(t)y(t)\,dt \to 1$, in order to prove $\|x_n - x\|_{(M)} \to 0$, by $M \in \Delta_2$ (see [2]), we only need to prove $x_n - x \xrightarrow{\mu} 0$. Noticing

$$0 \longleftarrow 1 + R_N(ky) - \int_G x_n(t)ky(t)\,dt = \int_G (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t))\,dt,$$

for any $F \subset G$, we have

$$\int_F (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t))\,dt \longrightarrow 0 \quad (n \to \infty).$$

Now we prove $x_n - x \xrightarrow{\mu} 0$ in three steps:

(1) $x_n(t) - x(t) \xrightarrow{\mu} 0$ on G_0.

Otherwise, there exist $\varepsilon > 0$, $\sigma > 0$, such that $\mu\{t \in G_0 : |x_n(t) - x(t)| > \varepsilon\} > \sigma$. Since

$$1 \geq R_M(x_n) \geq \int_{G(|x_n| \geq D)} M(x_n(t))\,dt \geq M(D)\mu G(|x_n(t)| \geq D),$$

we can take D large enough such that $\mu\{t \in G : |x_n(t) \geq D\} < \frac{\sigma}{4}$, $\mu\{t \in G : |x(t)| \geq \frac{D}{4}\} < \frac{\sigma}{4}$. Since for $t \in G_0$, $x(t) \neq r_n$, $x(t) \neq a_i$, there exist open segments δ_n, δ_i' such that $r_n \in \delta_n$, $a_i \in \delta_i'$, $\mu\{t \in G_0 : x(t) \in (\bigcup_n \delta_n) \cup (\bigcup_i \delta_i')\} < \frac{\sigma}{4}$. Denote

$$G_n = \left\{t \in G_0 : |x_n(t) - x(t)| \geq \varepsilon,\ 0 \leq x(t),\ x_n(t) \leq D,\ x(t) \notin \left(\textstyle\bigcup_n \delta_n\right) \cup \left(\textstyle\bigcup_i \delta_i'\right)\right\}$$

then $\mu G_n \geq \frac{\sigma}{4}$. Notice that the function $M(u) + N(v) - uv$ is continuous and positive on the closed bounded set $\{(u,v) : |u - v| \geq \varepsilon, 0 \leq u, v \leq D, v \in S_M \setminus \bigcup_n \delta_n \setminus \bigcup_i \delta_i'\}$, hence, there exists $\delta > 0$ such that for all (u,v) belonging to this set, we have

$$M(u) + N(v) - uv \geq \delta.$$

So, for all $t \in G_n$ we have

$$M(x_n(t)) + N(ky(t)) - x_n(t)ky(t) = M(x_n(t)) + N(p(x(t))) - x_n(t)p(x(t)) \geq \delta;$$

we arrive at a contradiction

$$0 \longleftarrow \int_{G_n} (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t))\,dt \geq \delta\mu G_n \geq \frac{\delta\sigma}{4}.$$

(2) $x_k - x(t) \xrightarrow{\mu} 0$ on e_n.

If there exist $\varepsilon > 0$, $\sigma > 0$, such that $e_{nk} = \mu\{t \in e_n : x_k(t) \geq r_n + \varepsilon\} \geq \sigma$, since $p(x_k(t)) \geq p(r_n) = p_-(r_n) + \varepsilon_n + \tau_n$ whenever $t \in e_{nk}$, we have

$$M(x_k(t)) + N(p_-(r_n) + \varepsilon_n) - x_k(t)(p_-(r_n) + \varepsilon_n) > \tau_n\varepsilon \quad \text{whenever} \quad t \in e_{nk}.$$

So we get a contradiction

$$\begin{aligned} 0 \longleftarrow & \int_{e_n} (M(x_k(t)) + N(ky(t)) - x_k(t)ky(t))\,dt \\ \geq & \int_{e_{nk}} (M(x_k(t)) + N(p_-(r_n) + \varepsilon_n) - x_k(t)(p_-(r_n) + \varepsilon_n))\,dt \geq \tau_n\varepsilon\sigma \quad (k \to \infty) \end{aligned}$$

hence $\mu\{t \in e_n : x_k(t) \geq r_n + \varepsilon\} \to 0$ $(k \to \infty)$. By the same argument, we can prove $\mu\{t \in e_n : x_k(t) \leq r_n - \varepsilon\} \to 0$ $(k \to \infty)$. So

$$x_k(t) - x(t) \xrightarrow{\mu} 0 \quad (\text{on } e_n).$$

(3) $x_n(t) - x(t) \xrightarrow{\mu} 0$ (on E_i).

