Whitney blocks in the hyperspace of a finite graph

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Abstract. Let X be a finite graph. Let $C(X)$ be the hyperspace of all nonempty subcontinua of X and let $\mu : C(X) \to \mathbb{R}$ be a Whitney map. We prove that there exist numbers $0 < T_0 < T_1 < T_2 < \cdots < T_M = \mu(X)$ such that if $T \in (T_{i-1}, T_i)$, then the Whitney block $\mu^{-1}(T_{i-1}, T_i)$ is homeomorphic to the product $\mu^{-1}(T) \times (T_{i-1}, T_i)$. We also show that there exists only a finite number of topologically different Whitney levels for $C(X)$.

Keywords: hyperspaces, Whitney levels, Whitney blocks, finite graphs
Classification: 54B20

Introduction

Throughout this paper X denotes a finite graph, i.e. a compact connected metric space which is the union of finitely many segments joined by their end points. A *segment* of X is one of those segments. A *subgraph* of X is a graph contained in X formed by some of those segments. Let $SG(X) = \{A \subset X : A$ is a subgraph of $X\}$.

The hyperspace of subcontinua of X is $C(X) = \{A \subset X : A$ is a nonempty, closed, connected subset of $X\}$ metrized with the Hausdorff metric. Let $F_1(X) = \{\{x\} \in C(X) : x \in X\}$. A *map* is a continuous function. A *Whitney map* for $C(X)$ (see [8, 0.50]) is a map $\mu : C(X) \to \mathbb{R}$ such that $\mu(\{x\}) = 0$ for every $x \in X$, $\mu(A) < \mu(B)$ if $A \subset B \neq A$ and $\mu(X) = 1$. A *Whitney level* is a set of the form $\mu^{-1}(t)$, where $t \in [0, 1]$. A *Whitney block* is a set of the form $\mu^{-1}(t, s)$, where $0 \leq t < s \leq 1$. From now on, μ will denote a Whitney map for $C(X)$.

In [1], R. Duda made a detailed study of the polyhedral structure of $C(X)$ by giving a good decomposition of $C(X)$ into balls. In [2], he gave a characterization of those polyhedra which are hyperspaces of acyclic finite graphs.

Whitney levels of finite graph have been studied by H. Kato. In [4] he showed that they are always polyhedra and that if $t_0 = \min\{\mu(A) : A$ is a simple closed curve contained in $X\}$ and $0 \leq t < t_0$, then $\mu^{-1}(t)$ is homotopically equivalent to X. In [4] and [6] he gave bounds for the fundamental dimension of Whitney levels of finite graphs and, in [5] he proved that Whitney levels of finite graphs admit all homotopy types of compact connected ANRs.

This paper was motivated by the following result of I. Puga ([10, Theorem 2.5]): "There exists $t \in [0, 1)$ and there exists a homeomorphism φ : (Cone over $\mu^{-1}(t)$)

*This paper was partially supported by CONACYT, 1085-E9201, Mexico.

$\to \mu^{-1}([t,1)$ such that $\varphi(A,0) = A$, $\varphi(A,1) = X$ and $s < t$ implies that $\varphi(A,s) \subset \varphi(A,t)$ for each $A \in \mu^{-1}(t)$". She expressed this property by saying that the hyperspace of subcontinua of a finite graph is conical pointed.

In this paper, we prove:

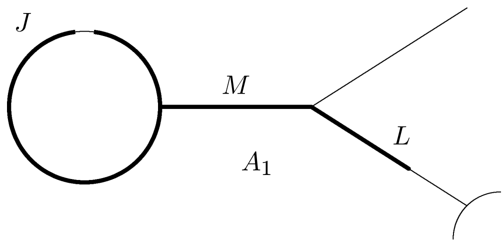
Theorem 1. *Suppose that* $\mu(SG(X)) \cup \{0\} = \{T_0, T_1, \dots, T_M\}$, *where* $0 = T_0 < T_1 < \dots < T_M = 1$. *If* $1 \le i \le M$ *and* $T \in (T_{i-1}, T_i)$, *then there exists a homeomorphism* $\varphi : \mu^{-1}(T) \times (T_{i-1}, T_i) \to \mu^{-1}(T_{i-1}, T_i)$ *such that* $\varphi(A,T) = A$ *and* $\varphi(A,s) \subset \varphi(A,t)$ *if* $s < t$ *for every* $A \in \mu^{-1}(T)$ *and, for each* $t \in (T_{i-1}, T_i)$, $\varphi \mid \mu^{-1}(T) \times \{t\}$ *is a homeomorphism from* $\mu^{-1}(T) \times \{t\}$ *onto* $\mu^{-1}(t)$.

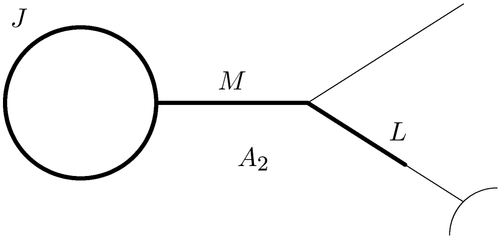
Theorem 2. *There is only a finite number of topologically different Whitney levels for* $C(X)$.

