A note on existence and uniqueness of solutions of neutral functional-differential equations with state-dependent delays

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Abstract. Existence and uniqueness theorem for state-dependent delay-differential equations of neutral type is given. This theorem generalizes previous results by Grimm and the author.

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Consider the scalar initial-value problem for state-dependent delay-differential equations of neutral type

$$
(1) \qquad \begin{aligned} y'(t) &= f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))), && t \in [a, b], \\ y(t) &= g(t), && t \in [\gamma, \alpha], \end{aligned}
$$

$\gamma \le a < b$, where $\gamma \le \alpha(t, y) \le t$, $\gamma \le \beta(t, y) \le t$, and g is a given initial function. We assume the following:

(i) g and g' are Lipschitz-continuous with constants L_g and $L_{g'}$ respectively;
(ii) $f(a, g(a), g(\alpha(a, g(a))),\ g'(\beta(a, g(a)))) = g'(a)$, where $g'(a)$ denotes the left hand side derivative.

Moreover, suppose that in their respective domains f, α and β satisfy the following conditions with nonnegative Lipschitz constants:

(iii) $|f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)|$
$\le L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2|z_1 - z_2|$, $L_2 < 1$;
(iv) $|\alpha(t_1, y_1) - \alpha(t_2, y_2)| \le A_1|t_1 - t_2| + A_2|y_1 - y_2|$;
(v) $|\beta(t_1, y_1) - \beta(t_2, y_2)| \le B_1|t_1 - t_2| + B_2|y_1 - y_2|$.

The problem (1) with $\gamma = a$ was studied by Grimm [1]. He proved an existence result for (1) assuming that f is bounded by some constant M, $L_2 < 1$, and $B_1 + B_2M \le 1$. He also proved a uniqueness result when β is independent of y. In the recent paper [2] the author relaxed this very restrictive assumption at the expense of the additional condition $L_2(1 + B_1 + B_2G) < 1$, where G is some constant depending on f and g. This condition means that the dependence of f on the last argument is not too strong. It is the purpose of this note to improve further the results given in [1] and [2]. We prove the existence and uniqueness theorem for (1) (with β depending both on t and y), where the inequality $L_2(1 + B_1 + B_2G) < 1$ is replaced by the weaker conditions $L_2 < 1$ and $B_1 + B_2G \leq 1$.

For any continuous functions y and z on $[\gamma, b]$, put

$$F(t, y, z) := f(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t)))),$$

and define

$$\begin{aligned}
M &:= \sup\{|F(t,0,0)| : t \in [a,b]\}; \qquad C_1 := (g'_{[\gamma,a]} + M)/(1 - L_2);\\
C_2 &:= 2L_1/(1 - L_2); \qquad Y := (g_{[\gamma,a]} + C_1/C_2)\exp((b-a)C_2);\\
Z &:= \max\{C_1 + C_2Y, (M + 2L_1Y)/(1 - L_2)\}; \qquad G := \max\{L_g, Z\};\\
D &:= \max\{L_{g'}, L_1(1 + G(1 + A_1 + A_2G))/(1 - L_2(B_1 + B_2G))\}.
\end{aligned}$$

Here $x_{[c,d]} := \sup\{|x(t)| : t \in [c,d]\}$ for any function x. We have the following:

Theorem. *Assume that* (i)–(v) *hold,* $L_2 < 1$, *and* $B_1 + B_2G \leq 1$. *Then* (1) *has a solution* y *whose derivative is Lipschitz-continuous. Moreover, this solution is unique in the space of continuously differentiable functions on* $[\gamma, a]$.

