A generalization of Magill's Theorem for non-locally compact spaces

Gary D. Faulkner, M. Cristina Vipera

Abstract. In the theory of compactifications, Magill's theorem that the continuous image of a remainder of a space is again a remainder is one of the most important theorems in the field. It is somewhat unfortunate that the theorem holds only in locally compact spaces. In fact, if all continuous images of a remainder are again remainders, then the space must be locally compact. This paper is a modification of Magill's result to more general spaces. This of course requires restrictions on the nature of the function.

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Introduction

The theorem known as Magill's theorem is the following:

Theorem [M1]. *For a locally compact space X, the following are equivalent:*

1. *Y is a remainder of X.*
2. *Y is the continuous image of a remainder of X.*
3. *Y is a continuous image of $\beta X \setminus X$.*

This theorem is a cornerstone of the theory of compactifications. It is easy to see that any space X having the property that each continuous image of $\beta X \setminus X$ is a remainder must have a one-point compactification. This forces the space to be locally compact. It is very difficult, however, to find hypotheses involving something weaker than local compactness and still have a robust theorem. The point of this paper is to try exactly that.

Results

All spaces considered are at least Tychonoff. Let X, Y be spaces. We will say that Y is a remainder of X, if there exists a Hausdorff compactification αX of X such that $Y = \alpha X \setminus X$. We denote by $\pi_{\beta\alpha}$ the natural map from βX onto αX. Because of the following easily established Proposition, we can without loss of generality restrict our considerations to perfect maps.

Proposition 1. *Let $Y = \alpha X \setminus X$ be a remainder of X. Then $\pi_{\beta\alpha}|_{\beta X\setminus X} : \beta X \setminus X \to Y$ is a (surjective) perfect map.*

Definition. For any map $f : X \to Y$, we put

$$M_f = \cup\{f^{-1}(p) : |f^{-1}(p)| > 1\}.$$

If X is compact, as would be the case if X were the remainder of a locally compact space, then $\overline{M_f}$ is compact. We will say that a closed continuous map f is *supercompact* if $\overline{M_f}$ is compact.

Clearly every supercompact map has compact fibers so it is perfect. One can prove that every quotient map such that $\overline{M_f}$ is compact, is also closed, hence supercompact.

For any map $f : X \to Y$, we will say that $E \subset X$ is *saturated with respect to* f if $f^{-1}(f(E)) = E$.

The following lemma uses arguments similar to [M2, Theorem 13].

Lemma 2. *Let K be a compact Hausdorff space and let q be a quotient map from K onto a space Z. Let F be a closed subset of K such that $M_q \subset F$ and suppose that $q(F)$ is Hausdorff. Then Z is Hausdorff.*

Proof: Let x_1, x_2 be distinct points of Z. The only nontrivial case is when x_1, x_2 are in $G = q(F)$. Since F is saturated, G is closed, in Z and hence normal (since it is Hausdorff and compact). Let U_1, U_2 be open subsets of G which contains x_1, x_2, respectively, and such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Let V_i be an open subset of Z such that $V_i \cap G = U_i$, $i = 1, 2$, and let W_1, W_2 be disjoint open subsets of K such that $q^{-1}(\overline{U_i}) \subset W_i$. Put $S_i = W_i \cap q^{-1}(V_i)$. One has $S_i \cap F = q^{-1}(U_i)$. Since q is injective on $K \setminus F$, the sets S_i are saturated, hence $q(S_1)$ and $q(S_2)$ are disjoint open subsets of Z which contain x_1, x_2 respectively. □

Let Y be a Hausdorff space and let $f : \beta X \setminus X \to Y$ be a quotient map. We denote by q_f the map $1_X \cup f : \beta X \to X \cup Y$ and by τ_f the quotient topology induced by q_f on $X \cup Y$ (the union is assumed to be disjoint). Then $(X \cup Y, \tau_f)$ is a compact (not necessarily Hausdorff) space in which the image of X is dense. Suppose $(X \cup Y, \tau_f)$ is Hausdorff. Then q_f is closed and so that $q_f|_X$ is a homeomorphism. Then it follows that Y is a remainder of X.

