Some new versions of an old game

Vladimir V. Tkachuk

Abstract. The old game is the point-open one discovered independently by F. Galvin [7] and R. Telgársky [17]. Recall that it is played on a topological space X as follows: at the n-th move the first player picks a point $x_n \in X$ and the second responds with choosing an open $U_n \ni x_n$. The game stops after ω moves and the first player wins if $\cup\{U_n : n \in \omega\} = X$. Otherwise the victory is ascribed to the second player.

In this paper we introduce and study the games θ and Ω. In θ the moves are made exactly as in the point-open game, but the first player wins iff $\cup\{U_n : n \in \omega\}$ is dense in X. In the game Ω the first player also takes a point $x_n \in X$ at his (or her) n-th move while the second picks an open $U_n \subset X$ with $x_n \in \overline{U}_n$. The conclusion is the same as in θ, i.e. the first player wins iff $\cup\{U_n : n \in \omega\}$ is dense in X.

It is clear that if the first player has a winning strategy on a space X for the game θ or Ω, then X is in some way similar to a separable space. We study here such spaces X calling them θ-separable and Ω-separable respectively. Examples are given of compact spaces on which neither θ nor Ω are determined. It is established that first countable θ-separable (or Ω-separable) spaces are separable. We also prove that
1) all dyadic spaces are θ-separable;
2) all Dugundji spaces as well as all products of separable spaces are Ω-separable;
3) Ω-separability implies the Souslin property while θ-separability does not.

Keywords: topological game, strategy, separability, θ-separability, Ω-separability, point-open game

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0. Introduction

The games we are going to study here are slight variations of the well known point-open game G which was discovered and studied independently by F. Galvin [7] and R. Telgársky [17]. Recall that the game G is played on a topological space X as follows: the n-th move of the first player (from here on denoted by **I**) consists in taking a point $x_n \in X$. The second player (called **II** in this paper) answers choosing an open $U_n \subset X$ with $x_n \in U_n$. The play is finished after ω moves and **I** is announced to be the winner if $\cup\{U_n : n \in \omega\} = X$. Otherwise **II** wins the game $\{(x_n, U_n) : n \in \omega\}$.

F. Galvin [7] proved that it is independent of ZFC whether G is determined on all subsets of the real line $\mathbf{R}$. Telgársky proved in [17] that if X is a σ-Čech-complete or pseudocompact space then G is determined on X. Later in [18] he gave a ZFC example of a space X on which G is undetermined. P. Daniels and G. Gruenhage [5] as well as S. Baldwin [4] studied the point-open game which does not end after ω moves.

The main purpose of this paper is to introduce two new games θ and Ω (it took the author a long time to try to invent good names for them, but all his attempts failed) and to study them as well as some of their derivatives. It is worth mentioning that the author first introduced them (under the names T and TT) in his book [19] (which is written in Russian and is hence generally unobtainable by Western readers) and formulated their simplest properties as exercises.

The games θ and Ω differ only a little from the point-open game G. The moves in θ are exactly the same as in G but the assessment of the play $\{(x_n, U_n) : n \in \omega\}$ is different: the player **I** wins if the set $U = \cup\{U_n : n \in \omega\}$ is dense in X. Otherwise wins the second player. In the game Ω the first player still has to pick a point $x_n \in X$ at his (or her) n-th move, while the second player has more freedom — he also chooses an open $U_n \subset X$ but only $x_n \in \overline{U_n}$ is required. And again **I** wins the play $\{(x_n, U_n) : n \in \omega\}$ iff $U = \cup\{U_n : n \in \omega\}$ is dense in X.

Once the definitions of θ and Ω are given, it is straightforward that for any separable space X the first player has a winning strategy on X in both θ and Ω. This is the reason why we call a space X θ-separable (or Ω-separable) if the first player has a winning strategy on X for the game θ (or Ω respectively). We also mimic the terminology of [17] in saying that a space X is θ-antiseparable (or Ω-antiseparable) if the second player has a winning strategy on X in θ (or Ω respectively). Now what we do in this paper can be reformulated in a very short way: we study $\theta(\Omega)$-(anti)separable spaces.

The results in the foregoing text are numerous, so let us mention only that
— any θ-separable (and hence Ω-separable) space is weakly Lindelöf;
— a first countable θ-separable (or Ω-separable) space is separable;
— any product of separable spaces is Ω-separable (and hence θ-separable);
— the games θ and Ω are both determined on metric spaces;
— there are compact first countable examples of indeterminacy for θ and Ω;
— any Eberlein compact Ω-separable space is metrizable;
— any Ω-separable space has the Souslin property.

1. Notations and terminology

Throughout this paper "a space" means "a Tychonoff space". If X is a space then $\mathcal{T}(X)$ is its topology and $\mathcal{T}^*(X) = \mathcal{T}(X)\backslash\{\emptyset\}$. If $A \subset X$ then $\mathcal{T}(A, X) = \{U \in \mathcal{T}(X) : A \subset U\}$ and $\mathcal{T}(x, X) = \mathcal{T}(\{x\}, X)$ for any $x \in X$.

The symbol **I** (**II**) stands for the first (second) player in a topological game. The phrase "the player **I** (**II**) picks a point $x_n \in X$" (an open $U_n \subset X$) is encoded by $\mathbf{I} \to x_n \in X$ (or $\mathbf{II} \to U_n \in \mathcal{T}(X)$ respectively). The end of a proof of a statement will be denoted by $\square$. If a substatement is proved inside a proof of some statement (which is not proved yet) we will use the symbol $\triangle$. For a space X and $A \subset X$ we denote by $\overline{A}$ the closure of A in X. If it might not be clear in which space the closure is taken, then we write $\mathrm{cl}_X(A)$ for the closure of A in X.

If we have a function f, then its domain is denoted by dom (f) and ran $(f) = f(\text{dom } (f))$.

A map $f : X \to Y$ is called d-open, if for every $U \in \mathcal{T}(X)$ there is a $V \in \mathcal{T}(Y)$ such that $f(U) \subset V \subset \overline{f(U)}$. The symbol $\mathfrak{c}$ stands for the power (as well as for the cardinal number) equal to continuum. If $f : X \to Y$ is a map, then $f^{\#}(A) = Y \backslash f(X \backslash A)$ is the small image of A for every $A \subset X$. A cardinal number τ is identified with the smallest ordinal number having power τ. If $X = \prod\{X_\alpha : \alpha \in \tau\}$ and $Y \subset X$, then for $T \subset \tau$ the map $\pi_T : X \to X_T = \prod\{X_\alpha : \alpha \in T\}$ is the natural projection and $Y_T = \pi_T(Y) \subset X_T$. If $T = \{\alpha\}$, then we write π_α instead of π_T. Also, if $S \subset T \subset \tau$, then $\pi^T_S : Y_T \to Y_S$ is the natural projection. A Luzin space (or a Luzin set) is an uncountable space with all its nowhere dense subsets countable.

All other notions are standard and can be found in [6].

2. The games θ and Ω. Basic properties and relevant classes of spaces

To make this paper readable for a non-specialist in topological games we will start with definitions.