From the result of (1) and (2), it is easy to know $x_n(t) - x(t) \xrightarrow{\mu} 0$ on $G \setminus \bigcup_i E_i$. So by Fatou theorem, it follows

$$\underline{\lim}_{n\to\infty} R_M(x_n(t)\chi_{G\setminus\bigcup_i E_i}) \geq R_M(x(t)\chi_{G\setminus\bigcup_i E_i});$$

in view of $R_M(x_n(t)) \leq 1$, we deduce

$$(*) \qquad \overline{\lim}_{n\to\infty} R_M(x_n(t)\chi_{\bigcup_i E_i}) \leq R_M(x(t)\chi_{\bigcup_i E_i}).$$

Notice that for all $t \in E_i$, $x(t) = a_i$, and a_i is a left extreme point of affine segments of $M(u)$, analogously to the proof of (2), we can get that for any $\varepsilon > 0$ $\mu\{t \in E_i : x_n(t) \leq x(t) - \varepsilon\} \to 0$ $(n \to 0)$. So

$$\underline{\lim}_{n\to\infty} R_M(x_n(t)\chi_{E_i}) \geq R_M(x(t)\chi_{E_i}) = M(a_i)\mu E_i.$$

If there exist i_0, $\varepsilon_0 > 0$, $\sigma_0 > 0$, such that $\mu\{t \in E_{i_0} : x_n(t) \geq x(t) + \varepsilon_0\} \geq \sigma_0$, noticing $M(u)$ is increasing monotonously, we deduce

$$\overline{\lim}_{n\to\infty} R_M(x_n(t)\chi_{E_{i_0}}) > R_M(x(t)\chi_{E_{i_0}}),$$

hence

$$\overline{\lim}_{n\to\infty} R_M(x_n(t)\chi_{\bigcup_i E_i}) > R_M(x(t)\chi_{\bigcup_i E_i})$$

which contradicts $(*)$, i.e. $x_n(t) - x(t) \xrightarrow{\mu} 0$ on E_i.

(II) $\mu\{t \in G : x(t) \in \{a_n\}\} = 0$, there exists $\tau > 0$ such that $\int_G N((1+\tau)p_-(x(t)))\,dt < \infty$. Denote the set of all discontinuous points of $p(u)$ as $\{r_n\}$, denote $e_n = \{t \in G : x(t) = r_n\}$, $E_j = \{t \in G : x(t) = b_j\}$. Take $\varepsilon_n > 0$ such that $p_-(r_n) + \varepsilon_n < p(r_n)$,

$$\int_{G\setminus\bigcup_n e_n} N((1+\tau)p_-(x(t)))\,dt + \sum_n N((1+\tau)(p_-(r_n)+\varepsilon_n))\mu e_n < \infty.$$

Put $G_0 = G \setminus \bigcup_j E_j \setminus \bigcup_n e_n$, $W(t) = p_-(x(t))\chi_{G\setminus\bigcup_n e_n} + \sum_n (p_-(r_n)+\varepsilon_n)\chi_{e_n}$, $y(t) = \frac{W(t)}{\|W(t)\|_N}$. Then $k = \|W(t)\|_N \in K_N(y(t))$ and $\int_G x(t)y(t)\,dt = 1$. For $x_n \in B(L_{(M)})$, $\int_G x_n(t)y(t)\,dt \to 1$ $(n \to \infty)$, it is enough to show $x_n(t) - x(t) \xrightarrow{\mu} 0$ on G.

First we prove

$$\lim_{\mu\delta\to 0}\{\sup_n R_N(x_n(t)\chi_\delta)\} = 0. \tag{1}$$

Otherwise, there exist $\varepsilon > 0$, $\delta_n \subset G$, $\mu\delta_n \to 0$, such that $R_M(x_n\chi_{\delta_n}) \geq \varepsilon > 0$, we get a contradiction

$$\begin{aligned}
0 &\longleftarrow \int_{\delta_n} (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t))\,dt \\
&\geq \int_{\delta_n} (M(x_n(t)) - \frac{1}{1+\tau}x_n(t)(1+\tau)ky(t))\,dt \\
&\geq \int_{\delta_n} (M(x_n(t)) - \frac{1}{1+\tau}(M(x_n(t)) + N((1+\tau)ky(t))))\,dt \\
&= \int_{\delta_n} \frac{1}{1+\tau}M(x_n(t))\,dt - \int_{\delta_n} \frac{1}{1+\tau}N((1+\tau)ky(t))\,dt \\
&\geq \frac{\tau\varepsilon}{1+\tau} - \frac{1}{1+\tau}R_N((1+\tau)ky(t)\chi_{\delta_n}) \longrightarrow \frac{\tau\varepsilon}{1+\tau}\,.
\end{aligned}$$