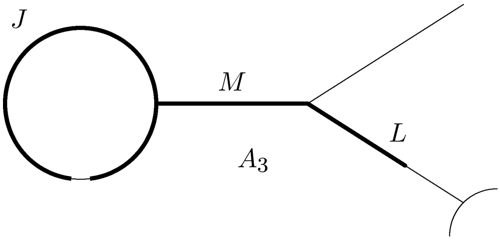
1. Preliminaries

The *vertices* of X are the end points of the segments of X. Notice that the set $SG(X)$ of subgraphs of X depends on the choice of the segments. We are interested in having as less subgraphs as possible, so we will suppose that X is not a simple closed curve and each vertex of X is either an end point of X or a ramification point of X. With this restriction two extremes of a segment of X may coincide and then such a "segment" would be a simple closed curve. The set of segments of X is denoted by $\mathcal{S}$. For each $J \in \mathcal{S}$, we fix an orientation and then we identify J with a closed interval $[(-1)_J, (1)_J]$. Notice that it is possible that $(-1)_J = (1)_J$. We write -1 (resp. 1) instead of $(-1)_J$ (resp. $(1)_J$) if no confusion arrives.

In order to define the map φ in Theorem 1, we will describe its action in each $J \in \mathcal{S}$. For each $A \in \mu^{-1}(T)$, we consider $A \cap J$ and we enlarge or shrink this set. To illustrate how this shrinking of $A \cap J$ has to be done, let us consider the following diagram:







Here, L and M are segments of X and J is a segment in X such that the end points of J coincide (that is, J is a simple closed curve). The subcontinua A_1, A_2 and A_3 have been outlined in thicker lines. The subcontinuum A_2 contains J and M and one half of L, $A_1 \cap L$ and $A_3 \cap L$ are a little bit larger that $A_2 \cap L$ while $A_1 \cap J$ and $A_3 \cap J$ are a little bit smaller than $A_2 \cap J$. In this example, $T_{i-1} = \mu(J \cup M)$.

If we shrink $A_2 \cap J$, then we have to cut it at some place of the circle J. Since A_1 is very close to A_2, the continuity of the shrinking implies that we have to cut $A_1 \cap J$ at a similar position as $A_2 \cap J$. Then, the connectedness of the shrinking of $A_1 \cap J$ implies that $A_2 \cap J$ has to be cut only on the upper part of J. But, since A_3 is very close to A_2, in the same way as above, $A_2 \cap J$ has to be cut only on the lower part of J. This contradiction shows that it is not possible to shrink $A_2 \cap J$.

However, we have to shrink the continuum A_2 and the shrinkings have to take all the sizes in the interval $(T_{i-1}, \mu(A_2)]$. Then, the shrinking of A_2 will be carried out by making the arc $A_2 \cap L$ shorter and shorter. Since A_1 and A_3 are very close to A_2, then the shrinking of $A_1 \cap J$ and $A_3 \cap J$ have to be almost imperceptible compared with the shrinking of $A_1 \cap L$ and $A_3 \cap L$, respectively.

The map φ in Theorem 1 will be an appropriate reparametrization and restriction of the following map F, so the behaviour of F will be similar to the behaviour of φ and the discussion concerning the shrinking of the subcontinua of X is applicable to F.

Observe that to get the effect of shrinking some intervals very slowly compared with others, we strongly use the asymptoteness of the graph of the map g to the lines $y = \pm 1$ in the Euclidean plane.

2. Auxiliary maps

Consider the map $f : (-1,1) \to \mathbb{R}$ given by $f(t) = tg(t\pi/2)$ and let $g : \mathbb{R} \to (-1,1)$ be the inverse map of f. Then $f(-t) = -f(t)$ for every $t \in (-1,1)$, $g(-s) = -g(s)$ for every $s \in \mathbb{R}$ and $-g$ is the inverse map of $-f$. Define $C^{\vee}(X) = C(X) - (SG(X) \cup F_1(X))$.

Define $F : C^{\vee}(X) \times \mathbb{R} \to C^{\vee}(X)$ by $F(A,t) = \bigcup\{F_J(A,t) : J \in \mathcal{S}\}$, where $F_J : C^{\vee}(X) \times \mathbb{R} \to \{E : E \text{ is a closed subset of } J\}$ is defined as follows:

$$F_J(A,t) = \begin{cases} \text{(a)} & A \cap J \quad \text{if } \ A \cap J = \emptyset, \{-1\}, \{1\}, \{-1,1\} \ \text{ or } \ J, \\ \text{(b)} & [-1, g(f(b)+t)] \quad \text{if } \ A \cap J = [-1,b] \ \text{ and } \ -1 < b < 1, \\ \text{(c)} & [g(f(a)-t), 1] \quad \text{if } \ A \cap J = [a,1] \ \text{ and } \ -1 < a < 1, \\ \text{(d)} & [a + e(m-a), b + e(m-b)], \quad \text{where } \ m = \frac{a+b}{2+a-b} \ \text{ and} \\ & e = 1 + \frac{1+g(f(b-a-1)+t)}{a-b} \ \text{ if } \ A \cap J = [a,b] \ \text{ and} \\ & -1 < a < b < 1 \ \text{ and }, \\ \text{(e)} & [-1, a + e(m-a)] \cup [b + e(m-b), 1], \\ & \text{where } \ m = \frac{a+b}{2+a-b} \ \text{ and} \\ & e = 1 + \frac{1+g(f(b-a-1)-t)}{a-b} \ \text{ if } \ A \cap J = [-1,a] \cup [b,1], \\ & -1 \le a < b \le 1 \ \text{ and } \ -1 < a \ \text{ or } \ b < 1. \end{cases}$$