2.1 Definition. *Given a space X we say that the game θ (or Ω) is played on X if and only if*

(0) *there are two players called* **I** *and* **II** *who make moves enumerated by natural numbers;*
(1) *for every $n \in \omega$ the n-th move is made first by* **I** *and then by* **II**;
(2) *the n-th move for* **I** *consists in choosing an $x_n \in X$ while* **II** *responds with a $U_n \in \mathcal{T}(X)$ such that $x_n \in U_n$ (or $x_n \in \overline{U}_n$ respectively);*
(3) *after all moves have been made, the player* **I** *is announced to be the winner if $\cup\{U_n : n \in \omega\}$ is dense in X;*
(4) *if $\cup\{U_n : n \in \omega\}$ is not dense in X, then* **II** *wins the play $\{(x_n, U_n) : n \in \omega\}$.*

2.2 Definition. *We say that s is a strategy for the player* **I** *in θ (or in Ω respectively) on a space X if*

(1) *s is a function with ran$(s) \subset X$ and $\emptyset \in$ dom(s);*
(2) *$\xi \in$ dom$(s)\backslash\{\emptyset\}$ if and only if there is an $n \in \omega$ such that $\xi = (U_0, \dots, U_n)$, where $x_0 = s(\emptyset) \in U_0 \in \mathcal{T}(x_0, X)$ (or $U_0 \in \mathcal{T}(X)$ and $x_0 \in \overline{U}_0$ respectively), $x_1 = s(U_0) \in U_1 \in \mathcal{T}(x_1, X)$ (or $U_1 \in \mathcal{T}(X)$ and $x_1 \in \overline{U}_1$ respectively), ... , $x_n = s(U_0, \dots, U_{n-1}) \in U_n \in \mathcal{T}(x_n, X)$ (or $U_n \in \mathcal{T}(X)$ and $x_n \in \overline{U}_n$ respectively). Such $(U_0, \dots, U_n)$ as in (2) are called admissible for s or s-admissible. It is clear from the definition, that if $(U_0, \dots, U_n)$ is s-admissible, then for every $k \leqslant n$ the $(k+1)$-tuple $(U_0, \dots, U_k)$ is also admissible for s.*

2.3 Definition. *A function t is called a strategy for the player* **II** *in θ (or in Ω) on X if $t : \cup\{X^n : n \in \omega\} \to \mathcal{T}^*(X)$ and $t(x_0, \dots, x_n) \ni x_n$ (or $\overline{t(x_0, \dots, x_n)} \ni x_n$ respectively) for all $n \in \omega$.*

2.4 Definition. *If s is a strategy for the first (or for the second) player on a space X, then we say that it is used by* **I** *(or by* **II** *respectively) in a play*

$P = \{(x_n, U_n) : n \in \omega\}$ if $x_0 = s(\emptyset)$, $x_{n+1} = s(U_0, \dots, U_n)$ (or $U_n = s(x_0, \dots, x_n)$ respectively) for all $n \geqslant 0$. The strategy s is called winning or WS for the player **I** *(or* **II***) on X if* **I** *(or* **II** *respectively) wins every play on X in which he (or she) uses the strategy s. A game is determined on a space X if one of the players has a winning strategy on X (in this game).*

2.5 Definition. *A space X is called θ-separable (or Ω-separable) if the first player has a WS on X in θ (or Ω respectively). A space X is θ-antiseparable (Ω-antiseparable) if the second player has a winning strategy on X in θ (or Ω respectively).*

Now that the reader has been bored enough with definitions, we set to prove the simplest facts about θ- and Ω-(anti)separability.

2.6 Proposition. (i) *If a space X is Ω-separable, then it is θ-separable;*

(ii) *if a space X is θ-antiseparable, then it is Ω-antiseparable;*

(iii) *if X is a space and Y is θ-separable (Ω-separable) and dense in X then X is θ-separable (Ω-separable);*

(iv) *if X is θ-antiseparable (Ω-antiseparable) and Y is dense in X then Y is θ-antiseparable (Ω-antiseparable);*

(v) *if X is θ-separable and $U \in \mathcal{T}^*(X)$ then $\overline{U}$ is θ-separable;*

(vi) *if X is Ω-separable and $U \in \mathcal{T}(X)$ then $X \setminus \overline{U}$ and $\overline{U}$ are Ω-separable;*

(vii) *a continuous image of a θ-separable space is θ-separable;*

(viii) *a d-open continuous image of an Ω-separable space is Ω-separable;*

(ix) *if X can be mapped continuously onto a θ-antiseparable space, then X is itself θ-antiseparable;*

(x) *if X can be d-openly and continuously mapped onto an Ω-antiseparable space, then X is Ω-antiseparable;*

(xi) *if a space X is θ-separable (or Ω-separable), then it is weakly Lindelöf and in particular, every discrete $\gamma \in \mathcal{T}^*(X)$ is countable;*

(xii) *if $X = \bigoplus\{X_n : n \in \omega\}$ and each X_n is θ-separable (respectively Ω-separable), then X itself is θ-separable (respectively Ω-separable).*

Proof: As (i) and (ii) are clear, let us start with (iii). Take a winning strategy s on Y. Let $s_1(\emptyset) = s(\emptyset) = x_0$. If moves $(x_0, U_0), \dots, (x_n, U_n)$ are made, then let

$$s_1(U_0, \dots, U_n) = s(U_0 \cap Y, \dots, U_n \cap Y).$$

Then s_1 is a WS on X and (iii) is done. $\varDelta$

To prove (iv) take any strategy s for the second player on X (in θ or Ω). If $\mathbf{I} \to y_0 \in Y$ (recall that this means that **I** picked a point $y_0 \in Y$) then let $U_0 = s(y_0)$ and $s_1(y_0) = V_0 = U_0 \cap Y$. After n moves we will have the points $y_0, \dots, y_{n-1} \in Y$ and the sets $V_0, \dots, V_{n-1}$; $U_0, \dots, U_{n-1}$ with $V_i = U_i \cap Y$, $i = 0, \dots, (n-1)$. If $\mathbf{I} \to y_n \in Y$, then let

$$U_n = s(y_0, \dots, y_n) \text{ and } s_1(y_0, \dots, y_n) = U_n \cap Y.$$

The strategy s_1 is a winning one for **II** on Y so we proved (iv). $\varDelta$

To prove (v) and (vi) we need the following

2.7 Lemma. *Let X be a θ-separable (Ω-separable) space. Then* **I** *has a winning strategy δ on X for θ (for Ω respectively) such that for every $n \in \omega$ and for any $(U_0, \dots, U_n) \in dom(\delta)$ we have*

$(*)$ $\quad \delta(U_0, \dots, U_n) \notin U_0 \cup \dots \cup U_n$ *if* $\overline{U_0 \cup \dots \cup U_n} \neq X$,

(*or we have*

$(**)$ $\quad \delta(U_0, \dots, U_n) \notin \overline{U_0 \cup \dots \cup U_n}$ *if* $\overline{U_0 \cup \dots \cup U_n} \neq X$,

respectively).

Proof of the lemma: Let s be a WS for the player **I** on X in θ (or Ω). We are going to construct a winning strategy δ, satisfying $(*)$ (or $(**)$ respectively). Without loss of generality we may define δ only for those $(n+1)$-tuples $\xi = (U_0, \dots, U_n)$ whose union is not dense in X, for if a "bad" ξ occurs for some n, then **II** loses the play at the n-th move and δ may be defined arbitrarily for all subsequent moves.

Let $\delta(\emptyset) = s(\emptyset) = x_0$. If the answer of **II** is U_0, then let

$$V_0 = U_0, \quad \text{and} \quad k_0 = \max\{p : (\underbrace{V_0, \dots, V_0}_{(p+1) \text{ times}}) \in \operatorname{dom}(s)\}.$$

The defining set for maximum is non-empty, because $(V_0) \in \operatorname{dom}(s)$, and the maximum exists for otherwise we would have a play $\{(x_n, V_0) : n \in \omega\}$ in which **I** uses s and loses.

Now if we put $x_1 = \delta(U_0) = s(\underbrace{V_0, \dots, V_0}_{(k_0+1) \text{ times}})$, then $x_1 \notin U_0$ (or respectively $x_1 \notin \overline{U_0}$), because otherwise $(\underbrace{V_0, \dots, V_0}_{(k_0+2) \text{ times}}) \in \operatorname{dom}(s)$ contradicting the choice of k_0.