Similarly to the proof of (I), we can get $x_n(t) - x(t) \xrightarrow{\mu} 0$ on $G \setminus \bigcup_j E_j = G_0 \cup (\bigcup_n e_n)$. Using (1) we deduce

$$\lim_{n\to\infty} R_M(x_n(t)\chi_{G\setminus\bigcup_j E_j}) = R_M(x(t)\chi_{G\setminus\bigcup_j E_j});$$

moreover, by $\int_G x_n(t)y(t)\,dt \to 1$, we know $\|x_n\|_{(M)} \to 1$, so $R_M(x_n) \to 1 = R_M(x)$, thus

$$\lim_{n\to\infty} R_M(x_n\chi_{\bigcup_j E_j}) = R_M(x\chi_{\bigcup_j E_j}). \tag{2}$$

Noticing b_j is a right extreme point of affine segments of $M(u)$, using the same method as above, we can get that for any $\varepsilon > 0$, $\mu\{t \in E_j : x_n(t) - x(t) \geq \varepsilon\} \to 0$ so

$$\varlimsup_{n\to\infty} R_M(x_n(t)\chi_{E_j}) \leq R_M(x(t)\chi_{E_j}).$$

If there exist j_0, $\varepsilon > 0$, $\sigma > 0$, such that $\mu\{t \in E_{j_0} : x_n(t) \leq x(t) - \varepsilon\} > \sigma$, then

$$\varliminf_{n\to\infty} R_M(x_n(t)\chi_{E_{j_0}}) < R_M(x(t)\chi_{E_{j_0}}),$$

combining with (1), we get a contradiction

$$\lim_{n\to\infty} R_M(x_n(t)\chi_{\bigcup_j E_j}) < R_M(x(t)\chi_{\bigcup_j E_j}).$$

$\square$

Corollary 1. *$L_{(M)}$ has strongly exposed property, i.e. all points in $S(L_{(M)})$ are strongly exposed points of $B(L_{(M)})$ if and only if*

(i) *$M(u) \in \Delta_2$;*
(ii) *$M(u)$ is strictly convex.*

Theorem 2. *$x \in S(L_M)$ is a strongly exposed point of $B(L_M)$ if and only if*

(i) *$M(u) \in \Delta_2$ and $K_M(x) = \{k\}$ is a singleton set;*
(ii) *$\mu\{t \in G : kx(t) \in (R \setminus S_M) \cup \{a'_i\} \cup \{b'_j\} = 0\}$, where $\{a'_i\}$, $\{b'_j\}$ denote the sets of all continuous left extreme points and right extreme points of affine segments of $M(u)$ respectively;*
(iii) *there exists $y(t) \in S(L_{(N)})$ and $\tau > 0$, such that $\int_G x(t)y(t)\,dt = 1$,*
$$R_N((1+\tau)y(t)) < \infty;$$
(iv) *$R_N(p_-(kx(t))) = 1$ implies $\mu\{t \in G : kx(t) \in \{b_j\}\} = 0$,*
$R_N(p(kx(t))) = 1$ implies $\mu\{t \in G : kx(t) \in \{a_i\}\} = 0$,
where $\{a_i\}$, $\{b_j\}$, denote the sets of all discontinuous left extreme points and right extreme points of affine segments of $M(u)$ respectively.