Proposition 3. *Let Y be a Hausdorff space and let $f : \beta X \setminus X \to Y$ be a quotient map. Then $(X \cup Y, \tau_f)$ is Hausdorff if and only if $q_f(Cl_{\beta X}(M_f))$ is Hausdorff.*

Proof: If $q_f(Cl_{\beta X}(M_f))$ is Hausdorff, then we can apply Lemma 2, putting $K = \beta X$, $Z = X \cup Y$, $q = q_f$ and $F = Cl_{\beta X}(M_f)$. □

From Proposition 3 it easily follows that:

Theorem 4. *Let X, Y be spaces and let f be a supercompact map from $\beta X \setminus X$ onto Y. Then Y is a remainder of X.*

In the proof of Theorem 3.7 in [T], it is proved that, if $\beta X \setminus X$ is realcompact and C^*-embedded in βX, then, for each compactification αX of X, $Cl_{\beta X \setminus X}(M_{\pi_{\beta\alpha}})$ must be compact. The proof of that theorem is based on the following fact

[T, Lemma 3.6]: if $\beta X \setminus X$ is C^*-embedded in βX and αX is any compactification of X, then $M_{\pi_{\beta\alpha}}$ does not contain any copy of $\mathbf{N}$ which is C-embedded in $\beta X \setminus X$. Now suppose that $\beta X \setminus X$ is normal and every noncompact closed subset of $\beta X \setminus X$ is not pseudocompact (for instance, let $\beta X \setminus X$ be paracompact). Also, suppose again that $\beta X \setminus X$ is C^*-embedded in βX. Then $Cl_{\beta X\setminus X}(M_{\pi_{\beta\alpha}})$ must be compact. Otherwise, $M_{\pi_{\beta\alpha}}$ would contain a copy of $\mathbf{N}$ C-embedded in $Cl_{\beta X\setminus X}(M_{\pi_{\beta\alpha}})$, hence in $\beta X \setminus X$. Then one has:

Corollary 5. *Suppose $\beta X \setminus X$ is C^*-embedded in βX and realcompact or paracompact. Then Y is a remainder of X if and only if Y it is a supercompact image of $\beta X \setminus X$.*

Corollary 6. *Suppose X is not locally compact. If $Z = \beta X \setminus X$ is C-embedded in βX and realcompact or paracompact, then there is a space Y and a perfect surjective map $f : Z \to Y$ which cannot be the restriction to Z of any $\pi_{\beta\alpha}$ from βX to a compactification αX of X.*

PROOF: The hypotheses imply that Z is not pseudocompact. Let $A = \{a_n\}_{n\in\mathbf{N}} \subset Z$ a C^*-embedded copy of $\mathbf{N}$. Let us identify a_{2n-1} and a_{2n} for every n and let Y be the quotient space and f be the quotient mapping. Clearly f is closed, because, for every closed subset F of Z, $f^{-1}(f(F)) = F \cup B$, where B is a subset of A, so it is closed. It is easy to see that Y is Hausdorff, hence f is perfect. But $Cl_Z(M_f)$ is not compact, so there is no compactification αX such that $f = \pi_{\beta\alpha}|_Z$. □

Thus, although supercompactness seems unnecessarily strong, in some cases it is necessary. It is also true [T] that given any space Y there is a space X such that $\beta X \setminus X = Y$ and $\beta X \setminus X$ is C^*-embedded in βX.

Now we will give more applications of Lemma 1 and Proposition 3.

For a compactification αX, we put $C_\alpha(X) = \{f \in C^*(X) : f \text{ extends to } \alpha X\}$. If $f \in C_\alpha(X)$, we denote by f^α the unique extension of f to αX. If $\mathcal{F} \subset C_\alpha(X)$, we denote by $\mathcal{F}^\alpha$ the set of the extensions.

We will say that a family $\mathcal{F} \subset C^*(X)$ *generates* the compactification αX if $\mathcal{F} \subset C_\alpha(X)$ and $\mathcal{F}^\alpha$ separates points of αX. Every $\mathcal{F} \subset C^*(X)$ which separates points from closed sets of X generates a compactification of X.