Now suppose that we defined δ for an n-tuple $(U_0, \dots, U_{n-1})$ in such a way that we have the sets $V_0, \dots, V_{n-1}$ and integers $k_0, \dots, k_{n-1}$ with the following properties:

(1) $V_0 = U_0,\ V_{k+1} = V_k \cup U_{k+1}$ for $k = 0, \dots, (n-2)$;

(2) k_m is maximal among the integers q for which

$$\xi_{m,q} = (\underbrace{V_0, \dots, V_0}_{(k_0+1) \text{ times}}, \dots, \underbrace{V_{m-1}, \dots, V_{m-1}}_{(k_{m-1}+1) \text{ times}}, \underbrace{V_m, \dots, V_m}_{(q+1) \text{ times}}) \in \operatorname{dom}(s).$$

(3) $x_m = \delta(U_0, \dots, U_m) = s(\xi_{m,k_m})$ for all $m = 1, \dots, (n-1)$.

Suppose that **II** $\to U_n$. Define the set V_n to be $V_{n-1} \cup U_n$ and let

$$k_n = \max\{q \in \omega : \xi_{n,q} = (\underbrace{V_0, \dots, V_0}_{(k_0+1) \text{ times}}, \dots, \underbrace{V_{n-1}, \dots, V_{n-1}}_{(k_{n-1}+1) \text{ times}}, \underbrace{V_n, \dots, V_n}_{(q+1) \text{ times}}) \in \operatorname{dom}(s)\}.$$

It is easy to see that k_n is correctly defined and we can put $\delta(U_0, \dots, U_n) = s(\xi_{n,k_n})$. It immediately follows from the definition of δ, that it has $(*)$ (or

$(**)$ respectively). Now the strategy δ is a winning one because for every play $P = \{(x_n, U_n) : n \in \omega\}$ in which **I** uses δ, there is a play $Q = \{(x_n, W_n) : n \in \omega\}$ such that **I** uses s in Q and $\cup\{U_n : n \in \omega\} = \cup\{W_n : n \in \omega\}$. The strategy s being winning we have $U = \cup\{U_n : n \in \omega\}$ is dense in X because $U = \cup\{W_n : n \in \omega\}$. Thus δ is a WS. $\triangle$

Returning to the proof of (v) (or (vi) respectively) let us take any $U \in \mathcal{T}(X)$. Suppose that s is a winning strategy on X in θ (or Ω respectively) having $(*)$ (or $(**)$ respectively). To construct a WS δ on $\overline{U}$ (or on $X\backslash\overline{U}$ respectively) let $x_0 = s(\emptyset)$. There are two possibilities: $x_0 \in \overline{U}$ (or $x_0 \in X\backslash\overline{U}$ respectively) or $x_0 \notin \overline{U}$ (or $x_0 \in \overline{U}$ respectively).

1) If $x_0 \in \overline{U}$ (or $x_0 \in X\backslash\overline{U}$ respectively), then let $\delta(\emptyset) = x_0$ and if moves $x_0, U_0, \ldots, x_n, U_n$ are made, then the $(n+1)$-tuple $\xi = (V_0, \ldots, V_n)$ is in the domain of s, where $V_i = U_i \cup (X\backslash\overline{U})$ (or $V_i = U_i \cup U$ respectively). It is clear, that $x_{n+1} = s(\xi) \in \overline{U}$ (or $x_{n+1} = s(\xi) \in X\backslash\overline{U}$ respectively) and if 1) takes place, we have our strategy δ constructed.

2) If $x_0 \notin \overline{U}$ (or $x_0 \notin X\backslash\overline{U}$ respectively), then let $V_0 = X\backslash\overline{U}$ (or $V_0 = U$ respectively). The strategy s has $(*)$ (or $(**)$ respectively), so $y_0 = s(V_0)$ has to belong to $\overline{U}$ (or $X\backslash\overline{U}$ respectively). Let $\delta(\emptyset) = y_0$ and repeat the construction of δ we carried out in 1). This completes the construction of the strategy δ.

To see that δ is a WS, note that $\cup\{U_n : n \in \omega\}$ is dense in $\overline{U}$ (or $X\backslash\overline{U}$ respectively) if and only if $\cup\{V_n : n \in \omega\}$ is dense in X which is true, because s is a WS. Therefore we proved (v) and the first part of (vi).

Now to establish that $\overline{U}$ is Ω-separable in case when so is X, observe that $X\backslash\overline{X\backslash\overline{U}}$ is dense in $\overline{U}$ and it suffices to apply (iii) and the proved part of (vi). $\triangle$

Now let $f : X \to Y$ be a continuous (d-open) onto map. If s is a WS for **I** on X in θ (or Ω respectively) then let $x_0 = s(\emptyset)$ and $\tilde{s}(\emptyset) = f(x_0) = y_0$.

For every $n \in \omega$ if $\xi = (V_0, \ldots, V_n) \in \operatorname{dom}(\tilde{s})$, then let

$$\tilde{s}(\xi) = f(s(f^{-1}(V_0), \ldots, f^{-1}(V_n))).$$

It is clear that

$$(f^{-1}(V_0), \ldots, f^{-1}(V_n)) \in \operatorname{dom}(s)$$

(by d-openness of f) so the strategy $\tilde{s}$ is well defined. Evidently, $\tilde{s}$ is a WS on Y for θ (resp. Ω) so we finished with (vii) and (viii). $\triangle$

To prove (ix) and (x) take a WS s for **II** on Y in θ (or Ω respectively). If $\mathbf{I} \to x_0 \in X$ then let $y_0 = f(x_0)$ and $\tilde{s}(x_0) = f^{-1}(s(y_0))$. If we defined $\tilde{s}$ for all $(n-1)$-tuples and $\xi = (x_0, \ldots, x_{n-1}, x_n)$ then let

$$U_n = \tilde{s}(\xi) = f^{-1}(s(f(x_0), \ldots, f(x_{n-1}), f(x_n))).$$

The strategy $\tilde{s}$ is well defined because $x_n \in U_n$ (or $x_n \in \overline{U}_n$ by d-openness of f) by inductive hypothesis. It is straightforward that $\tilde{s}$ is a winning strategy, so (ix) and (x) are proved. $\triangle$

Assume that X is θ-separable. If X is not weakly Lindelöf, then there is an open cover γ of X such that for every countable $\gamma_1 \subset \gamma$ the set $\cup\gamma_1$ is not dense in X. Now **II** has the following winning strategy: if **I** $\to x_n$, then **II** $\to U_n$, where U_n is any element of γ, containing x_n. This gives a contradiction, so that (xi) is proved. ⊿

Finally, let X_n have a winning strategy s_n for **I** in θ (or Ω respectively). There exists a bijection $b : \omega\backslash\{0\} \to \omega \times \omega$ such that

1) $n > m + k$ as soon as $b(n) = (m, k)$;

2) if $b(n) = (m, k)$ and $l < k$, then $b^{-1}((m, l)) < n$.

We are going to construct a winning strategy s on X for the first player and the relevant game.

Let $s(\emptyset) = x_0 = s_0(\emptyset)$. Observe that without loss of generality we may define s only on n-tuples $(U_0, \dots, U_n)$ such that $U_i \subset X_{p(i)}$ for all $i = 0, \dots, n$. Take any $(U_0, \dots, U_n) \in \mathrm{dom}(s)$ and let $b(n) = (m, k)$. If $k = 0$, then let $s(U_0, \dots, U_n) = s_m(\emptyset)$. If $k > 0$, then by the choice of b we have $(U_{i_0}, \dots, U_{i_{k-1}}) \in \mathrm{dom}(s_m)$ for some $i_0, \dots, i_{k-1} \subset \{0, \dots, n\}$. Let $s(U_0, \dots, U_n) = s_m(U_{i_0}, \dots, U_{i_{k-1}})$. The strategy s being constructed let us check that it is a WS. Indeed, if in a play $P = \{(x_n, U_n) : n \in \omega\}$ the first player used s, then for every $m \in \omega$ there is a subsequence $P_m = \{(x_{i(j,m)}, U_{i(j,m)}) : j \in \omega\}$ of P which is a play on X_m with **I** using s_m. Hence $U = \cup\{U_n : n \in \omega\}$ intersects every X_m in a dense set, so U is dense in X and we proved (xii). □

2.8 Remarks to some clauses of 2.6. (i) We could have put 2.6 (i) in a formally stronger way saying that any WS for **I** in Ω is also a winning strategy for **I** in θ. Another observation to this clause is that a θ-separable space X need not be Ω-separable. The simplest example of such X is the one point compactification of a discrete space of power ω_1. The space X is θ-separable because **I** can choose the unique non-isolated point x_0 of X as his (or her) first move. If **II** answers with a U_0, then the set $X\backslash U_0$ is finite so **I** wins after a finite number of moves.