Proof: Necessity. Without loss of generality, we assume $x(t) \geq 0$, since a strongly exposed point is a strongly extreme point, we get (i) and $\mu\{t \in G : kx(t) \in R \setminus S_M\} = 0$. If there exists b'_j satisfying $\mu G_{b'_j} = \mu\{t \in G : kx(t) = b'_j\} > 0$, take $a < b'_j$, $E \subset G_{b'_j}$, such that $p(a) = p(b'_j)$, $0 < \mu(G_{b'_j} \setminus E) < \mu G_{b'_j}$. Put
$$x'(t) = x(t)\chi_{G\setminus E} + \frac{a}{k}\chi_E,$$
noticing $R_N(p(kx'(t))) = R_N(p(kx(t))) \geq 1$ and for any $\varepsilon > 0$, $R_N(p(1-\varepsilon)kx'(t)) \leq R_N(p((1-\varepsilon)kx(t))) \leq 1$, we have $k \in K_M(x')$. Take $y(t) \in S(L_{(N)})$ with $\int_G x(t)y(t)\,dt = 1$, obviously
$$\begin{aligned} &p_-(kx'(t)) = p_-(kx(t)) \leq y(t) \leq p(kx(t)) = p(kx'(t)) &&\text{whenever } t \in G \setminus E\\ &y(t) = p(kx(t)) = p(b'_j) = p(a) = p(kx'(t)) &&\text{whenever } t \in E. \end{aligned}$$
So $y(t)$ is a supporting functional of $x'(t)$, in view of $\frac{x'}{\|x'\|_M} \neq x$, which contradicts that $x(t)$ is an exposed point of $B(L_M)$. So we have
$$\mu\{t \in G : kx(t) \in \{b'_j\}\} = 0.$$
If there exists a'_i such that $\mu G_{a'_i} = \mu\{t \in G : kx(t) = a'_i\} > 0$, take $b > a'_i$, $E \subset G_{a'_i}$ satisfying $p_-(a'_i) = p(a'_i) = p(b)$, $0 < \mu(G_{a'_i} \setminus E) < \mu G_{a'_i}$. Put
$$x'(t) = x(t)\chi_{G\setminus E} + \frac{b}{k}\chi_E,$$

for any $\tau > 0$, we have

$$\begin{aligned} R_N(p((1-\tau)kx')) &\le \int_{G\setminus E} N(p((1-\tau)kx))\,dt + N(p(a_i'))\mu E \\ &\le \lim_{m\to\infty}\Big(\int_{G\setminus E} N(p((1-\tau/m)kx))\,dt\Big) + \lim_{m\to\infty} N(p((1-\tau/m)a_i'))\mu E \\ &= \lim_{m\to\infty}\int_G N(p((1-\tau/m)kx))\,dt \le 1. \end{aligned}$$

Notice that $R_N(p(kx')) = R_N(p(kx)) \ge 1$, we get $k \in K_M(x')$. For any $y \in S(L_{(N)})$ with $\int_G x(t)y(t)\,dt = 1$, similarly to the above we can get $\int_G(\frac{x'(t)}{\|x'(t)\|_M})y(t)\,dt = 1$. Noticing $\frac{x'}{\|x'\|_M} \ne x$, and the arbitrariness of $y(t)$, we get a contradiction that $x(t)$ is not an exposed point of $B(L_M)$, so $\mu\{t \in G : kx(t) \in \{a_i'\}\} = 0$. Thus we have showed that the condition (ii) is necessary.

If (iii) is not necessary, then for any $y(t) \in S(L_{(N)})$ with $\int_G x(t)y(t)\,dt = 1$ and $\varepsilon > 0$, $R_N((1+\varepsilon)y(t)) = \infty$. Hence $\xi_N(y) = 1$ and $\lim_{n\to\infty}\|y\chi_{G\setminus G_n}\|_{(N)} = 1$, where $G_n = \{t \in G : |y(t)| \le n\}$. By Hahn-Banach theorem, there exist $u_n(t) = u_n(t)\chi_{G\setminus G_n}$ satisfying $\|u_n\|_M = 1$, $\int_{G\setminus G_n} u_n(t)y(t)\,dt \to 1$. Put

$$x_n(t) = \frac{1}{2}(x(t)\chi_{G_n} + u_n(t)),$$

then

$$\begin{aligned} 1 &\ge \frac{1}{2}(\|x\chi_{G_n}\|_M + \|u_n\|_M) \ge \|x_n\|_M \ge \int_G x_n(t)y(t)\,dt \\ &= \frac{1}{2}\Big(\int_{G_n} x(t)y(t)\,dt + \int_{G\setminus G_n} u_n y\,dt\Big) \longrightarrow 1. \end{aligned}$$

So $\|x_n\|_M \to 1$, $\int_G x_n(t)y(t)\,dt \to 1$, noticing $\|x - x_n\|_M \ge \|\frac{1}{2}x\chi_{G_n}\|_M \to \frac{1}{2}\|x\|_M = \frac{1}{2}$, we obtain that $y(t)$ is not a strongly exposed functional of $x(t)$.

Now we prove that (iv) is necessary. Otherwise, we only need to consider the following two cases.