Theorem 7. *Let Y be a Hausdorff space. Then Y is a remainder of X if and only if there is a continuous map f from $\beta X \setminus X$ onto Y such that the family $\mathcal{G}(f) = \{g \in C^*(X) \,|\, g^\beta \text{ is constant on the fibers of } f\}$ separates points from closed subsets of X.*

PROOF: If $Y = \alpha X \setminus X$ is a remainder of X, then $\mathcal{G}(f) = C_\alpha(X)$ separates points from closed subsets of X.
Conversely we will prove that $(X \cup Y, \tau_f)$ is Hausdorff. Put $G = q_f(Cl_{\beta X}(\beta X \setminus X))$. It is easy to see that $G = Cl_{(X\cup Y,\tau_f)}(Y)$. By Lemma 2, it suffices to prove that G is Hausdorff. First, let $y_1, y_2 \in Y$ and let V_1, V_2 be disjoint open subsets of Y with $y_i \in V_i$. Put $H_i = Y \setminus V_i$. Then $Cl_{(X\cup Y,\tau_f)}(H_1), Cl_{(X\cup Y,\tau_f)}(H_2)$

are (closed) subsets of G, whose union is G, and such that $y_i \notin Cl_{(X\cup Y,\tau_f)}(H_i)$, $i=1,2$. This proves that y_1, y_2 are separated by disjoint open subsets of G.
Now, let αX be the compactification generated by $\mathcal{G}(f)$. Let $j : X \cup Y \to \alpha X$ be defined by $j(x) = \pi_{\beta\alpha}(q_f^{-1}(x))$. Clearly $\pi_{\beta\alpha}$ is constant on the fibers of f, hence of q_f, and so j is well defined. Since $\pi_{\beta\alpha} = j \circ q_f$, j is continuous and surjective. Let y_1, y_2 be points of G such that at least one of them is not in Y. Then $j(y_1), j(y_2)$ are distinct points of αX, hence they are separated by disjoint open subsets W_1, W_2 of αX. Then $j^{-1}(W_i) \cap G$, $i=1,2$ are open subsets of G which separates y_1, y_2. □

Theorem 8. *Let $f : \beta X \setminus X \to Y$ be a perfect surjective map such that $Cl_{\beta X}(M_f) \cap X = \{x_1, \dots, x_n\}$.*
Then the following are equivalent:

(i) *Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_{\beta X\setminus X}$.*
(ii) *There exist pairwise disjoint open sets U_i in $Cl_{\beta X}(M_f)$, such that $x_i \in U_i$, $i = 1, dots, n$ and the $f(U_i \setminus X)$ are still pairwise disjoint.*

Proof: First, we prove that (i) implies (ii). If $Y = \alpha X \setminus X$, then we can take, in αX, pairwise disjoint open neighborhoods $V_1, \dots, V_n$ of $x_1, \dots, x_n$, respectively, and put $U_i = \pi_{\beta\alpha}^{-1}(V_i) \cap Cl_{\beta X}(M_f)$, for each i.
Now, we will prove the converse. Let q_f, $(X\cup Y, \tau_f)$ be defined as in Proposition 3. We need only to prove that $q_f(Cl_{\beta X}(M_f))$ is Hausdorff. Put $q = q_f|_{Cl_{\beta X}(M_f)}$ and $S = Cl_{\beta X\setminus X}(M_f)$, so that $Cl_{\beta X}(M_f) = S \cup \{x_1, \dots, x_n\}$. Then S is locally compact and, since $f|_S$ is perfect, then $T = f(S)$ is also locally compact. Clearly, one has $q(Cl_{\beta X}(M_f) = T \cup \{x_1, \dots, x_n\}$. Let $U_i' = U_i \cap S$, $i = 1, \dots, n$, and let $K = S \setminus \bigcup_{i=1}^n U_i'$, $K_1 = f^{-1}(f(K))$. K is clearly compact, hence K_1 is also compact. If we put $W_i = U_i' \setminus K_1$, for each i, then $W_i = f^{-1}(f(W_i))$, so that the $f(W_i)$ are open and pairwise disjoint in T. Furthermore, the $f(W_i)$ are not relatively compact in T. Then $f(K), f(W_1), \dots, f(W_n)$ determine an n-point compactification $\kappa T = T \cup \{y_1, \dots, y_n\}$ where $f(W_i) \cup \{y_i\}$ is a neighborhood of y_i for each i. We can suppose that $y_i = x_i$, i.e. that the underlying set of κT is $q_f(Cl_{\beta X}(M_f)$. We want to prove that the topology of κT is less fine than the (quotient) topology induced by q. Since the two topologies coincide on T, we need only to prove that, for each i and for each basic neighborhood of x_i in κT, the inverse image with respect to q is open. Let H be any compact subset of T. One has $q^{-1}((f(W_i) \cup \{x_i\}) \setminus H) = (W_i \cup \{x_i\}) \setminus f^{-1}(H) = U_i \setminus (K_1 \cup f^{-1}(H))$, which is open. Then $q_f(Cl_{\beta X}(M_f)$, with the quotient topology, is Hausdorff. □