The space X is not Ω-separable (and is in fact Ω-antiseparable) for a winning strategy for **II** on X in Ω could be described as follows: after any move x_n of the first player **II** picks a countable $U_n \in \mathcal{T}(X)$ with $x_n \in \overline{U_n}$. After ω moves there will be an isolated point outside $\cup\{U_n : n \in \omega\}$ so **II** wins using this strategy;

(ii) a little bit stronger (but still trivial) version of 2.6 (ii) could be stated as follows: any winning strategy for the second player in θ is also a WS for **II** in Ω. The same X as in (i) is an example of an Ω-antiseparable space which is not θ-antiseparable;

(iii) if a space X is θ-separable and Y is dense in X then Y need not be θ-separable — the example is still the same X from (i). Indeed, X is θ-separable, but has a dense uncountable discrete subspace which is not θ-separable by 2.6 (xi);

(iv) the space X from (i) has a dense θ-antiseparable subspace but is not θ-antiseparable;

(v)–(vi) the Tychonoff cube $\mathbf{I}^{\mathfrak{c}}$ is separable and hence Ω-separable but it contains a closed subset Y homeomorphic to $\beta\omega\backslash\omega$, which is not θ-separable, because **II** has the following WS on Y: at every move he (or she) chooses a clopen set so as to guarantee that the (finite) union of already chosen sets is not equal to Y. It is clear that it is possible to stick to it and this strategy will be winning, because any non-empty G_δ in Y has a non-empty interior. This proves that neither θ- nor Ω-separability are hereditary with respect to arbitrary closed sets. A space Y can be θ-separable with θ-antiseparable subset $Y\backslash\overline{U}$ for some $U \in \mathcal{T}(Y)$. Indeed, if X is the space from (i) then let Y be a space obtained by identifying non-isolated points in $X \bigoplus X$. If U is any infinite set of isolated points of Y lying in one of the copies of X in $X \bigoplus X$, then $Y\backslash\overline{U}$ is not weakly Lindelöf and hence not θ-separable by 2.6 (xi);

(xi) we have in fact proved a stronger version of 2.6 (xi), namely: if a space is not weakly Lindelöf then it is θ-antiseparable.

2.9 Corollary.

(i) *If a space X is a countable union of its θ-separable subspaces, then X is θ-separable.*

(ii) *if X is a space and $X = \cup\{X_i : i \in \omega\}$, where X_i is Ω-separable and $X_i \subset \mathrm{Int}_X(\overline{X_i})$ for all $i \in \omega$ (in particular, if X_i is open in X for all $i \in \omega$) then X is Ω-separable.*

Proof: To prove (i), use 2.6 (vii) and 2.6 (xii). If all X_i's are as in (ii) it is easy to see that the natural map $u : \bigoplus\{X_i : i \leq\} \to X$ is d-open so all there is to do is to use 2.6 (xii) and 2.6 (viii). □

2.10 Definition. *Given a space X and $x \in X$ we say that $\Delta\pi\chi(x, X) \leqslant \omega$ if there exists a countable π-base $\mathcal{B}$ at x in X such that $x \in \overline{U_n}$ for all $n \in \omega$. Such a π-base is called $\Delta\pi$-base at x in X.*

2.11 Theorem. (i) *A first countable θ-separable space is separable;*

(ii) *if $\Delta\pi\chi(x, X) \leqslant \omega$ for every $x \in X$ and X is Ω-separable, then it is separable.*

Proof: We are going to prove (i) and (ii) simultaneously. For every $x \in X$ let $\mathcal{B}_x = \{U_n^x : n \in \omega\}$ be a ($\Delta\pi$-)base at x in X. Let s be a winning strategy for the first player in θ (or Ω respectively) on X. Let $y = s(\emptyset)$ and $y(n_0) = s(U_{n_0}^y)$ for all $n_0 \in \omega$. Suppose that for all $k < m$ and for any $(k+1)$-tuple $(n_0, \ldots, n_k) \in \omega^{k+1}$ we have a point $y(n_0, \ldots, n_k) \in X$. Fix an $(m+1)$-tuple $(n_0, \ldots, n_m) \in \omega^{m+1}$ and let

$$y(n_0, \ldots, n_m) = s(U_{n_0}^y, U_{n_1}^{y(n_0)}, \ldots, U_{n_m}^{y(n_0,\ldots,n_{m-1})}).$$

Thus we have a countable set $Y = \{y\} \cup \{y(n_0, \ldots, n_m) : (n_0, \ldots, n_m) \in \omega^{m+1},\ m \in \omega\}$. We claim that Y is dense in X.

Indeed, if there is a $U \in \mathcal{T}^*(X)$ with $\overline{U} \cap Y = \emptyset$, then $y \notin \overline{U}$ so there is an $n_0 \in \omega$ with $U_{n_0}^y \cap \overline{U} = \emptyset$. If we have $n_0, \ldots, n_k \in \omega$ such that $U_{n_{i+1}}^{y(n_0,\ldots,n_i)} \cap \overline{U} = \emptyset$ for all

$i \leqslant (k-1)$, then $y(n_0, \dots, n_k) \notin \overline{U}$ so there is an $n_{k+1} \in \omega$ with $U_{n_{k+1}}^{y(n_0,\dots,n_k)} \cap \overline{U} = \emptyset$.

Having got the sequence $(n_0, n_1, \dots)$ let $x_0 = y$, $x_{k+1} = y(n_0, \dots, n_k)$ and $U_k = U_{n_{k+1}}^{x_k}$ for $k \in \omega$. Then the play $\{(x_k, U_k) : k \in \omega\}$ is played by **I** with the use of s. However $W = \cup\{U_n : n \in \omega\}$ is not dense in X because $W \cap U = \emptyset$, which is a contradiction. □

2.12 Corollary. *If a space X is Ω-separable and every $x \in X$ is a limit of a sequence of non-empty open subsets of X then X is separable.*

Proof: Recall that a sequence $S = \{U_n : n \in \omega\}$ converges to a point $x \in X$ if every $U \in \mathcal{T}(x, X)$ contains all but finitely many elements of S. It is clear that if S converges to x, then $\mathcal{B} = \{\cup\{U_k : k > n\} : n \in \omega\}$ is a $\Delta\pi$-base at x so we may apply 2.11 (ii). □

2.13 Corollary. *Within the class of first countable spaces, θ-separability and Ω-separability coincide with separability.*

2.14 Corollary. *A metric space is θ-separable iff it is Ω-separable iff it is separable.*

2.15 Corollary. *Both games θ and Ω are determined on the class of all metric spaces.*

Proof: We need to prove only that on a non-separable metric space, **II** has a winning strategy in θ (which of course will be a WS in Ω). If M is metrizable and non-separable then it is not weakly Lindelöf. Now use 2.8 (xi). □

2.16 Corollary. *If X is an Ω-separable Eberlein compact space, then it is metrizable.*

Proof: Every $x \in X$ is a limit of a sequence of non-empty open subsets of X [14]. Therefore X is separable by 2.12 and metrizable because any separable Eberlein compact space is metrizable [9]. □

2.17 Proposition. *Let $f : X \to Y$ be a closed surjective irreducible map. Then*

(i) *If Y is Ω-separable, then so is X;*

(ii) *if X is Ω-antiseparable, then so is Y.*

Proof: (i) Let s be a winning strategy for the first player on Y (in the game Ω). Let $y_0 = s(\emptyset)$. Pick any $x_0 \in f^{-1}(y_0)$ and put $t(\emptyset) = x_0$. Suppose that for all $k \leqslant n$ we have defined the strategy t for all t-admissible k-tuples $\xi = (U_0, \dots, U_{k-1})$ in such a way that $\pi = (f^{\#}(U_0), \dots, f^{\#}(U_{k-1}))$ is s-admissible and $s(\pi) = f(t(\xi))$.