In case (e), $a(1+a) \le b(1+a)$ and $a(1-b) \le b(1-b)$, then $2a + a^2 - ab \le a + b \le 2b + ab - b^2$, so $a \le m \le b$, where $a < m$ or $b < m$. Notice that e is a strictly increasing function of t. If $t \to \infty$, $e \to 1$, $a + e(m-a) \to m$ and $b + e(m-b) \to m$. If $t \to -\infty$, $e \to 1 + \frac{2}{a-b}$ $a + e(m-a) \to -1$ and $b + e(m-b) \to 1$. Thus $F_J(A,t)$ is a proper subset of J, $\{-1,1\} \subset F_J(A,t) \neq \{-1,1\}$; if $t < s$, then $F_J(A,t) \subset F_J(A,s) \neq F_J(A,t)$, $F_J(A,t) \to J$ as $t \to \infty$ and $F_J(A,t) \to \{-1,1\}$ as $t \to -\infty$.

Similarly, in case (d), $F_J(A,t)$ is a proper subset of J, $-1, 1 \notin F_J(A,t)$, $m \in F_J(A,t)$; if $t < s$, then $F_J(A,t) \subset F_J(A,s) \neq F_J(A,t)$, $F_J(A,t) \to J$ as $t \to \infty$ and $F_J(A,t) \to \{m\}$ as $t \to -\infty$.

In all the cases, if $A \cap J$ is a nonempty proper subset of J, then $F_J(A,t)$ is a nonempty proper subset of J. Moreover, -1 (resp. 1) belongs to A if and only if -1 (resp. 1) belongs to $F_J(A,t)$. It follows that, for each t, a vertex p of X belongs to A if and only if p belongs to F(A,t) and $F(A,t) \in C^{\vee}(X)$. Therefore F is well defined.

We will need the following properties of function F:

I. If $t < s$, then $F(A,t) \subset F(A,s) \neq F(A,t)$.
It follows from the fact that in cases (b), (c), (d) and (e), if $t < s$, then $F_J(A,t) \subset F_J(A,s) \neq F_J(A,t)$.
II. For a fixed $A \in C^\vee(X)$, if $t \to -\infty$, $F(A,t)$ tends to a one-point set or to a subgraph of X which is contained in A and, if $t \to \infty$, then $F(A,t)$ tends to a subgraph of X which contains A.
III. F is continuous.
Let $((A_n,t_n))n$ be a sequence in $C^\vee(X) \times \mathbb{R}$ which converges to an element (A,t) in $C^\vee(X) \times \mathbb{R}$. We may suppose that if $J \in \mathcal{S}$ and $A \cap J = \emptyset$, then $A_n \cap J = \emptyset$ for every n. Let $\mathcal{S}^* = \{J \in \mathcal{S} : A \cap J \neq \emptyset\}$. Since $F(A,t)$ has no isolated points, if we can find a finite set E such that $F(A_n,t_n) \cup E \to F(A,t)$, then we will have that $F(A_n,t_n) \to F(A,t)$. In order to find such a set E, it is enough to show that, for each $J \in \mathcal{S}^*$, there exists a finite set E_J such that $F_J(A_n,t_n) \cup E_J \to F_J(A,t)$. Then take $J \in \mathcal{S}^*$. Here it is necessary to consider the following cases:

1. $A \cap J = J$,
2. $A \cap J = [-1,b]$ with $-1 < b < 1$,
3. $A \cap J = [a,1]$ with $-1 < a < 1$,
4. $A \cap J = [a,b]$ with $-1 < a < b < 1$,
5. $A \cap J = [-1,a] \cup [b,1]$ with $-1 < a < b < 1$,
6. $A \cap J = [-1,a] \cup \{1\}$ with $-1 < a < 1$,
7. $A \cap J = \{-1\} \cup [a,1]$ with $-1 < a < 1$,
8. $A \cap J = \{-1\}$,
9. $A \cap J = \{1\}$ and,
10. $A \cap J = \{-1,1\}$.

We only check cases 1 and 6; the others are similar. For case 1, the sequence $(A_n)n$ can be partitioned into subsequences $(B_k)k$ where each B_k lies in one of the following subcases:

(a) $B_k \cap J = J$. Then $F_J(B_k,t_{n_k}) = J \to F_J(A,t)$.
(b) $B_k \cap J = [-1,b_k]$ with $-1 < b_k < 1$. Since $B_k \to A$, $b_k \to 1$, then $F_J(B_k,t_{n_k}) = [-1, g(f(b_k)+t_{n_k})] \to [-1,1] = F_J(A,t)$.
(c) $B_k \cap J = [a_k,1]$ with $-1 < a_k < 1$. It is similar to (b).
(d) $B_k \cap J = [a_k,b_k]$ with $-1 < a_k < b_k < 1$. Then $a_k \to -1$ and $b_k \to 1$, so $e_k = 1 + [1 + g(f(b_k - a_k - 1) + t_{n_k})]/(a_k - b_k) \to 0$. Thus $b_k + e_k(m_k - b_k) - (a_k + e_k(m_k - a_k)) = (b_k - a_k)(1 - e_k) \to 2$. Therefore $F_J(B_k,t_{n_k}) = [a_k + e_k(m_k - a_k), b_k + e_k(m_k - b_k)] \to [-1,1] = F_J(A,t)$.
(e) $B_k \cap J = [-1,a_k] \cup [b_k,1]$, with $-1 < a_k < b_k < 1$ and $-1 < a_k$ or $b_k < 1$. Then $b_k - a_k \to 0$. Thus $b_k + e_k(m_k - b_k) - (a_k + e_k(m_k - a_k)) = (b_k - a_k)(1 - e_k) = (b_k - a_k)([1 + g(f(b_k - a_k - 1) + t_{n_k})]/(a_k - b_k)) \to 0$. Thus $F_J(B_k,t_{n_k}) \to J = F_J(A,t)$.