Corollary 9. *If $Z = \beta X \setminus X$ and $|Cl_{\beta X}(Z) \setminus Z| = 1$, then every perfect image of Z is a remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_{\beta X\setminus X}$.*

Note that Corollary 9 provides a new proof to Corollary 2.3 in [R].

Even if the space X satisfies the hypotheses of Corollary 6, it can happen that every perfect image of $\beta X \setminus X$ is a remainder of X. For instance, if $\beta X \setminus X \cong \mathbf{N}$, every perfect image of $\beta X \setminus X$ is homeomorphic to $\mathbf{N}$. But, in general, there exist perfect images of $\beta X \setminus X$ which are not remainders of X (see the following example).

Example. As we indicated earlier it is known that, for every Tychonoff space Z, there exists a space X such that $Z \cong \beta X \setminus X$ and $\beta X \setminus X$ is C^*-embedded in βX. So suppose $\beta X \setminus X \cong \mathbf{R}$ and C^*-embedded in βX. Then $[0,+\infty)$ is a perfect image of $\beta X \setminus X$, but it is not a remainder of X, because it is not a supercompact image of $\beta X \setminus X$. In fact, suppose $f : \mathbf{R} \to [0,+\infty)$ is supercompact. Let $(a,b) \subset \mathbf{R}$ be an open interval containing M_f. Then $f(-\infty,a]$ and $f([b,+\infty)$ are closed, connected and noncompact and they must be disjoint. But this is not possible in $[0,+\infty)$.

Example. In Corollary 5 we cannot drop the hypothesis that $\beta X \setminus X$ is C^*-embedded. In fact, let W be a space such that $\beta W \setminus W \cong \mathbf{I}$ and let $p \in \beta W \setminus W$ be the point corresponding to 1 in the homeomorphism. Let $X = W \cup \{p\}$, so that $Z = \beta X \setminus X \cong [0,1)$. Clearly $|Cl_{\beta X}(Z) \setminus Z| = 1$, hence every perfect image of $[0,1)$ is a remainder of X. But, clearly, there exist perfect images of $[0,1)$ which are not supercompact images. For instance, let $Y = \{(x,0) : 0 \leq x < 1\} \cup (\bigcup_{n\geq 2}\{(1-1/n,y) : 0 \leq y \leq 1/n\} \subset \mathbf{R}^2$. It is easy to see that Y is a perfect image of $[0.1)$. Suppose $f : [0,1) \to Y$ is supercompact and surjective. Let $z = (1-1/m,0) \notin f(\overline{M_f})$, which is compact. Let K be a compact and connected neighborhood of z which misses $f(\overline{M_f})$ and $K_1 = f^{-1}(K)$. Then $f|_{K_1}$ is a homeomorphism from K_1 to K. Therefore K_1 must be a closed interval, while K cannot be homeomorphic to an interval, a contradiction.

Example. Suppose $\beta X \setminus X$ is C^*-embedded in βX and homeomorphic to $[0,\omega_1)$. Let Y be the quotient space of $[0,\omega_1)$ obtained by identifying $\tau+1$ and $\tau+2$, for each limit ordinal $\tau < \omega_1$. Clearly the quotient space is again $[0,\omega_1)$ and the quotient map f is perfect, so, by Corollary 9, there exists a compactification αX of X with $\alpha X \setminus X = [0,\omega_1)$ and $f = \pi_{\beta\alpha}|_{\beta X \setminus X}$. Clearly f is not supercompact but $\alpha X \setminus X$ is still a supercompact image of $\beta X \setminus X$.

Now we rephrase a theorem by Rayburn into a form more relevant to our needs.