Let $t(\xi) = x_n$ and $t(\pi) = y_n$. We know that $y_n = f(x_n)$. Suppose that $\mathbf{II} \to U_n$. Then $y_n \in \overline{f^{\#}(U_n)}$ because f is irreducible, so that

$$\tilde{\pi} = (f^{\#}(U_0), \dots, f^{\#}(U_n)) \in \operatorname{dom}(s).$$

For $y_{n+1} = s(\tilde{\pi})$ pick any $x_{n+1} \in f^{-1}(y_{n+1})$ and put

$$t(U_0, \dots, U_n) = x_{n+1}.$$

The strategy t being defined let us prove that it is a winning strategy. If in a play $P = \{(x_n, U_n) : n \in \omega\}$ the strategy t has been used, then in the play $Q = \{(f(x_n), f^{\#}(U_n)) : n \in \omega\}$ the strategy s was applied so that the set $\cup\{f^{\#}(U_n) : n \in \omega\}$ is dense in Y. Use irreducibility of f once more to assure that $\cup\{U_n : n \in \omega\}$ is dense in X so we are done.

(ii) Let t be a strategy for the second player on X in Ω. Suppose that for all $l < n$ we defined a strategy s on Y for all $(l+1)$-tuples $\xi = (y_0, \dots, y_l) \in Y^{l+1}$ in such a way that for every such ξ there are $x_0, \dots, x_l$ with $f(x_i) = y_i$, $i = 0, \dots, l$. If we have an $(n+1)$-tuple $\pi = (y_0, \dots, y_{n-1}, y_n)$ and corresponding points $x_0, \dots, x_{n-1}$, then pick any $x_n \in f^{-1}(y_n)$ and let

$$s(y_0, \dots, y_n) = f^{\#}(t(x_0, \dots, x_n)).$$

The strategy s being defined let us prove that it is a winning strategy. If in a play $P = \{(y_n, V_n) : n \in \omega\}$ the strategy s has been used, then there is a play $Q = \{(x_n, U_n) : n \in \omega\}$ in which the strategy t was applied and such that $V_n = f^{\#}(U_n)$ for each $n \in \omega$. The set $\cup\{U_n : n \in \omega\}$ is dense in X because t is a WS. Therefore $\cup\{V_n : n \in \omega\}$ is dense in Y so we are done. □

2.18 Theorem. *If a space X is Ω-separable, then $c(X) = \omega$, i.e. X has the Souslin property.*

Proof: The space βX is Ω-separable by 2.6 (iii). Therefore the absolute Z of the space βX is also Ω-separable by 2.17 and $c(Z) = c(X)$. The space Z is extremally disconnected, so if $c(Z) > \omega$, then there is a disjoint family $\gamma = \{U_\alpha : \alpha \in \omega_1\} \subset \mathcal{T}^*(Z)$ such that U_α is a clopen set for all $\alpha < \omega_1$.

Let $Z_1 = \overline{\cup\gamma}$. Then Z_1 is an extremally disconnected compact space which is also Ω-separable by 2.6 (vi) and hence θ-separable. The set $U = \cup\gamma$ is dense in Z_1 so $Z_1 = \beta U$.

Let $D = D_0 \cup D_1$ be the Alexandroff duplicate of the unit segment $I = [0,1]$, where D_0 is (as a subspace) homeomorphic to I and all points of D_1 are isolated. Let $E = \{e_\alpha : \alpha \in \omega_1\} \subset D_1$ be such that its copy in D_0 is dense in D_0 and $e_\alpha \neq e_\beta$ for different α and β. The space $Y = D_0 \cup E$ is a first countable non-separable compact space.

The map $g : U_1 \to Y$ defined by $g(U_\alpha) = \{e_\alpha\}$ is continuous so there is a continuous $h : Z_1 \to Y$ with $h \restriction U = g$. But then $g(Z_1) = Y$ which is impossible by 2.11 (i), because Z_1 is θ-separable and Y is not. This contradiction proves that $c(Z) = c(X) \leqslant \omega$. □

2.19 Remark. *Closed irreducible preimages do not preserve θ-separability. Indeed, let X be any θ-separable space with $c(X) > \omega$, e.g. the space from 2.8 (i). If its absolute Y were θ-separable, then it would be Ω-separable, because these notions are clearly the same for extremally disconnected spaces. But this is a contradiction with 2.18 for $c(Y) = c(X) > \omega$.*

2.20 Definition. *From this moment on the letter b is reserved for a bijection from $\omega\backslash\{0\}$ to $\omega \times \omega$ such that*

(1) if $b(n) = (m,k)$, then $n > m + k$;

(2) if $b(n) = (m,k)$ and $l < k$, then $b^{-1}((m,l)) < n$.

It is clear that such a bijection exists.

2.21 Theorem. *Let X_α be a separable space for all $\alpha < \tau$. Let Y be a dense subset of $X = \prod\{X_\alpha : \alpha \in \tau\}$ such that there is a retraction $r : Y \to Z$ and for any countable $T \subset \tau$ the set $Z_T = \pi_T(Z) \subset X_T = \prod\{X_\alpha : \alpha \in T\}$ is separable. Then Z is Ω-separable.*

Proof: We are going to define a winning strategy s for the first player on Z. Let $s(\emptyset) = z_0$, where the point $z_0 \in Z$ is chosen arbitrarily. Suppose that for every $l \leqslant n$ we have defined the strategy s for all s-admissible l-tuples $\xi = (U_0, \dots, U_{l-1})$ in such a way that for each ξ as above we have sets T_i and Z_i, $(i \leqslant (l-1))$ with the following properties:

(1) $T_0 \subset T_1 \subset \dots \subset T_{l-1} \subset \tau$ and $|T_{l-1}| \leqslant \omega$;

(2) $Z_i = \{z^i_k : k \in \omega\} \subset Z$ and $\pi_{T_i}(Z_i)$ is dense in $\pi_{T_i}(Z)$ for all $i \leqslant (l-1)$;

Let $z_n = s(U_0, \dots, U_{n-1})$. If **II** $\to U_n$, then the set $\mathrm{cl}_X(r^{-1}(U_n))$ depends on countably many coordinates so let T_n be the relevant countable set containing T_{n-1}. The set Z_{T_n} is separable, so there is a $Z_n = \{z^n_l : l \in \omega\} \subset Z$ such that $\pi_{T_n}(Z_n)$ is dense in Z_{T_n}. Let $b(n) = (m,k)$, where b is the function defined in 2.20. We have to define $z_{n+1} = s(U_0, \dots, U_n)$. Let $z_{n+1} = z^k_m$.

Our inductive construction is accomplished, so we have a strategy s for the first player on Z. Let us prove that s is a WS.

For any play $P = \{(z_n, U_n) : n \in \omega\}$ we have defined the sets T_n and Z_n. Let $T = \cup\{T_n : n \in \omega\}$. Fix an $O \in \mathcal{T}^*(Z)$. We may assume that $O = V \cap Z$ where V is open in X, depends on finitely many coordinates and $V \cap Y \subset r^{-1}(O)$. Let B be the (finite) set of coordinates the set V depends on. Then $B = B_0 \cup B_1$, where $B_0 = B \cap T$, $B_1 = B\backslash B_0$. There is a $k \in \omega$ such that $B_0 \subset T_k$. The set $\pi_{T_k}(Z_k)$ is dense in $\pi_{T_k}(Z)$ so $\pi_{T_k}(z^k_m) \in \pi_{T_k}(V)$ for some $m \in \omega$. Now $(m,k) = b(n)$ for some $n > 0$ so that $z^k_m \in \mathrm{cl}_Z(U_n)$ and $z^k_m \in \mathrm{cl}_X(r^{-1}(U_n))$.