Therefore $F_J(A_n,t_n) \to F_J(A,t)$.

In case 6, define $E_J = \{1\}$. Note that $F_J(A,t) = [-1, g(f(a)+t)] \cup \{1\}$. We must consider the following subcases:

(a) $B_k \cap J = [-1,b_k]$ with $-1 < b_k < 1$. Since $B_k \to A$, $b_k \to a$, then

$F_J(B_k, t_{n_k}) \cup E_J = [-1, g(f(b_k) + t_{n_k})] \cup \{1\} \to [-1, g(f(a) + t)] \cup \{1\} = F_J(A, t)$.

(b) $B_k \cap J = [a_k, b_k]$ with $-1 < a_k < b_k < 1$. Then $a_k \to -1$ and $b_k \to a$. This implies that $m_k = (a_k + b_k)/(2 + a_k - b_k) \to -1$ and $e_k \to 1 + [1 + g(f(a) + t)]/(-1 - a)$. Thus $F_J(B_k, t_{n_k}) \cup E_J = [a_k + e_k(m_k - a_k), b_k + e_k(m_k - b_k)] \cup E_J \to [-1, g(f(a) + t)] \cup \{1\} = F_J(A, t)$.

(c) $B_k \cap J = [-1, a_k] \cup [b_k, 1]$, with $-1 \leq a_k < b_k \leq 1$ and $-1 < a_k$ or $b_k < 1$. Then $a_k \to a$, $b_k \to 1$, $m_k \to 1$ and $e_k \to (a - g(f(a) + t))/(a - 1)$. Thus, $F_J(B_k, t_{n_k}) \cup E_J = [-1, a_k + e_k(m_k - a_k)] \cup [b_k + e_k(m_k - b_k), 1] \to [-1, g(f(a) + t)] \cup \{1\} = F_J(A, t)$.

Hence, $F_J(A_n, t_n) \cup E_J \to F_J(A, t)$.

Therefore, F is continuous.

IV. If $(A, t), (B, s) \in C^\vee(X) \times \mathbb{R}$ are such that $A - B \neq \emptyset$ and $F(A, t) = F(B, s)$, then $t < s$.

To prove this, choose a point $p \in A - B$, let $J \in \mathcal{S}$ be such that $p \in J$. If p is a vertex of X, then $p \in F(A, t) = F(B, s)$, so $p \in B$. This contradiction proves that p is not a vertex of X. Then J is the unique segment of X which contains p. We consider some cases:

(a) $A \cap J = J$. Then $J \subset F(B, s)$. This implies that $B \cap J = J$ and $p \in B$. This contradiction shows that this case is not possible.

(b) $A \cap J = [-1, b]$ with $-1 < b < 1$. Since $F(A, t) = F(B, s)$, then $B \cap J$ is of the form $B \cap J = [-1, b_1]$ with $-1 < b_1 < b$ and $[-1, g(f(b) + t)] = [-1, g(f(b_1) + s)]$. Then $f(b) + t = f(b_1) + s$. Thus $t < s$.

(c) $A \cap J = [a, 1]$ with $-1 < a < 1$. This case is similar to case (b).

(d) $A \cap J = [-1, a] \cup [b, 1]$ with $-1 \leq a < b \leq 1$ and $-1 < a$ or $b < 1$. Since $F(A, t) = F(B, s)$, then $B \cap J$ is of the form $B \cap J = [-1, a_1] \cup [b_1, 1]$, with $-1 \leq a_1 < b_1 \leq 1$ and $-1 < a_1$ or $b_1 < 1$. Moreover, $a + e(m - a) = a_1 + e_1(m_1 - a_1) \dots (1)$ and $b + e(m - b) = b_1 + e_1(m_1 - b_1) \dots (2)$, where $m = (a + b)/(2 + a - b)$, $m_1 = (a_1 + b_1)/(2 + a_1 - b_1)$, $e - 1 = (1 + g(f(b - a - 1) - t))/(a - b)$ and $e_1 - 1 = (1 + g(f(b_1 - a_1 - 1) - s))/(a_1 - b_1) \dots (3)$.

From (1) and (2), $(1 - e)a - (1 - e_1)a_1 = (1 - e)b - (1 - e_1)b_1$, then $(1 - e)(a - b) = (1 - e_1)(a_1 - b_1) \dots (4)$. Using (3) we have $s + f(b - a - 1) = t + f(b_1 - a_1 - 1) \dots (5)$.