Theorem 10 [R, Theorem 1.3]**.** *Let $Z = \beta X \setminus X$ and let $f : Z \to Y$ be a perfect surjective map. Then the following are equivalent:*

(i) *Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_{\beta X \setminus X}$.*

(ii) *There exists a compactification γY of Y and an extension $\tilde{f} : Cl_{\beta X}(Z) \to \gamma Y$ such that $\tilde{f}|_{Cl_{\beta X}(Z)\setminus Z}$ is injective.*

Note that Corollary 9 also follows from Theorem 10. The hypothesis implies that Z is locally compact. If $f : Z \to Y$ is perfect and surjective, then Y is also locally compact. Then we can take as γY the one-point compactification of Y. It is easy to see that the natural extension $\tilde{f}$ of f to $Cl_{\beta X}(Z)$ is continuous and, obviously, injective on $Cl_{\beta X}(Z) \setminus Z$.

Corollary 11. *Let $Z = \beta X \setminus X$ be C^*-embedded in βX and let $f : Z \to Y$ be a perfect surjective map. Then the following are equivalent:*

(i) *Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_{\beta X\setminus X}$.*

(ii) *The extension $\tilde{f} : Cl_{\beta X}(Z) \to \beta Y$ is injective on $Cl_{\beta X}(Z) \setminus Z$.*

Proof: It is easy to see that, if $f : Z \to Y$ is perfect and surjective and there exists a compactification γY of Y such that the extension of f from βZ to γY is injective on the remainder, then $\gamma Y = \beta Y$. We can then use the above theorem. □

These results and the theorem of Rayburn (Theorem 9) suggest the following general problem: given a perfect surjective map $f : Z \to Y$ and a compactification κZ of Z, find conditions which ensure that there exists a compactification γY of Y and an extension of f, $\tilde{f} : \kappa Z \to \gamma Y$ such that $f|_{\kappa Z\setminus Z}$ is injective. In particular: find conditions on f for $\tilde{f} : \beta Z \to \beta Y$ to be injective on $\beta Z \setminus Z$. This question is certainly central to our results.

Lemma 12. *Let $f : Z \to Y$ be a perfect surjective map and let $\tilde{f} : \beta Z \to \beta Y$ be its extension. Suppose that*

(i) *For all closed completely separated subsets A and B of Z, there exists a relatively compact open subset U of Z such that $f(A \setminus U)$ and $f(B \setminus U)$ are completely separated.*

Then

(ii) *$\tilde{f}|_{\beta Z\setminus Z}$ is injective.*

If Z is locally compact, then (ii) *implies* (i).

Proof: For a closed set A in Z, let us denote by A^* the set $Cl_{\beta Z}(A) \setminus A$. Since f is perfect, $\tilde{f}(A^*) \subset \beta Y \setminus Y$. Now, suppose f satisfies (i) and let p, q be distinct points of $\beta Z \setminus Z$. Let $p \in V$, $q \in W$, where V, W are open in βZ and have disjoint closures. If we put $A = Cl_{\beta Z}(V) \cap Z$ and $B = Cl_{\beta Z}(W) \cap Z$, then A and B are completely separated, $p \in A^*$, $q \in B^*$. Let U be as in the hypothesis, so that $p \in (A \setminus U)^*$, $q \in (B \setminus U)^*$. Furthermore, $\tilde{f}((A \setminus U)^*) \cap \tilde{f}((B \setminus U)^*) \subset \tilde{f}(Cl_{\beta Z}(A\setminus U)) \cap \tilde{f}(Cl_{\beta Z}(B\setminus U)) = Cl_{\beta Y}(f(A\setminus U)) \cap Cl_{\beta Y}(f(B\setminus U)) = \emptyset$. Then $\tilde{f}(p) \neq \tilde{f}(q)$.