We claim that $U_n \cap O \neq \emptyset$. Indeed, it suffices to show that $r^{-1}(U_n) \cap V \neq \emptyset$. If, on the contrary, $r^{-1}(U_n) \cap V = \emptyset$, then $F \cap V = \emptyset$, where $F = \mathrm{cl}_X(r^{-1}(U_n))$. Therefore $F_B \cap V_B = \emptyset$. But $(\pi^B_{B_0})^{-1}(F_{B_0}) = F_B$ so that $F_{B_0} \cap V_{B_0} = \emptyset$. As $z^k_m \in F$ we have $\pi_{B_0}(z^k_m) \in F_{B_0} \cap V_{B_0}$, because $\pi_{T_k}(z^k_m) \in \pi_{T_k}(V)$ and $B_0 \subset T_k$. The obtained contradiction proves our theorem. $\square$

2.22 Corollary. *If X_α is a separable space for every $\alpha \in \tau$ then $X = \prod\{X_\alpha : \alpha \in \tau\}$ is Ω-separable (and hence θ-separable).*

2.23 Corollary. *If every X_α has a countable network for all $\alpha \in \tau$ then any dense subset of $X = \prod\{X_\alpha : \alpha \in \tau\}$ is Ω-separable (and hence θ-separable).*

2.24 Corollary. *Every dyadic space is θ-separable.*

The author did not succeed to clarify whether or not every dyadic space is Ω-separable. However in case of Dugundji compact spaces this is true. Recall that a compact space X is Dugundji if for every zero-dimensional compact space Y and for every continuous map f defined on a closed subset of Y the map f has continuous extension $f_1 : Y \to X$. It is well known that any Dugundji compact space is dyadic [16].

2.25 Corollary. *Any Dugundji compact space X is Ω-separable.*

Proof: Use V.V.Uspenskii's characterization [20] of Dugundji compact spaces: a compact space X is Dugundji iff X is a retract of some dense subset of $I^{w(X)}$. Now use Theorem 2.21. □

2.26 Corollary. *Any compact topological group is Ω-separable.*

Proof: Any compact topological group is a Dugundji space [16]. Now use 2.25. □

The following two results are concerned with hereditary θ- and Ω-separability. These properties are very close to hereditary separability, because they imply countable spread ($\equiv$ all discrete subspaces are countable). However, at least under continuum hypothesis hereditary separability and hereditary Ω-separability do not coincide.

2.27 Example. *If the continuum hypothesis* (*CH*) *holds then there is a hereditarily Ω-separable space X which is not hereditarily separable.*

Proof: Let $\Sigma = \{x \in 2^{\omega_1} : x(\alpha) \neq 0$ only for countably many $\alpha \in \omega_1\}$. It is known [1] that under CH the space Σ contains a dense Luzin subspace X, where "Luzin" means all nowhere dense subsets of X are countable. Let us prove that X is hereditarily Ω-separable.

Take any $Y \subset X$. If Y is countable, then everything is clear. Otherwise let $V = \mathrm{Int}_X \mathrm{cl}_X Y$. The set $Y \setminus V$ is nowhere dense in Y and thus countable, because X is a Luzin space. The set $Y \cap V$ is dense in an open subset of X and hence in an open subset U of 2^{ω_1}. Let $\gamma = \{U_n : n \in \omega\}$ be a disjoint family of standard open subsets of 2^{ω_1} with $\cup\{U_n : n \in \omega\} \subset U \subset \overline{\cup\{U_n : n \in \omega\}}$. Every U_n is homeomorphic to 2^{ω_1} so $V_n = V \cap U_n$ can be densely embedded in 2^{ω_1}. Therefore V_n is Ω-separable by 2.23. The set V contains a dense subset homeomorphic to $\bigoplus\{V_n : n \in \omega\}$ so that V is Ω-separable by 2.6 (xii). Hence Y is Ω-separable. □

In case X is compact the situation is different.

2.28 Theorem. *If X is a compact hereditarily θ-separable space, then X is hereditarily separable.*

Proof: We need the following lemma which seems to be of interest in itself.

2.28 Lemma. *A hereditarily θ-separable Corson compact space is metrizable.*

Proof: If Y is a Corson compact space, then Y has a dense subset Z with $\chi(Z) \leqslant \omega$ [3]. By assumption of the lemma the space Z is θ-separable and hence separable by 2.11 (i). Consequently Y is separable. But any separable Corson compact space is metrizable [9]. ⊿

Now let X be a compact hereditarily θ-separable space. Then $t(X) \leqslant \omega$ because $s(X) \leqslant \omega$ [2]. Therefore X can be continuously and irreducibly mapped onto a Corson compact space Y [15]. We know that Y is separable by 2.28, so that X itself is separable. The same reasoning proves that each closed subspace of X is separable. Hence X is hereditarily separable [3]. □

3. Some game-theoretical results on θ and Ω

Quite a few topological games have been introduced and studied in the last twenty years (see [8], [10], [12], [13] and [21] for the games different from the point-open one). Usually, the main question about every game under consideration was whether it was determined or not, and if it was not then what were some good classes of spaces it is determined on. So far we have only proved (see 2.15) that θ and Ω are determined on the class of metric spaces. R. Telgársky has shown in ZFC that the point-open game was not determined on the class of Lindelöf P-spaces. Assuming Martin's axiom F. Galvin showed that there are undetermined subsets of the real line for the point open game. R. Telgársky proved [17] that the point open game is determined on the class of countably compact spaces. In this section we are going to prove that there are compact spaces on which neither θ nor Ω are determined. We also introduce some games equivalent to θ and Ω.

3.1 Definition. *We say that the game θ_* (or Ω_* respectively) is played on X if two players called $\mathbf{I}_*$ and $\mathbf{II}_*$ take turns playing. At the n-th move $\mathbf{I}_*$ chooses a family $\gamma_n \subset \mathcal{T}(X)$ such that $\cup\gamma_n = X$ (or $\cup\{\overline{U} : U \in \gamma_n\} = X$ respectively) and $\mathbf{II}_*$ picks a $U_n \in \gamma_n$. After ω moves the play stops and $\mathbf{II}_*$ is announced to be the winner in the play $P = \{(\gamma_n, U_n) : n \in \omega\}$ if $\cup\{U_n : n \in \omega\}$ is dense in X. Otherwise $\mathbf{I}_*$ wins.*

In what follows we are going to use the notion of winning strategy for one of the players in θ_* (or Ω_*) without giving definitions. An interested reader can easily restore them repeating the reasoning in 2.1–2.5.

3.2 Definition. *If X is a space, then a family $\gamma \in \mathcal{T}(X)$ is called a weak cover of X if $\cup\{\overline{U} : U \in \gamma\} = X$.*

The following theorem explains why we used the letters θ and Ω for defining the games in 3.1.

3.3 Theorem. *The game θ_* (or Ω_* respectively) is equivalent to the game θ (or Ω respectively) i.e. for any space X:*

(1) *the player $\mathbf{I}_*$ has a winning strategy in θ_* (or Ω_* respectively) on the space X iff $\mathbf{II}$ has a winning strategy in θ (or Ω respectively) on the space X;*

(2) *the player* $\mathbf{II}_*$ *has a winning strategy in* θ_* (*or* Ω_* *respectively*) *on the space* X *iff* $\mathbf{I}$ *has a winning strategy in* θ (*or* Ω *respectively*) *on the space* X.

PROOF: The proof we give here is obtained by making obvious changes in F. Galvin's proof of an analogous theorem for the point-open game [7, Theorem 1]. That's why it will be pretty concise — with necessary strategies constructed but without proofs that they are winning.