Let $r = 1 + g(f(b - a - 1) - t) = 1 + g(f(b_1 - a_1 - 1) - s) > 0$. Then $e = 1 + r/(a - b)$ and $e_1 = 1 + r/(a_1 - b_1)$. So, (1) and (2) imply: $m + r(m - a)/(a - b) = m_1 + r(m_1 - a_1)/(a_1 - b_1)$ and $m + r(m - b)/(a - b) = m_1 + r(m_1 - b_1)/(a_1 - b_1)$. Using definitions of m and m_1, $m - r(1 + a)/(2 + a - b) = m_1 - r(1 + a_1)/(2 + a_1 - b_1)$ and $m + r(1 - b)/(2 + a - b) = m_1 + r(1 - b_1)/(2 + a_1 - b_1) \dots (6)$. Then $m - m_1 = r[(1 + a)/(2 + a - b) - (1 + a_1)/(2 + a_1 - b_1)]$. Hence $m - m_1 = r(a - a_1 + b - b_1 - ab_1 + ba_1)/(2 + a - b)(2 + a_1 - b_1)$. While, from definitions of m and m_1, $m - m_1 = 2(a - a_1 + b - b_1 - ab_1 + ba_1)/(2 + a - b)(2 + a_1 - b_1)$. Since $r < 2$, $(a - a_1 + b - b_1 - ab_1 + ba_1)/(2 + a - b)(2 + a_1 - b_1) = 0$. Therefore $m = m_1$.

From (6) we have $(1+a)/(2+a-b) = (1+a_1)/(2+a_1-b_1)$ and $(1-b)/(2+a-b) = (1-b_1)/(2+a_1-b_1)$. Since $p \in (A \cap J) - (B \cap J)$, then $a_1 < a$ or $b < b_1$. In the first case, $1+a_1 < 1+a$, so $2+a-b > 2+a_1-b_1$ and $f(b-a-1) < f(b_1-a_1-1)$, then (5) implies $t < s$. Analogously, in the second case, $t < s$.

(e) $A \cap J = [a, b]$ with $-1 < a < b < 1$. This case is similar to case (d). Then $t < s$.

This completes the proof of Property IV.

Define $G : C^\vee(X) \times \mathbb{R} \to C^\vee(X)$ by $G(B,t) = \bigcup\{G_J(B,t) : J \in \mathcal{S}\}$, where $G_J : C^\vee(X) \times \mathbb{R} \to \{E : E \text{ is a closed subset of } J\}$ is defined as follows:

$$G_J(B,t) = \begin{cases} \text{(a)} \quad B \cap J \quad \text{if } B \cap J = \emptyset, \{-1\}, \{1\}, \{-1,1\} \text{ or } J, \\ \text{(b)} \quad [-1, g(f(b)-t)] \quad \text{if } B \cap J = [-1,b] \text{ and } -1 < b < 1, \\ \text{(c)} \quad [g(f(a)+t), 1] \quad \text{if } B \cap J = [a,1] \text{ and } -1 < a < 1, \\ \text{(d)} \quad [(a-e'm)/(1-e'), (b-e'm)/(1-e')], \text{ where } m = \frac{a+b}{2+a-b} \\ \qquad \text{and } e' = 1 + \frac{b-a}{-1+g(t-f(b-a-1))} \text{ if } B \cap J = [a,b] \text{ and} \\ \qquad -1 < a < b < 1 \text{ and,} \\ \text{(e)} \quad [-1, (a-e'm)/(1-e')] \cup [(b-e'm)/(1-e'), 1], \text{ where} \\ \qquad m = \frac{a+b}{2+a-b} \text{ and } e' = 1 + \frac{b-a}{-1+g(-t-f(b-a-1))} \text{ if } B \cap J = \\ \qquad [-1,a] \cup [b,1],\ -1 \le a < b \le 1 \text{ and } -1 < a \text{ or } b < 1. \end{cases}$$

In case (e), let $a_1 = (a-e'm)/(1-e')$ and $b_1 = (b-e'm)/(1-e')$, then $a_1 < b_1$. Note that e' is an increasing continuous function of t. If $t \to \infty$, $e' \to (2+a-b)/2$, if $t \to -\infty$, $e' \to -\infty$. Then $e' < (2+a-b)/2$ for every $t \in \mathbb{R}$. Thus $e'(1+m) = e'2(1+a)/(2+a-b) \le 1+a$ and $e'(1-m) = e'2(1-b)/(2+a-b) \le 1-b$. This implies that $-1 \le (a-e'm)/(1-e') = a_1$ (equality holds if and only if $-1 = a$) and $b_1 = (b-e'm)/(1-e') \le 1$ (equality holds if and only if $b = 1$). If $t \to \infty$, $a_1 \to -1$ and $b_1 \to 1$. If $t \to -\infty$, $a_1 \to m$ and $b_1 \to m$. Since $a+b-2e'm = m(2+a-b-2e')$, $m = (a-e'm+b-e'm)/(2(1-e')+a-b) = (a_1+b_1)/(2+a_1-b_1)$. Therefore $m = \frac{a_1+b_1}{2+a_1-b_1}$. Define $e = 1+\frac{1+g(f(b_1-a_1-1)+t)}{a_1-b_1}$. Note that $b_1 - a_1 - 1 = (b-a-(1-e'))/(1-e') = -g(-t-f(b-a-1))$. This implies that $e = e'$. Thus $a_1 + e(m-a_1) = a$ and $b_1 + e(m-b_1) = b$.