(1) $R_N(p_-(kx)) = 1$ and there exist b_j satisfying $p_-(b_j) < p(b_j)$, and $\mu G_{b_j} = \mu\{t \in G : kx(t) = b_j\} > 0$. Take $a < b_j$, $E \subset G_{b_j}$ such that $p_-(a) = p(a) = p_-(b_j)$, $0 < \mu(G_{b_j} \setminus E) < \mu G_{b_j}$, put

$$x'(t) = x(t)\chi_{G\setminus E} + \frac{a}{k}\chi_E.$$

From $R_N(p_-(kx')) = R_N(p_-(kx)) = 1$, we derive that $k \in K_M(x')$, and $y = p_-(kx) = p_-(kx')$ is the unique support functional of $x(t)$. Obviously $\int_G \frac{x'(t)}{\|x'\|_M}y(t)\,dt = 1$ and $\frac{x'}{\|x'\|_M} \ne x$, so $x(t)$ is not a strongly exposed point.

(2) $R_N(p(kx)) = 1$ and there exist a_i satisfying $p_-(a_i) < p(a_i)$, $\mu G_{a_i} = \mu\{t \in G : kx(t) = a_i\} > 0$. Take $b > a_i$, $E \subset G_{a_i}$ such that $p(b) = p_-(b) = p(a_i)$, $0 < \mu E < \mu G_{a_i}$. Put $x'(t) = x(t)\chi_{G\setminus E} + \frac{b}{k}\chi_E$. From $R_N(p(kx')) = R_N(p(kx)) = 1$, we derive that $k \in K_M(x')$, and $y = p(kx') = p(kx)$ is the unique support functional of $x(t)$. Obviously we have $\frac{x'}{\|x'\|_M} \neq x$, and $\int_G \frac{x'(t)}{\|x'\|_M} y(t)\,dt = 1$, so $x(t)$ is not a strongly exposed point of $B(L_M)$.

Sufficiency. First we prove that if $y \in S(L_M)$ with $\int_G x(t)y(t)\,dt = 1$ and for some $\tau > 0$, $R_N((1+\tau)y) < \infty$, there exist $\{x_n\}_{n=1}^\infty \subset S(L_M)$ such that $\int_G x_n(t)y(t)\,dt \to 1$, then

$$\lim_{\mu e\to 0} \sup_n R_M(k_n x_n \chi_e) = 0, \tag{3}$$

$$\lim_{\mu e\to 0} \sup_n R_N(p(k_n x_n \chi_e)) = 0, \tag{4}$$

where $k_n \in K_M(x_n)$.

Otherwise, there exist e_i, satisfying $\mu e_i \to 0$ and $\varepsilon > 0$, such that for some $\{x_{ni}\}_{i=1}^\infty \subset \{x_n\}_{n=1}^\infty$, $R_M(k_{ni}x_{ni}\chi_{e_i}) \geq \varepsilon$, so it follows a contradiction:

$$\begin{aligned}
0 &\longleftarrow 1 + R_M(k_{ni}x_{ni}) - k_{ni}\int_G x_{ni}(t)y(t)\,dt \\
&= \int_G (M(k_{ni}x_{ni}(t)) + N(y(t)) - k_{ni}x_{ni}y(t))\,dt \\
&\geq \int_{e_i} (M(k_{ni}x_{ni}(t)) + N(y(t)) - k_{ni}x_{ni}(t)y(t))\,dt \\
&\geq \int_{e_i} \Big(M(k_{ni}x_{ni}(t)) - \frac{1}{1+\tau}k_{ni}x_{ni}(1+\tau)y(t)\Big)\,dt \\
&\geq \int_{e_i} \Big(M(k_{ni}x_{ni}(t)) - \frac{1}{1+\tau}(M(k_{ni}x_{ni}(t)) + N((1+\tau)y(t)))\Big)\,dt \\
&= \frac{\tau}{1+\tau}R_M(k_{ni}x_{ni}\chi_{e_i}) - \frac{1}{1+\tau}R_N((1+\tau)y\chi_{e_i}) \\
&\geq \frac{\tau\varepsilon}{1+\tau} - \frac{R_N((1+\tau)y\chi_{e_i})}{1+\tau} \longrightarrow \frac{\tau\varepsilon}{1+\tau},
\end{aligned}$$

the contradiction shows that (3) is true. Noticing that $M \in \Delta_2$, and $\lim_{\mu e\to 0}\sup_n \int_e k_n x_n(t)p(k_n x_n(t))\,dt = 0$, we get (4).