Let $\mathbf{I}_*$ have a winning strategy s_* on X in θ_* (or Ω_* respectively). We must construct a winning strategy s for the second player in θ (or Ω respectively) on X. Let $\mathbf{I} \to x_0$. We have the cover $\gamma_0 = s_*(\emptyset)$. Choose any $U_0 \in \gamma_0$ such that $x_0 \in U_0$ (or $x_0 \in \overline{U_0}$ respectively). Let $s(x_0) = U_0$. If after n moves $\mathbf{I} \to x_n$ and we have $x_0, U_0, \dots, x_{n-1}, U_{n-1}$ and covers $\gamma_0, \dots, \gamma_{n-1}$ such that $U_i \in \gamma_i$ let $\gamma_n = s_*(U_0, \dots, U_{n-1})$ and pick a $U_n \in \gamma_n$ with $x_n \in U_n$ (or $x_n \in \overline{U_n}$ respectively). Then define $s(x_0, \dots, x_n)$ to be the set U_n. The strategy s thus constructed is the needed WS.

Now let the second player have a winning strategy s on X in θ (or Ω respectively). Announce the cover $\gamma_0 = \{s(x_0) : x_0 \in X\}$ to be $s_*(\emptyset)$. Suppose that for all $l < n$ we have constructed $s_*(\xi)$ for all s_*-admissible $(l+1)$-tuples $\xi = (U_0, \dots, U_l)$ in such a way that for every such ξ we have $(x_0, \dots, x_l) \in \mathrm{dom}(s)$. Let $(x_0, \dots, x_{n-1})$ correspond to $(U_0, \dots, U_{n-1})$ and assume that $\mathbf{II}_*$ has chosen a $U_n \in \gamma_n$. Let $s_*(U_0, \dots, U_n) = \gamma_{n+1} = \{s(x_0, \dots, x_{n-1}, x_n) : x_n \in X\}$.

The strategy s_* is thus constructed and it is a routine to check that it is winning. $\triangle$

Suppose that $\mathbf{II}_*$ has a winning strategy s_* on X in θ_* (or Ω_* respectively). Then there is a point $x_0 \in X$ such that for every $U \in \mathcal{T}(x_0, X)$ (or for every $U \in \mathcal{T}(X)$ with $x_0 \in \overline{U}$) there is a (weak) cover γ with $U = s_*(\gamma)$. Such x_0 exists because otherwise we would have a "bad" open set $U_x \in \mathcal{T}(x, X)$ for every $x \in X$. Then $\gamma = \{U_x : x \in X\}$ is a (weak) cover of X and $s_*(\gamma) = U_y$ for some $y \in X$ which is a contradiction by "badness" of U_y. Therefore the promised x_0 exists so let $s(\emptyset) = x_0$.

Suppose that for all $l < n$ and s-admissible $(l+1)$-tuples $\xi = (U_0, \dots, U_l)$ we defined what $s(\xi)$ is in such a way that for every such ξ there are covers $\gamma_0, \dots, \gamma_l$ with $(\gamma_0, \dots, \gamma_l) \in \mathrm{dom}(s_*)$. Given $\xi_i = (U_0, \dots, U_{i-1})$ let $x_i = s(\xi_i)$ for $0 < i < n$. Assume that the second player chose a set U_n. There exists a point $x_{n+1} \in X$ such that for every $U \in \mathcal{T}(x_{n+1}, X)$ (or for every $U \in \mathcal{T}(X)$ with $x_{n+1} \in \overline{U}$) there is a (weak) cover γ such that $U = s_*(\gamma_0, \dots, \gamma_{n-1}, \gamma)$. Such x_{n+1} exists because otherwise we would have a "bad" open set $U_x \in \mathcal{T}(x, X)$ for every $x \in X$. Then $\gamma = \{U_x : x \in X\}$ is a (weak) cover of X and $s_*(\gamma_0, \dots, \gamma_{n-1}, \gamma) = U_y$ for some $y \in X$ which is a contradiction by "badness" of U_y. Therefore the promised x_{n+1} exists so let $s(U_0, \dots, U_n) = x_{n+1}$. This completes the construction of a WS for $\mathbf{I}$ on X in θ (or Ω respectively).

Finally, let $\mathbf{I}$ have a winning strategy s on X in θ (or Ω respectively). If $\mathbf{I}_* \to \gamma_0$ then let $x_0 = s(\emptyset)$, pick a $U_0 \in \gamma_0$ with $x_0 \in U_0$ (or $x_0 \in \overline{U_0}$ respectively) and let $s_*(\gamma_0) = U_0$.

If in the process of playing we have $\gamma_0, U_0, \dots, \gamma_{n-1}, U_{n-1}, \gamma_n$ and $x_0 = s(\emptyset), \dots, x_n = s(U_0, \dots, U_{n-1})$, pick an element U_n from γ_n with $x_n \in U_n$ (or $x_n \in \overline{U_n}$ respectively) and announce U_n to be $s_*(\gamma_0, \dots, \gamma_n)$. Thus the winning strategy s_* for the second player on X in θ_* (or Ω_* respectively) is constructed. □

3.4 Theorem. *A space X is θ-antiseparable (or Ω-antiseparable respectively) if there exist a cardinal number τ and a family $\Gamma = \{U(\alpha_0, \dots, \alpha_n) : \alpha_i \in \tau, i \in (n+1), n \in \omega\} \subset \mathcal{T}^*(X)$ with the following properties:*

(1) *$\{U(\alpha_0) : \alpha_0 \in \tau\}$ is a (weak) cover of X;*
(2) *if $\alpha_0, \dots, \alpha_n \in \tau$, then $\Gamma(\alpha_0, \dots, \alpha_n) = \{U(\alpha_0, \dots, \alpha_n, \alpha) : \alpha \in \tau\}$ is a (weak) cover of X;*
(3) *For any sequence $(\alpha_i : i \in \omega) \in \tau^\omega$ the set $\cup\{U(\alpha_0, \dots, \alpha_n) : n \in \omega\}$ is not dense in X.*

Proof: It is analogous to the proof of Theorem 6.3 in [18] so we will not go into details. If Γ is a family with (1)-(3) and moves $x_0, U_0, \dots, x_{n-1}, U_{n-1}, x_n$ are made in such a way that there are $\alpha_0, \dots, \alpha_{n-1} \in \tau$ with $U_i = U(\alpha_0, \dots, \alpha_i)$ for $i \in n$, then take any $U_n \in \mathcal{T}(x_n, X) \cap \Gamma(\alpha_0, \dots, \alpha_{n-1})$ (or $U_n \in \Gamma(\alpha_0, \dots, \alpha_{n-1})$, $\overline{U_n} \ni x_n$ respectively) and let $s(x_0, \dots, x_n) = U_n$. The strategy s thus constructed is a winning one. ⊿

If a strategy s on X is given, then let $\Gamma = \{s(x_0, \dots, x_n) : x_i \in X,\ i \in (n+1),\ n \in \omega\}$ is as required after an evident identification of X with $\tau = |X|$. □

3.5 Corollary. *If X is a Lindelöf space, then it is θ-antiseparable iff there is a family $\Gamma = \{U(k_0, \dots, k_n) : k_i \in \omega, i \in (n+1), n \in \omega\} \subset \mathcal{T}^*(X)$ such that*

(1) *$\Gamma_0 = \{U(k_0) : k_0 \in \omega\}$ is a cover of X;*
(2) *$\Gamma(k_0, \dots, k_n) = \{U(k_0, \dots, k_n, k) : k \in \omega\}$ is a cover of X for every $(k_0, \dots, k_n) \in \omega^{n+1}$;*
(3) *for every sequence $(k_n : n \in \omega) \in \omega^\omega$ the set $\cup\{U(k_0, \dots, k_n) : n \in \omega\}$ is not dense in X.*

3.6 Corollary. *If X is a Ω-antiseparable space with $c(X) = \omega$, then there is a family $\Gamma = \{U(k_0, \dots, k_n) : k_i \in \omega, i \in (n+1), n \in \omega\} \subset \mathcal{T}^*(X)$ such that*

(1) *for the family $\Gamma_0 = \{U(k_0) : k_0 \in \omega\}$ we have $\overline{\cup\Gamma} = X$;*
(2) *$\overline{\cup\Gamma(k_0, \dots, k_n)} = X$, where $\Gamma(k_0, \dots, k_n) = \{U(k_0, \dots, k_n, k) : k \in \omega\}$;*
(3) *for every sequence $(k_n : n \in \omega) \in \omega^\omega$ the set $\cup\{U(k_0, \dots, k_n) : n \in \omega\}$ is not dense in X.*

3.7 Example. *There exists a Lindelöf P-space X on which θ is undetermined, i.e. neither of players has a WS.*

Proof: Let X be the space used by R. Telgársky [18, Theorem 7.1] to prove that the point-open game is undetermined on X. We do not need to know exactly what the structure of X is. It suffices for us to know that X is a Lindelöf P-space which has a dense set of isolated points with the property that whatever a strategy s of **I** is there is a play $\{(x_n, U_n) : n \in \omega\}$ on X (in point-open game, but remember that θ has the same moves and the same definitions of strategies!) in which **I** used s and $\cup\{U_n : n \in \omega\}$ did not cover an isolated point from X. This of course means that the first player has no winning strategy on X in θ.