Now suppose Z is locally compact, and $\tilde{f}|_{\beta Z\setminus Z}$ is injective. Let A, B be closed and completely separated, so that, $\tilde{f}(A^*) \cap \tilde{f}(B^*) = \emptyset$. Then $F = \tilde{f}(Cl_{\beta Z}(A)) \cap \tilde{f}(Cl_{\beta Z}(B)$ is a compact subset of Y. If $K = f^{-1}(F)$, K is also compact. Let U be a relatively compact neighbourhood of K. Then $Cl_{\beta Y}(f(A\setminus U)) \cap Cl_{\beta Y}(f(B\setminus U)) = \tilde{f}(Cl_{\beta Z}(A \setminus U) \cap \tilde{f}(Cl_{\beta Z}(B \setminus U) = \emptyset$. □

Thus we have:

Theorem 13. *Let $Z = \beta X \setminus X$ be C^*-embedded in βX and let $f : Z \to Y$ be perfect and surjective. Suppose that f satisfies the condition* (i) *in the above* *lemma. Then there is a compactification αX of X with $Y = \alpha X \setminus X$ and $f = \pi_{\beta\alpha}|_{\beta X \setminus X}$. If Z is locally compact, the converse is also true.*

Lemma 14. *Let Z be paracompact and let $f : Z \to Y$ be a perfect map. Then the following are equivalent:*

(i) *f is supercompact.*
(ii) *For every pair A, B of disjoint closed subsets of Z, $f(A) \cap f(B)$ is compact.*

PROOF: If f is supercompact and A, B are closed and disjoint, then $f(A) \cap f(B)$ is closed and it is a subset of $f(Cl_Z(M_f))$, which is compact.
Conversely, suppose that f satisfies (ii), but it is not supercompact. Then M_f contains a copy N of $\mathbf{N}$ which is C-embedded in $Cl_Z(M_f)$, hence closed in Z. Since the fibers of f are compact, N meets infinitely many fibers, so we can choose an infinite set $A = \{a_n\}_{n \in \mathbf{N}} \subset F$ such that $N \cap f^{-1}(f(a_n)) = \{a_n\}$ for each n. Clearly A is closed in Z. Let $B = \{b_n\}_{n \in \mathbf{N}}$ such that, for each n, $f(b_n) = f(a_n)$ and $b_n \neq a_n$. Then B is also closed. Suppose not and let $\{b_{n_\lambda}\}$ be an ultranet in B converging to $z \in Z \setminus B$. The corresponding ultranet $\{a_{n_\lambda}\}$ is nontrivial, so it converges to a point $p \in \beta Z \setminus Z$. If $\tilde{f} : \beta Z \to \beta Y$ is the extension of f, one has $\tilde{f}(p) = \tilde{f}(z) \in Y$. This is impossible because f is perfect. Then A and B are disjoint closed sets such that $f(A) \cap f(B) = f(A)$ is not compact, a contradiction. □

Example. The above lemma is not true for normal spaces. If $Z = [0, \omega_1)$, then every closed continuous map from Z into a space Y satisfies (ii). In fact, if A and B are closed and disjoint, one of them is compact. However we have earlier seen that there are mappings $f : [0, \omega_1) \to [0, \omega_1)$ which are not supercompact.

Note. In the proof of Lemma 12 we have seen that, if f satisfies (ii), then, for A and B closed and completely separated, $f(A) \cap f(B)$ is compact. To prove that, we did not use local compactness.

By the above note, Corollary 5, and Lemma 14, one has:

Theorem 15. *Suppose $Z = \beta X \setminus X$ is paracompact and C^*-embedded in βX and let $f : Z \to Y$ be a perfect surjective map. Then the following are equivalent:*

(i) *Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_Z$.*
(ii) *If A, B are disjoint closed subsets of Z, then $f(A) \cap f(B)$ is compact.*

Now suppose $\mathcal{F} \subset C^*(X)$, where X is locally compact. For each $h \in \mathcal{F}$ let I_h be a closed interval containing the image of h. It is known that the smallest compactification to which every element of $\mathcal{F}$ extends, denoted by $\omega_{\mathcal{F}} X$, can be constructed as follows: put on the disjoint union $X \cup \prod_{h \in \mathcal{F}} I_h$ the topology generated by sets of the form $W \cup \left(e_{\mathcal{F}}^{-1}(W) \setminus F\right)$, where W is a basic open set of $\prod_{h \in \mathcal{F}} I_h$, $e_{\mathcal{F}}$ is the diagonal map of $\mathcal{F}$ and F is any compact subset of X; $\omega_{\mathcal{F}} X$ is the closure of X in that space, which is compact [CFV, Proposition 1.2].