Therefore, $G_J(B,t)$ is a continuous function of t, $G_J(B,t) \to J$ as $t \to -\infty$, $G_J(B,t) \to \{-1,1\}$ as $t \to \infty$, $G_J(B,0) = B \cap J$ and supposing that $G(B,t) \in C^\vee(X)$, we have that $F_J(G(B,t),t) = [-1,a] \cup [b,1] = B \cap J$ for every $t \in \mathbb{R}$.

The analysis of cases (a), (b), (c) and (d) is similar and we conclude that $G(B,t) \in C^\vee(X)$ for each $t \in \mathbb{R}$, $F_J(G(B,t),t) = B \cap J$ for every $t \in \mathbb{R}$, then $F(G(B,t),t) = B$ for every $t \in \mathbb{R}$, $G(B,t)$ depends continuously on t, $G(B,t)$ tends to one-point set or to a subgraph of X which is contained in B as $t \to \infty$ and $G(B,t)$ tends to a subgraph of X which contains B as $t \to -\infty$.

3. Proof of Theorem 1

Define $\mathcal{A} = \mu^{-1}(T) \subset C^{\vee}(X)$ and $\mathcal{B} = \mu^{-1}(T_{i-1}, T_i)$. For each $A \in \mathcal{A}$, let $r(A) = \inf\{t \in \mathbb{R} : F(A,t) \in \mathcal{B}\}$ and $R(A) = \sup\{t \in \mathbb{R} : F(A,t) \in \mathcal{B}\}$. Since $F_J(A,0) = A \cap J$ for every $J \in \mathcal{S}$, we have that $F(A,0) = A \in \mathcal{B}$ for each $A \in \mathcal{A}$. Then $r(A)$ and $R(A)$ are defined and $-\infty \leq r(A) < 0 < R(A) \leq \infty$. Let $\mathcal{C} = \{(A,t) \in \mathcal{A} \times \mathbb{R} : r(A) < t < R(A)\}$. We will prove that the function $F_0 = F \mid C$ is a homeomorphism from $\mathcal{C}$ onto $\mathcal{B}$.

Property I implies that $F_0(A,t) \in \mathcal{B}$ for ever $(A,t) \in \mathcal{C}$. In order to prove that F_0 is injective, suppose that $F_0(A,t) = F_0(B,s)$. If $A \neq B$, since $\mu(A) = \mu(B)$, then $A - B \neq \emptyset$ and $B - A \neq \emptyset$. Property IV implies that $t < s$ and $s < t$. This contradiction implies that $A = B$. Thus, by Property I, $(A,t) = (B,s)$. Therefore F_0 is injective. To prove that F_0 is onto, let $B \in \mathcal{B} \subset C^{\vee}(X)$. Since $G(B,t)$ tends to one-point set or to a subgraph of X which is contained in B as $t \to \infty$ and $G(B,t)$ tends to a subgraph of X which contains B as $t \to -\infty$. Then $\lim_{t\to\infty} \mu(G(B,t)) \leq T_{i-1}$ and $\lim_{t\to-\infty} \mu(G(B,t)) \geq T_i$. Thus there exists $t \in \mathbb{R}$ such that $A = G(B,t) \in \mathcal{A}$. The continuity of F implies that $r(A) < t < R(A)$. Then $F_0(A,t) = B$. Therefore F_0 is surjective.

Let $K : \mathcal{B} \to \mathcal{C}$ be the inverse function of F_0. We will show that K is continuous. It is enough to prove that if $(B_n)n$ is a sequence in $\mathcal{B}$ which is convergent to an element $B \in \mathcal{B}$ and the sequence $(K(B_n))n$ converges to an element $(A_0, t_0) \in \mathcal{A} \times [-\infty, \infty]$, then $(A_0, t_0) = K(B)$.

Let $(A,t) = K(B)$ and, for each n, let $(A_n, t_n) = K(B_n)$. Then $(A_n, t_n) \to (A_0, t_0)$. If $r(A_0) < t_0 < R(A_0)$, then $F_0(A,t) = B = \lim_{n\to\infty} B_n = \lim_{n\to\infty} F_0(A_n, t_n) = F_0(A_0, t_0)$, so $(A_0, t_0) = K(B)$. If $t_0 \leq r(A_0)$, take a number $t^* > r(A_0)$. Then there exists N such that $t_n < t^*$ for each $n \geq N$. Then $B_n \subset F(A_n, t_n) \subset F(A_n, t^*)$ for each $n \geq N$. Thus $B \subset F(A_0, t^*)$ for every $t^* > r(A_0)$. If $r(A_0) > -\infty$, then $B \subset F(A_0, r(A_0)) \subset F(A_0, 0) = A_0$. Thus $T_{i-1} < \mu(B) \leq \mu(F(A_0, r(A_0))) \leq \mu(A_0) < T_i$. Then there exists $r < r(A_0)$ such that $T_{i-1} < \mu(F(A_0, r)) < T_i$ which is a contradiction with the definition of $r(A_0)$. If $r(A_0) = -\infty$, then $B \subset \lim_{n\to\infty} F(A_0, -n)$ which is a subgraph of X or a one-point set contained in A_0. Thus $\mu(B) \leq T_{i-1}$ which is a contradiction. Similar contradictions are obtained supposing that $t_0 \geq R(A_0)$. This completes the proof that $(A_0, t_0) = K(B)$. Therefore K is continuous.