In the following, we prove the sufficiency in three cases.

(I) $R_N(p_-(kx)) = 1$, $\mu\{t \in G : |kx(t)| \in \{b_j\}\} = 0$.
In these cases we have that $y(t) = p_-(kx(t))$ is the unique support functional of $x(t)$. For any $\{x_n(t)\}_{n=1}^\infty \subset S(L_M)$ with $\int_G x_n(t)y(t)\,dt \to 1$, take $k_n \in K_M(x_n)$, and denote $E_i = \{t \in G : kx(t) = a_i\}$. Analogously to the proof of the sufficiency of Theorem 1, we can get $k_n x_n(t) - kx(t) \xrightarrow{\mu} 0$ on $G \setminus \bigcup_i E$. Since $p_-(t)$ is not

decreasing and continuous on the left hand, we have

$$\varliminf_{n\to\infty} R_N(p_-(k_n x_n)\chi_{G\setminus\bigcup_i E_i}) \geq R_N(p_-(kx(t))\chi_{G\setminus\bigcup_i E_i}),$$

moreover, by $R_N(p_-(k_n x_n)) \leq R_N(p_-(kx))$, we have

$$\varlimsup_{n\to\infty} R_N(p_-(k_n x_n)\chi_{\bigcup_i E_i}) \leq R_N(p_-(kx)\chi_{\bigcup_i E_i}). \tag{5}$$

For every i and any $\varepsilon > 0$, in view of that a_i is a left extreme point of affine segment of $M(u)$, we have

$$\mu\{t \in E_i : k_n x_n(t) \leq kx(t) - \varepsilon\} \longrightarrow 0 \quad (n \to \infty),$$

hence

$$\varlimsup_{n\to\infty} R_N(p_-(k_n x_n)\chi_{E_i}) \geq R_N(p_-(kx)\chi_{E_i}).$$

If there exist i_0, $\varepsilon > 0$, $\sigma > 0$, such that

$$\mu\{t \in E_{i_0} : k_n x_n(t) \geq kx(t) + \varepsilon\} \geq \sigma,$$

then

$$\varlimsup_{n\to\infty} R_N(p_-(k_n x_n)\chi_{E_{i_0}}) > R_N(p_-(kx)\chi_{E_{i_0}}),$$

and hence

$$\varlimsup_{n\to\infty} R_N(p_-(k_n x_n)\chi_{\bigcup_i E_i}) > R_N(p_-(kx)\chi_{\bigcup_i E_i}).$$

This is in contradiction with (5). So $k_n x_n(t) - kx(t) \xrightarrow{\mu} 0$ on $\bigcup_i E_i$. Combining (3) we get

$$k_n = 1 + R_M(k_n x_n) \longrightarrow 1 + R_M(kx) = k \quad (n \to \infty),$$

hence $x_n(t) - x(t) \xrightarrow{\mu} 0$. By $M(u) \in \Delta_2$, we deduce $\|x_n - x\|_M \to 0$ $(n \to \infty)$.

(II) $R_N(p(kx)) = 1$, $\mu\{t \in G : kx(t) \in \{a_i\}\} = 0$.
In this case, $y(t) = p(kx)$ is the unique support functional of $x(t)$. For any $\{x_n(t)\}_{n=1}^{\infty} \subset S(L_M)$ with $\int_G x_n(t)y(t)\,dt \to 1$, take $k_n \in K_M(x_n)$, and denote $F_j = \{t \in G : kx(t) = b_j\}$. Similarly, we can get $k_n x_n \xrightarrow{\mu} kx$ on $G \setminus \bigcup_j F_j$. Since $p(u)$ is not decreasing and continuous on the right hand, by (4) it follows

$$\varlimsup_{n\to\infty} R_N(p(k_n x_n)\chi_{G\setminus\bigcup_j F_j}) \leq R_N(p(kx)\chi_{G\setminus\bigcup_j F_j}).$$

Noticing $R_N(p(k_n x_n)) \geq 1 = R_N(p(kx))$, we have

$$\varliminf_{n\to\infty} R_N(p(k_n x_n)\chi_{\bigcup_j F_j}) \geq R_N(p(kx)\chi_{\bigcup_j F_j}). \tag{6}$$