PROOF: For $h \in J := \{h \mid h = (b-a)/n,\ n \geq n_0\}$, where n_0 is a positive integer, put $t_i = a + ih$, $i = 0, 1, \dots, n$, and as in [2] define the modified Euler sequences $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ by

$$\begin{aligned}
&y_h(t_i + rh) = y_h(t_i) + rhz_h(t_i),\\
(2)\qquad &z_h(t_i + rh) = (1-r)z_h(t_i) + rz_h(t_{i+1}),\\
&z_h(t_{i+1}) = F(t_{i+1}, y_h, z_h),
\end{aligned}$$

$i = 0, 1, \dots, n-1$, $r \in (0,1]$, where $y_h(t) = g(t)$ and $z_h(t) = g'(t)$ for $t \in [\gamma, a]$. Note that (2) is, in general, implicit in z_h, but in view of $L_2 < 1$ it has a unique solution (y_h, z_h) for any $h \in J$. We will first show that $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are relatively compact in the space $C[\gamma, b]$ of continuous functions on $[\gamma, b]$. Proceeding as in [2] it follows that $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are uniformly bounded by Y and Z, respectively, and that $\{y_h\}_{h\in J}$ are uniformly Lipschitz-continuous with the constant G. The proof that $\{z_h\}_{h\in J}$ are also uniformly Lipschitz-continuous is more delicate than in [2]. The proof is by induction. Assume that

$$(3)\qquad |z_h(t_1) - z_h(t_2)| \leq D|t_1 - t_2|, \qquad t_1, t_2 \in [\gamma, t_i],$$

and we will show that this inequality is also true for $t_1, t_2 \in [\gamma, t_{i+1}]$ (obviously (3) holds for $t_1, t_2 \in [\gamma, t_0]$). Define on $[\gamma, t_{i+1}]$ the iterations $z_h^{[\nu]}(t) = z_h(t)$ for $t \in [\gamma, t_i]$, $\nu = 0, 1, \dots$, and

$$\begin{aligned}
&z_h^{[0]}(t_i + rh) = z_h(t_i),\\
&z_h^{[\nu+1]}(t_{i+1}) = F(t_{i+1}, y_h, z_h^{[\nu]}),\\
&z_h^{[\nu+1]}(t_i + rh) = (1-r)z_h(t_i) + rz_h^{[\nu+1]}(t_{i+1}),
\end{aligned}$$

$r \in (0,1]$, $\nu = 0,1,\dots$. It follows by the induction with respect to ν that $\{z_h^{[\nu]}\}_{\nu=0}^{\infty}$ are uniformly bounded by Z and uniformly Lipschitz-continuous on $[\gamma, t_{i+1}]$ with the same constant D. Indeed, this is true for $\nu = 0$ and, assuming that it is true for ν, routine manipulations yield

$$|z_h^{[\nu+1]}(t_{i+1})| \le M + 2L_1Y + L_2Z \le Z,$$

and

$$\begin{aligned}|z_h^{[\nu+1]}(t_{i+1}) - z_h^{[\nu+1]}(t_i)| \\ \le L_1(1 + G(1 + A_1 + A_2G))h + L_2D(B_1 + B_2G)h \le Dh.\end{aligned}$$

The last inequality follows from the definition of D. In view of the Ascoli-Arzela theorem the sequence $\{z_h^{[\nu]}\}_{\nu=0}^{\infty}$ is relatively compact in $C[\gamma, t_{i+1}]$ and since the solution (y_h, z_h) of (2) is unique, we have $z_h^{[\nu]} - z_{h[\gamma, t_{i+1}]} \to 0$ as $\nu \to \infty$. Therefore, $\{z_h\}_{h\in J}$ are uniformly Lipschitz-continuous on $[\gamma, t_{i+1}]$ with the same constant D. By induction with respect to i, this is also true on $[\gamma, b]$. Consequently, $\{y_h\}_{h\in J}$ and $\{z_h\}_{h\in J}$ are relatively compact in $C[\gamma, b]$ and from this point the proof is exactly the same as the proof of Theorem 2 in [2]. We prove the existence by showing that there is a subsequence of $\{y_h\}_{h\in J}$ convergent to the solution y of (1) and we prove the uniqueness by contradiction. □

References

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