Hence F is a homeomorphism.

In order to define φ, let $\varrho_1 : \mathcal{A} \times \mathbb{R} \to \mathcal{A}$ and $\varrho_2 : \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ be the respective projection maps. Define $\psi : \mathcal{B} \to \mathcal{A} \times (T_{i-1}, T_i)$ by $\psi(B) = (\varrho_1(K(B)), \mu(B))$. Then ψ is continuous.

Let $(A,t) \in \mathcal{A} \times (T_{i-1}, T_i)$. Since $F(A,n)$ converges to a subgraph of X which contains A, then $\lim_{n\to\infty} \mu(F(A,n)) \geq T_i$. Thus there exists $n_1 > 1$ such that $\mu(F(A,n_1)) > t$. Similarly, there exists $n_2 > 1$ such that $\mu(F(A,-n_2)) < t$. Hence there exists a unique $s \in \mathbb{R}$ such that $\mu(F(A,s)) = t$. Define $\varphi(A,t) = F(A,s)$.

Property I implies that if $t_1 < t_2$, then $\varphi(A,t_1) \subset \varphi(A,t_2)$. Note that

$\psi(\varphi(A,t)) = \psi(F(A,s)) = (A,t)$. Since $\mu(F(\varrho_1(K(B)),\varrho_2(K(B)))) = \mu(B)$, then $\varphi(\psi(B)) = \varphi((\varrho_1(K(B)),\varrho_2(K(B)))) = F(K(B)) = B$. Then ψ is the inverse map of φ. Since $\mu(F(A,0)) = \mu(A) = T$, then $\varphi(A,T) = A$ for every $A \in \mathcal{A}$.

To prove that φ is continuous, it is enough to prove that if $((A_n,t_n))n$ is a sequence in $\mathcal{A}\times(T_{i-1},T_i)$ which converges to an element (A,t) in $\mathcal{A}\times(T_{i-1},T_i)$ and $\varphi(A_n,t_n)$ converges to an element $B \in C(X)$, then $B = \varphi(A,t)$. Set $\varphi(A_n,t_n) = F(A_n,s_n)$, where $\mu(F(A_n,s_n)) = t_n$ and set $\varphi(A,t) = F(A,s)$ where $\mu(F(A,s)) = t$. Then $t_n = \mu(\varphi(A_n,t_n)) \to \mu(B)$, so $\mu(B) = t \in (T_{i-1},T_i)$. Thus $B \in \mathcal{B}$. Set $K(B) = (A^*,r)$. Then $(A^*,r) = \lim_{n\to\infty} K(\varphi(A_n,t_n)) = \lim_{n\to\infty} K(F(A_n,s_n)) = \lim_{n\to\infty}(A_n,s_n)$. Thus $A_n \to A^*$ and $s_n \to r$. Hence $A^* = A$. Since $t_n = \mu(F(A_n,s_n)) \to \mu(F(A,r))$, then $t = \mu(F(A,r))$. Hence $B = \varphi(A,t)$.

This completes the proof that φ is a homeomorphism and the proof of Theorem 1.

□

Corollary ([10, Theorem 2.5]). *$C(X)$ is conical pointed. That is, for each Whitney map $\mu : C(X) \to \mathbb{R}$ there exists $T \in (0,1)$ such that $\mu^{-1}([T,1])$ is homeomorphic to the topological cone of $\mu^{-1}(T)$.*

4. Proof of Theorem 2

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be two Whitney levels for $C(X)$ and let $C \in C(X)$. We say that C is *placed between $\mathcal{A}$ and $\mathcal{B}$* if there exists $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \subset C \subset B \neq A$ or $B \subset C \subset A \neq B$.

Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Whitney levels. Suppose that no element in $SG(X)\cup F_1(X)$ is placed between $\mathcal{A}$ and $\mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic.*

Proof: Set $\mathcal{A} = \mu^{-1}(t)$ and $\mathcal{B} = \nu^{-1}(s)$ where $\mu,\nu : C(X) \to \mathbb{R}$ are Whitney maps and $t,s \in [0,1]$. Let $A \in \mathcal{A}-\mathcal{B}$, we will prove that there exists a unique $r \in \mathbb{R}$ such that $\nu(F(A,r)) = s$. If $\nu(A) < s$, taking an order arc from A to X (see [8, Theorem 1.8]), there exists $B_0 \in \mathcal{B}$ such that $A \subset B_0 \neq A$, then $A \notin SG(X)\cup F_1(X)$. Therefore $A \in C^\vee(X)$. Let $D = \lim_{n\to\infty} F(A,n)$. Then D is a subgraph of X which contains A. If $\nu(D) \leq s$, there exists $B \in \mathcal{B}$ such that $D \subset B$. Then $\nu(A) < \nu(B)$ and $A \subset D \subset B \neq A$ which contradicts our assumption. Thus $\nu(D) > s$. Then $\nu(F(A,0)) = \nu(A) < s = \lim_{n\to\infty}\nu(F(A,n))$. This proves the existence of r in this case. The case $\nu(A) > s$ is similar. In both cases r is unique by Property I.

Analogously, for each $B \in \mathcal{B}-\mathcal{A}$, $B \in C^\vee(X)$ and there exists a $z \in \mathbb{R}$ such that $\mu(G(B,z)) = t$.