Lemma 16. *Let $f : Z \to Y$ be a perfect surjective map and let κZ be a compactification of Z. Put*

$$\mathcal{F} = \{g \circ f : g \in C^*(Y) \text{ and } g \circ f \in C_\kappa(Z)\}.$$

Then

(i) *There exists a compactification γY of Y and an extension of f, $\tilde{f} : \kappa Z \to \gamma Y$ such that $\tilde{f}|_{\kappa Z \setminus Z}$ is injective*

implies

(ii) *$\mathcal{F}^\kappa$ separates points of $\kappa Z \setminus Z$.*

If Z is locally compact, then the converse is also true.

Proof: First we prove that (i) implies (ii). Let $p, q \in \kappa Z \setminus Z$ be distinct. Then $\tilde{f}(p) \neq \tilde{f}(q)$. Therefore there exists $g \in C_\gamma(Y)$ such that $g^\gamma(\tilde{f}(p)) \neq g^\gamma(\tilde{f}(q))$. But, clearly, $g^\gamma \circ \tilde{f}$ is an extension of $g \circ f$ to κZ, then one has $g \circ f \in \mathcal{F}$ and $(g \circ f)^\kappa(p) \neq (g \circ f)^\kappa(q)$.
Now suppose that Z is locally compact and that $\mathcal{F}^\kappa$ separates points of $\kappa Z \setminus Z$. Put $\mathcal{G} = \{g \in C^*(Y) : g \circ f \in \mathcal{F}\}$. Since f is surjective, $g \mapsto g \circ f$ is an injective map from $\mathcal{G}$ onto $\mathcal{F}$. Let γY be the smallest compactification to which every element of $\mathcal{G}$ extends. Then γY can be obtained as the closure of Y in $Y \cup \prod_{g \in \mathcal{G}} I_g$ with the topology described above. The hypothesis implies that κZ is the smallest compactification to which every element of $\mathcal{F}$ extends. Therefore κZ can be obtained as the closure of Z in $Z \cup \prod_{g \circ f \in \mathcal{F}} I_{g \circ f} = Z \cup \prod_{g \in \mathcal{G}} I_g$. We can define $\hat{f} : Z \cup \prod_{g \in \mathcal{G}} I_g \to Y \cup \prod_{g \in \mathcal{G}} I_g$ by $\hat{f}(p) = f(p)$ if $p \in Z$ and $\hat{f}(p) = p$ otherwise. We will prove that $\hat{f}$ is continuous. Let $U = W \cup \left(e_{\mathcal{G}}^{-1}(W) \setminus F\right)$ a basic open subset of $Y \cup \prod_{g \in \mathcal{G}} I_g$. Then $\hat{f}^{-1}(U) = W \cup \left((e_{\mathcal{G}} \circ f)^{-1}(W) \setminus f^{-1}(F)\right)$. One has $e_{\mathcal{G}} \circ f = e_{\mathcal{F}}$ and, since f is perfect, $f^{-1}(F)$ is compact. Then $\hat{f}^{-1}(U)$ is a basic open set. Clearly, $\hat{f}(\kappa Z) = \gamma Y$. Then we can take $\tilde{f} = \hat{f}|_{\kappa Z}$. □

Note. If $\mathcal{F}$ is defined as in the above theorem, then it is easy to see that $\mathcal{F} = \{h \in C_\kappa(Z) : h \text{ is constant on the fibers of } f\}$.

The following is true for any Z.

Lemma 17. *Let $f : Z \to Y$ be a perfect surjective map and let $\tilde{f} : \beta Z \to \beta Y$ be its extension. Put*

$$\mathcal{F} = \{h \in C^*(Z) : h \text{ is constant on the fibers of } f\}.$$

Then the following are equivalent:

(i) *$\tilde{f}|_{\beta Z \setminus Z}$ is injective.*

(ii) *$\mathcal{F}^\beta$ separates points of $\beta Z \setminus Z$.*

Proof: We need only to prove that (ii) implies (i). Let us identify βY with $\overline{e(Y)}$, where $e : Y \to \prod_{g\in C^*(Y)} I_g$ is the diagonal map of $C^*(Y)$. Let $\tilde{f} : \beta Z \to \prod_{g\in C^*(Y)} I_g$ be the extension of f to βZ, which is defined by $\tilde{f}(x) = \{(g \circ f)^\beta(x)\}_{g\in C^*(Y)}$. Then $\tilde{f}(\beta Z) = \beta Y$. Furthermore, since $\{(g\circ f)^\beta : g \in C^*(Y)\} = \mathcal{F}^\beta$ separates points of $\beta Z \setminus Z$, $\tilde{f}|_{\beta Z\setminus Z}$ is injective. □

