A note on convex sublattices of lattices

Václav Slavík

Abstract. Let $CSub\,(\mathbf{K})$ denote the variety of lattices generated by convex sublattices of lattices in $\mathbf{K}$. For any proper variety $\mathbf{V}$, the variety $CSub\,(\mathbf{V})$ is proper. There are uncountably many varieties $\mathbf{V}$ with $CSub\,(\mathbf{V}) = \mathbf{V}$.

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Let A be a lattice. Denote by $Int\,(A)$ the lattice of all intervals of A and by $CSub\,(A)$ the lattice of all convex sublattices of A. The empty set is considered to be in both $Int\,(A)$ and $CSub\,(A)$. For a variety $\mathbf{K}$ of lattices, let $CSub\,(\mathbf{K})$ denote the variety of lattices generated by $\{CSub\,(A); A \in \mathbf{K}\}$.
The aim of the paper is to show that there exist uncountably many varieties of lattices $\mathbf{K}$ with $CSub\,(\mathbf{K}) = \mathbf{K}$ and that for any proper subvariety $\mathbf{K}$ of lattices $CSub\,(\mathbf{K})$ is also proper. Thus we give a partial answer to the problem I. 10 posed in G. Grätzer [1].

Lemma 1. *If p is a lattice term in k variables and $A_1, \dots, A_k$ convex sublattices of a lattice A, then*

$$p(A_1, \dots, A_k) = \bigcup \{p(I_1, \dots, I_k); I_j \subseteq A_j \ \text{ and } \ I_j \in Int\,(A)\}.$$

Proof: (By induction on the length of p) Evidently, $p(I_1, \dots, I_k) \subseteq p(A_1, \dots, A_k)$ for any intervals $I_j \subseteq A_j$. We must show that every element of $p(A_1, \dots, A_k)$ belongs to $p(I_1, \dots, I_k)$ for some intervals I_j of A_j. If p is a variable, it is clear. Let x be an element of $t_1(A_1, \dots, A_k) \vee t_2(A_1, \dots, A_k)$ for some terms t_1, t_2. We shall show that $x \in t_1(I_1, \dots, I_k) \vee t_2(I_1, \dots, I_k)$ for some intervals $I_j \subseteq A_j$. If either $t_1(A_1, \dots, A_k) = \emptyset$ or $t_2(A_1, \dots, A_k) = \emptyset$, then we get it by induction. In the opposite case there exist elements a_1, $b_1 \in t_1(A_1, \dots, A_k)$ and a_2, $b_2 \in t_2(A_1, \dots, A_k)$ such that $a_1 \wedge a_2 \leq x \leq b_1 \vee b_2$. By assumption there exist intervals J_j, K_j, L_j, M_j of A_j such that $a_1 \in t_1(J_1, \dots, J_k)$, $b_1 \in t_1(K_1, \dots, K_k)$, $a_2 \in t_2(L_1, \dots, L_k)$, $b_2 \in t_2(M_1, \dots, M_k)$. It is evident that a_1, a_2, b_1, b_2, $x \in t_1(I_1, \dots, I_k) \vee t_2(I_1, \dots, I_k)$, where $I_j = J_j \vee K_j \vee L_j \vee M_j$. The rest is easy. $\square$

For any lattice A, $Int\,(A)$ is a sublattice of $CSub\,(A)$. Combining this fact with Lemma 1 we obtain the following proposition.

Proposition 1. *Let A be a lattice and $\mathbf{V}$ a variety of lattices. Then $Int\,(A) \in \mathbf{V}$ if and only if $CSub\,(A) \in \mathbf{V}$.*

For a bounded lattice B (with the least element u and the greatest element v), denote by $\mathscr{L}(B)$ the lattice pictured in Fig. 1. Denote by $\mathscr{L}_0\,(B)$ the lattices obtained from $\mathscr{L}(B)$ by excluding its least element.

Let $\mathbf{V}$ be a variety of lattices. Denote by $\mathscr{L}(\mathbf{V})$ the class of all lattices L such that whenever $\mathscr{L}(B)$ is a sublattice of L, then $B \in \mathbf{V}$.

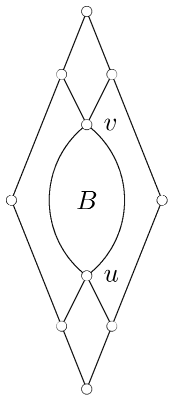

Figure 1: $\mathscr{L}(B)$

It is shown in [2] that $\mathscr{L}(\mathbf{V})$ is a variety of lattices. Moreover, $\mathscr{L}(\mathbf{V})$ is proper if $\mathbf{V}$ is and $\mathscr{L}(\mathbf{V}) \neq \mathscr{L}(\mathbf{W})$ for any pair of varieties $\mathbf{V} \neq \mathbf{W}$. If variety $\mathbf{V}$ is self-dual, then $\mathscr{L}(\mathbf{V})$ is self-dual, too.

Proposition 2. *Let A be a lattice and $\mathbf{V}$ a self-dual variety of lattices. Then $A \in \mathscr{L}(\mathbf{V})$ if and only if $Int\,(A) \in \mathscr{L}(\mathbf{V})$.*

Proof: Without loss of generality we may assume that A is a bounded lattice. The mapping h of A into $Int\,(A)$ defined by $h(a) = [0, a]$ (0 is the least element of A) is an embedding of A into $Int\,(A)$. So any variety containing $Int\,(A)$ must contain also the lattice A. Now suppose that $A \in \mathscr{L}(\mathbf{V})$. Let $\mathscr{L}(B)$ be a sublattice of $Int\,(A)$ for some bounded lattice B. For any element $b = [b_1, b_2] \in \mathscr{L}(B) \subseteq Int\,(A)$ different from the least element of $\mathscr{L}(B)$, denote $h(b) = (b_1, b_2)$. Clearly, the mapping h is an embedding of the partial lattice $\mathscr{L}_0\,(B)$ into $A^* \times A$, where A^* denotes the lattice dual to A. One can easily show that the sublattice of $A^* \times A$ generated by $h(\mathscr{L}_0\,(B))$ is isomorphic to $\mathscr{L}(B)$ (see [2]). Since $A^* \times A \in \mathscr{L}(\mathbf{V})$, we get $B \in \mathbf{V}$ and thus $Int\,(A) \in \mathscr{L}(\mathbf{V})$.

Since any proper variety $\mathbf{V}$ of lattices is a subvariety of a proper self-dual variety $\mathbf{W}$ and $\mathbf{W}$ is a subvariety of a proper variety $\mathscr{L}(\mathbf{W})$ which is, by Propositions 1 and 2, closed under the formation of lattice of all convex sublattices, we get the following result.

Theorem 1. *For any proper variety* $\mathbf{V}$ *of lattices, the variety* $CSub\,(\mathbf{V})$ *is proper.*

Since there exist uncountably many proper self-dual varieties of lattices (see [3], [4]) and $\mathscr{L}(\mathbf{V}) \neq \mathscr{L}(\mathbf{W})$ if $\mathbf{V} \neq \mathbf{W}$, we have, by Propositions 1 and 2, the following theorem.

Theorem 2. *There exist uncountably many self-dual varieties* $\mathbf{V}$ *of lattices such that* $CSub\,(\mathbf{V}) = V$.

References

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Agriculture University, Department of Mathematics, 165 21 Praha 6 – Suchdol, Czech Republic

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