Since b_j is a right extreme point of affine segment of $M(u)$, for any $\varepsilon > 0$ and every j, we have

$$\mu\{t \in F_j : k_n x_n(t) \geq kx(t) + \varepsilon\} = 0,$$

hence

$$\varlimsup_{n\to\infty} R_N(p(k_n x_n(t))\chi_{F_j}) \leq R_N(p(kx(t))\chi_{F_j}).$$

If there exist j_0, $\varepsilon > 0$, $\sigma > 0$, such that

$$\mu\{t \in F_{j_0} : k_n x_n(t) \leq kx(t) - \varepsilon\} \geq \sigma,$$

then

$$\varlimsup_{n\to\infty} R_N(p(k_n x_n(t))\chi_{F_{j_0}}) < R_N(p(kx(t))\chi_{F_{j_0}}),$$

and hence

$$\varlimsup_{n\to\infty} R_N(p(k_n x_n(t))\chi_{\bigcup_j F_j}) < R_N(p(kx(t))\chi_{\bigcup_j F_j}),$$

which contradicts (6). So $k_n x_n - kx \xrightarrow{\mu} 0$. From (4), it follows $k_n \to k$, and hence $x_n - x \xrightarrow{\mu} 0$. Noticing $M \in \Delta_2$,we get $\|x_n - x\|_M \to 0$.

(III) $R_N(p_-(kx)) < 1 < R_N(p(kx))$.

By the condition (iii) of this theorem, there exist $y(t) \in S(L_{(N)})$, and $\tau > 0$, such that $\int_G x(t)y(t)\,dt = 1$, $R_N((1+\tau)y) < \infty$. Denote all discontinuous points of $p(u)$ as $\{r_n\}$ (including $\{a_i\}$, $\{b_j\}$), denote $e_n = \{t \in G : kx(t) = r_n\}$. By $R_N(y) = 1$, for all $t \in G$, $p_-(kx(t)) \leq y(t) \leq p(kx(t))$, and

$$y(t) = p_-(kx(t)) = p(kx(t)) \quad \text{whenever} \quad t \in G \setminus \textstyle\bigcup_n e_n,$$

we have

$$\mu(\textstyle\bigcup_n \{t \in e_n : y(t) > p_-(r_n)\}) > 0; \quad \mu(\textstyle\bigcup_n \{t \in e_n : y(t) < p(r_n)\}) > 0.$$

Denote $e'_n = \{t \in e_n : y(t) = p_-(r_n)\}$. For r_n take $\varepsilon_n > 0$, such that $p_-(r_n)+\varepsilon_n < p(r_n)$ and

$$\int_{G\setminus\bigcup_n e'_n} N((1+\tau)y)\,dt + \sum_n N(p_-(r_n) + \varepsilon_n)\mu e'_n = 1.$$

Constructing a function $z(t)$ satisfying the following conditions

$$\begin{aligned}
&z(t) = y(t) && \text{whenever} \quad t \in G \setminus \textstyle\bigcup_n e_n,\\
&z(t) = p_-(r_n) + \varepsilon_n && \text{whenever} \quad t \in e'_n,\\
&p_-(r_n) < z(t) \leq y(t) && \text{whenever} \quad t \in e_n \setminus e'_n,
\end{aligned}$$

and such that $R_N(z) = R_N(y) = 1$, we can get $R_N((1+\tau)z) < \infty$ and

$$\mu\{t \in e_n : z(t) = p_-(r_n)\} = 0.$$

Similarly to the above, we can construct a function $u(t)$ satisfying $R_N(u(t)) = R_N(z) = R_N(y) = 1$, $R_N((1+\tau)u(t)) < \infty$, and

$$p_-(r_n) < u(t) < p(r_n) \quad \text{whenever } \; t \in e_n.$$

Obviously, $u(t)$ is a support functional of $x(t)$. For $\{x_n(t)\}_{n=1}^{\infty} \subset S(L_M)$ with $\int_G x_n(t)u(t)\,dt \to 1$, take $k_n \in K_M(x_n)$, then $k_n x_n(t)$ satisfies (3), (4). Analogously to the proof of the sufficiency of Theorem 1, we can get

$$k_n x_n(t) - kx(t) \xrightarrow{\mu} 0.$$

By (4), we have $k_n \to k$, hence we have $x_n(t) - x(t) \xrightarrow{\mu} 0$. In view of $M(u) \in \Delta_2$, we deduce $\|x_n - x\|_M \to 0$. □