Theorem 18. *Let $f : Z = \beta X \setminus X \to Y$ be perfect and surjective. Put*

$$\mathcal{E} = \{h \in C(Cl_{\beta X\setminus X}(Z)) : h \text{ is constant on the fibers of } f\}.$$

If Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_Z$, then $\mathcal{E}$ separates points of $Cl_{\beta X}(Z)\setminus Z$. If Z is locally compact, the converse is also true.

Theorem 19. *Let $f : Z = \beta X \setminus X \to Y$ be perfect and surjective. Also, suppose that Z is C^*-embedded in βX. Put*

$$\mathcal{F} = \{h \in C^*(Z) : h \text{ is constant on the fibers of } f\}.$$

Then the following are equivalent:

(i) *Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_Z$.*
(ii) *$\mathcal{F}^\beta$ separates points of $\beta Z \setminus Z$.*

Lemma 20. *Let $f : Z \to Y$ be a perfect surjective map, and let κZ be a compactification of Z. Then the following are equivalent:*

(i) *Each $p \in \kappa Z \setminus Z$ has a local basis $\mathcal{W}_p$ such that for each $W \in \mathcal{W}_p$, $W \cap Z$ is saturated with respect to f.*
(ii) *There exists a compactification γY of Y and an extension $\tilde{f}$ of f, $\tilde{f} : \kappa Z \to \gamma Y$ such that $\tilde{f}|_{\kappa Z\setminus Z}$ is injective.*

Proof: First we prove that (i) implies (ii). Put $S = \kappa Z \setminus Z$ and consider the disjoint union $Y \cup S$ with the quotient topology induced by $\tilde{f} = f \cup 1_S$. We need only to prove that space is Hausdorff. First let $p, q \in S$ and let $W \in \mathcal{W}_p$, $W' \in \mathcal{W}_q$ be disjoint. Then they are saturated with respect to $\tilde{f}$, hence their images are disjoint open neighborhoods of p and q, respectively, in $Y \cup S$. Now, suppose $p \in S$ and $q \in Y$. Since f is perfect, $f^{-1}(q)$ is closed in κZ. Let V and V' be disjoint open subsets of κZ which contain p and $f^{-1}(q)$ respectively. Let $W \in \mathcal{W}_p$ such that $W \subset V$. Since f is closed, $V' \cap Z$ contains an open subset U of Z, containing $f^{-1}(q)$, which is saturated with respect to f. Let W' be an open subset of κZ such that $U = W' \cap Z$. We can choose W' contained in V'. Therefore, both W and W' are saturated with respect to $\tilde{f}$, and their images contain p and q respectively. Finally, if $p, q \in Y$, then we can take disjoint open subsets V, V' of κZ which contain $f^{-1}(p)$ and $f^{-1}(q)$ respectively. As before, V and V' contain open subsets of κZ, saturated with respect to $\tilde{f}$, which contain $f^{-1}(p)$ and $f^{-1}(q)$ respectively.

Conversely, let $p \in \kappa Z \setminus Z$ and let V be a neighborhood of p in κZ. We need to prove that there exists a neighborhood W, of p such that $p \in W \subset V$ and $W \cap Z$ is saturated with respect to f. Put $F = \kappa Z \setminus V$, $G = \tilde{f}(F)$. Then G is closed and $\tilde{f}(p) \notin G$. Then we can take $W = \kappa Z \setminus \tilde{f}^{-1}(G)$. □

And finally we have:

Theorem 21. *Let $f : Z = \beta X \setminus X \to Y$ be perfect and surjective. Then the following are equivalent:*

1. *Y is the remainder of a compactification αX of X and $f = \pi_{\beta\alpha}|_Z$.*
2. *For every $p \in Cl_{\beta X}(Z) \setminus Z$ there is a local basis $\mathcal{W}_p$ of p in $Cl_{\beta X}(Z)$ such that, for each $W \in \mathcal{W}_p$, $W \cap Z$ is saturated with respect to f.*

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Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

Dipartimento di Matematica, Università di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy

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