Define $\gamma : \mathcal{A} \to \mathcal{B}$ by $\gamma(A) = A$ if $A \in \mathcal{A}\cap\mathcal{B}$ and $\gamma(A) = F(A,r) \in \mathcal{B}$ if $A \in \mathcal{A}-\mathcal{B}$.

Note that $A \subset \gamma(A)$ or $\gamma(A) \subset A$. To prove that γ is surjective, let $B \in \mathcal{B}$. If $B \in \mathcal{A}$, then $B = \gamma(B)$. If $B \in \mathcal{B}-\mathcal{A}$, let $z \in \mathbb{R}$ be such that $\mu(G(B,z)) = t$.

Then $F(G(B,z),z) = B$ and $G(B,z) \in \mathcal{A}$. Thus $\gamma(G(B,z)) = B$. Hence γ is surjective. To prove that γ is injective, let $A_1, A_2 \in \mathcal{A}$ with $A_1 \neq A_2$. If $A_1, A_2 \in \mathcal{B}$, then $\gamma(A_1) = A_1 \neq A_2 = \gamma(A_2)$. If $A_1 \in \mathcal{B}$ and $A_2 \notin \mathcal{B}$, then $A_2 \subset \gamma(A_2) \neq A_2$ or $\gamma(A_2) \subset A_2 \neq \gamma(A_2)$, so $\gamma(A_2) \notin \mathcal{A}$, and $\gamma(A_2) \neq A_1 = \gamma(A_1)$. If $A_1, A_2 \notin \mathcal{B}$, since $A_1 - A_2 \neq \emptyset$ and $A_2 - A_1 \neq \emptyset$, Property IV implies that $F(A_1, r_1) \neq F(A_2, r_2)$ for every $r_1, r_2 \in \mathbb{R}$. Hence $\gamma(A_1) \neq \gamma(A_2)$. Therefore γ is injective.

Finally, we will prove that γ is continuous. It is enough to prove that if $(A_n)n$ is a sequence in $\mathcal{A}$ which converges to an element $A \in \mathcal{A}$ and $\gamma(A_n) \to B \in \mathcal{B}$, then $\varphi(A) = B$. We may suppose that $A_n \in \mathcal{B}$ for each n or $A_n \notin \mathcal{B}$ for each n. The first case is immediate. In the second case, set $\gamma(A_n) = F(A_n, r_n)$. We consider two subcases:

(a) $A \in \mathcal{A} - \mathcal{B}$, set $\gamma(A) = F(A, r)$. We suppose, for example, that $r \leq r_n$ for each n. Then $F(A_n, r) \subset F(A_n, r_n) = \gamma(A_n)$, then $\gamma(A) = F(A, r) = \lim_{n\to\infty} F(A_n, r) \subset \lim_{n\to\infty} \gamma(A_n) = B$. Since $\gamma(A)$, $B \in \mathcal{B}$, we have that $\gamma(A) = B$.

(b) $A \in \mathcal{B}$. Since $A_n \subset \gamma(A_n)$ or $\gamma(A_n) \subset A_n$ for every n, then $A \subset B$ or $B \subset A$ and $A, B \in \mathcal{B}$. Thus $A = B$. This completes the proof that γ is continuous.

Therefore γ is a homeomorphism. □

Proof of Theorem 2: Let $\mathfrak{A} = \{\mathcal{A} \subset C(X) : \mathcal{A}$ is a Whitney level for $C(X)$, $\mathcal{A} \neq F_1(X)$ and $\mathcal{A} \neq \{X\}\}$. Let $\mathfrak{P} = \{E : E \subset SG(X)\}$. Then $\mathfrak{P}$ is finite.

Define $\sigma : \mathfrak{A} \to \mathfrak{P} \times \mathfrak{P} \times \mathfrak{P}$ by:

$\sigma(\mathcal{A}) = (\{E \in SG(X) :$ there exists $A \in \mathcal{A}$ such that $E \subset A \neq E\}$,

$SG(X) \cap \mathcal{A}$, $\{E \in SG(X) :$ there exists $A \in \mathcal{A}$ such that $A \subset E \neq A\})$.

In order to prove Theorem 2, it is enough to show that if $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$, then $\mathcal{A}$ is homeomorphic to $\mathcal{B}$.

Suppose then that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$. By the previous theorem, it is enough to prove that no element in $SG(X)$ is placed between $\mathcal{A}$ and $\mathcal{B}$. Suppose, for example, that there exists $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $E_0 \in SG(X)$ such that $A \subset E_0 \subset B \neq A$. If $A = E_0$, then $E_0 \in SG(X) \cap \mathcal{A} = SG(X) \cap \mathcal{B} \subset \mathcal{B}$, so $E_0, B \in \mathcal{B}$ and $E_0 \subset B \neq E_0$ which is a contradiction. If $A \neq E_0$, $F(\mathcal{A}) = F(\mathcal{B})$ implies that there exists $B_1 \in \mathcal{B}$ such that $B_1 \subset E_0 \neq B_1$. Thus $B_1 \subset B \neq B_1$ which is also a contradiction.

Therefore $\mathcal{A}$ is homeomorphic to $\mathcal{B}$. □

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(*Received* April 6, 1993, *revised* September 4, 1994)