Corollary 2. *L_M has the strongly exposed property if and only if*

(i) *$M \in \Delta_2$;*
(ii) *$M(u)$ is strictly convex;*
(iii) *there exist $u_0 > 0$, $\tau > 0$, $D > 0$, when $u \geq u_0$, $N((1+\tau)p(u)) \leq DN(p(u))$.*

Proof: Sufficiency. For $x(t) \in S(L_M)$, by (i) and (ii) of the corollary, it immediately follows that (i), (ii) and (iv) of the Theorem 2 hold. Notice that when $u > u_0$, $N((1+\tau)p(u)) \leq DN(p(u))$, it follows $N((1+\tau)p_-(u)) \leq DN(p_-(u))$. If $R_N(p_-(kx)) = 1$, then $x(t)$ has the unique support functional $y(t) = p_-(kx(t))$, so

$$\begin{aligned}\int_G N((1+\tau)y(t))\,dt &= \int_G N((1+\tau)p_-(kx(t)))\,dt \\ &\leq N((1+\tau)p_-(u_0))\mu G + D\int_G N(p_-(kx(t)))\,dt \\ &= N((1+\tau)p_-(u_0))\mu G + D < \infty.\end{aligned}$$

If $R_N(p(kx)) = 1$, for the same reason, we have

$$\int_G N((1+\tau)y(t))\,dt = \int_G N((1+\tau)p(kx(t)))\,dt < \infty.$$

If $R_N(p_-(kx)) < 1 < R_N(p(kx))$, take $G_0 < G$ such that

$$\int_{G\setminus G_0} N(p_-(kx(t)))\,dt + \int_{G_0} N(p(kx(t)))\,dt = 1,$$

put

$$y(t) = p_-(kx(t))\chi_{G\setminus G_0} + p(kx(t))\chi_{G_0},$$

then $y(t)$ is a support functional of $x(t)$, and

$$\begin{aligned} R_N((1+\tau)y) &= \int_{G\setminus G_0} N((1+\tau)p_-(kx(t)))\,dt + \int_{G_0} N((1+\tau)p(kx(t)))\,dt \\ &\le N((1+\tau)p(u_0))\mu G + D < \infty. \end{aligned}$$

Combining the above, we get that the (iii) of Theorem 2 is true, so $x(t)$ is a strongly exposed point of $B(L_M)$.

Necessity. By (i) and (ii) of Theorem 2, it immediately follows that (i) and (ii) of the corollary hold. If (iii) is not true, then there exist $u_n \nearrow \infty$, such that $N((1+\frac{1}{n})p(u_n)) > 2^n N(p(u_n))$. Take a sequence $\{G_n\}_{n=1}^{\infty}$ of subsets of G with $G_i \cap G_j = \emptyset$ whenever $i \neq j$, such that $N(p(u_n))\mu G_n = \frac{1}{2^n}$. Put

$$y(t) = \sum_{n=1}^{\infty} p(u_n)\chi_{G_n}.$$

For any $\varepsilon > 0$, take n_0 so that $\frac{1}{n_0} < \varepsilon$, then

$$\begin{aligned} R_N((1+\varepsilon)y) &= \sum_{n=1}^{\infty} N((1+\varepsilon)p(u_n))\mu G_n \ge \sum_{n\ge n_0}^{\infty} N((1+\frac{1}{n})p(u_n))\mu G_n \\ &\ge \sum_{n=n_0}^{\infty} 2^n N(p(u_n))\mu G_n = \infty. \end{aligned}$$

But

$$R_N(y) = \sum_{n=1}^{\infty} N(p(u_n))\mu G_n = 1,$$

put

$$x(t) = \frac{\sum_{n=1}^{\infty} u_n\chi_{G_n}}{\|\sum_{n=1}^{\infty} u_n\chi_{G_n}\|_M}, \qquad \text{then } \ x(t) \in S(L_M).$$

By $R_N(p(\|\sum_{n=1}^{\infty} u_n\chi_{G_n}\|_M x)) = R_N(\sum_{n=1}^{\infty} p(u_n)\chi_{G_n}) = R_N(y) = 1$, we know $k_x = \|\sum_{n=1}^{\infty} u_n\chi_{G_n}\|_M$, since $R_N(p(k_x x)) = 1$, $x(t)$ has the unique support functional $y(t)$, but $y(t)$ does not satisfy (iii) of Theorem 2, so $x(t)$ is not a strongly exposed point of $B(L_M)$. $\square$

Remark. Under $M \in \Delta_2$, the condition (iii) is equivalent to $M \in \nabla_2$.

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