If **II** had a WS on X in θ, then this strategy would be winning in the point-open game which is impossible, because the point-open game is undetermined on X [18]. □

3.8 Theorem. *If the continuum hypothesis ($\equiv$CH) holds, then for any space X with $c(X) > \omega$ the second player has a winning strategy in Ω on X.*

Proof: By 2.17 (ii) and 2.6 (iii) it suffices to prove 3.8 for extremally disconnected compact spaces. If X is such a space, then pick a disjoint family $\gamma = \{U_\alpha : \alpha < \omega_1\}$ of non-empty clopen subsets of X. Let $D = D_0 \cup D_1$ be the Alexandroff duplicate of the unit segment $I = [0, 1]$, where D_0 is (as a subspace) homeomorphic to I and all points of D_1 are isolated. Then D is a first countable non-separable compact space. Use CH to enumerate all points of D_1 with countable ordinals: $D_1 = \{d_\alpha : \alpha < \omega_1\}$.

Let $Z_1 = \overline{\cup\gamma}$. Then Z_1 is an extremally disconnected compact space. The set $U = \cup\gamma$ is dense in Z_1 so $Z_1 = \beta U$. The map $g : Z_1 \to D$ defined by $g(U_\alpha) = \{d_\alpha\}$ is continuous so there is a continuous $h : Z_1 \to D$ with $h \restriction U = g$.

It is clear that if Z_1 is θ-antiseparable, then so is X. The space X being extremally disconnected in this case we will have it Ω-antiseparable, so it suffices by 2.6 (ix) to prove that D is θ-antiseparable.

To obtain a WS for **II** in θ on D suppose that $\mathbf{I} \to x_n$. Let U_n be the copies in D_0 and D_1 of the set $(x_n - 4^{-n}, x_n + 4^{-n}) \cap I$. It is evident that $\cup\{U_n : n \in \omega\}$ cannot cover D_1 so the strategy thus defined is a winning one for **II**. □

3.9 Remark. *Theorem 3.8 shows that under CH the space X from 3.7 is antiseparable being an uncountable Lindelöf P-space. It is known that the point-open game is determined on the class of compact spaces [17]. Although the space X from 3.7 cannot serve as an example of indeterminacy for both games θ and Ω, we are going to produce such an example (and even compact one) under the negation of Souslin hypothesis.*

3.10 Example. *If a Souslin continuum exists, then both θ and Ω are undetermined on it.*

Proof: Let X be a Souslin continuum. Then it is first countable and non-separable, so **I** cannot have a WS in θ (and hence in Ω) on X by 2.11.

All there is to do is to prove that **II** cannot have a WS on X in Ω. If there were such a strategy then by 3.6 we would have a family $\Gamma = \{U(k_0, \dots, k_n) : k_i \in \omega,\ i \in (n+1),\ n \in \omega\} \subset \mathcal{T}^*(X)$ such that

(1) for the family $\Gamma_0 = \{U(k_0) : k_0 \in \omega\}$ we have $\overline{\cup\Gamma} = X$;

(2) $\overline{\cup\Gamma(k_0,\ldots,k_n)} = X$, where $\Gamma(k_0,\ldots,k_n) = \{U(k_0,\ldots,k_n,k) : k \in \omega\}$;
(3) for every sequence $(k_n : n \in \omega) \in \omega^\omega$ the set $\cup\{U(k_0,\ldots,k_n) : n \in \omega\}$ is not dense in X.

Every element of Γ is a countable disjoint union of intervals. Let S be the closure of the set of ends of all those intervals. Then S is nowhere dense in X and $X\backslash S = \cup\{W_n : n \in \omega\}$, where every W_n is an interval.

Now, for every $n \in \omega$ let **II** pick some $U(k_0,\ldots,k_n)$ containing some point $x_n \in W_n$. The set $U(k_0,\ldots,k_n)$ must contain W_n, because otherwise some endpoint of an interval which is clopen in $U(k_0,\ldots,k_n)$ would be inside W_n, which is impossible. Hence $\cup\{U(k_0,\ldots,k_n) : n \in \omega\} \supset \cup\{W_n : n \in \omega\}$, so some $\cup\{U(k_0,\ldots,k_n) : n \in \omega\}$ is dense in X, which is a contradiction. □

3.11 Example. *If Martin's axiom and the negation of CH hold, then the Alexandroff duplicate D of the unit segment $I = [0,1]$ contains a compact subspace on which the game θ is not determined.*

Proof: Let $D = D_0 \cup D_1$, where D_0 and D_1 are like in 3.8 and let E be any subset of D_1 of cardinality ω_1. The space $X = D_0 \cup E$ is as required.

Indeed, **I** does not have a WS on X because X is first countable and non-separable (see 2.11 (i)). Suppose that X is θ-antiseparable. Let Q be the set of rational points of D_0. Fix a family Γ like in 3.5. For any $f \in \omega^\omega$ let $W_f = \cup\{U(f(0),\ldots,f(n)) : n \in \omega\}$ and if $x \in Q \cup E$, then $G_x = \{f \in \omega^\omega : W_f \ni x\}$.

Assume, that $\xi = (m_0,\ldots,m_n) \in \omega^{n+1}$ and let

$$O(\xi) = \{f \in \omega^\omega : f(i) = m_i \text{ for all } i \in n\}$$

be an arbitrary standard open subset of ω^ω. The family $\Gamma(\xi)$ is a cover of X so $x \in U(m_0,\ldots,m_n,m_{n+1})$ for some $m_{n+1} \in \omega$. It is clear that if $f(i) = m_i$ for all $i \leqslant (n+1)$, then $W_f \ni x$ so that the set G_x is open and intersects any $O(\xi)$. This implies G_x dense in ω^ω.

It follows from Martin's axiom [11, Theorem 2.20] that $F = \cap\{G_x : x \in Q\cup E\} \neq \emptyset$. Take any $f \in F$. Then the set W_f covers $Q \cup E$ and hence is dense in X which gives a contradiction with 3.5 (3). □

4. Open questions

In this section the author collected most of the problems he was unable to solve while working on games θ and Ω. The given list shows that there is still a lot to be done on the topic developed in this paper.

4.1 Question. *Let X be Ω-separable and $f : X \to Y$ a continuous onto map. Must then Y be Ω-separable?*

4.2 Question. *Let X be Ω-separable and $f : X \to Y$ a quotient map. Must then Y be Ω-separable?*

4.3 Question. *Let X be Ω-separable and $f : X \to Y$ a closed onto map. Must then Y be Ω-separable?*

4.4 Question. *Let X be Ω-separable and $f : X \to Y$ a perfect onto map. Must then Y be Ω-separable?*

4.5 Question. *Let X be Ω-separable and $f : X \to Y$ a retraction. Must then Y be Ω-separable?*

4.6 Question. *Let $X = X_1 \cup X_2$ and X_i is Ω-separable for $i = 1, 2$. Is then X Ω-separable?*

4.7 Question. *Let $X = \cup\{X_n : n \in \omega\}$ and X_i is Ω-separable for all $i \in \omega$. Is then X Ω-separable?*

4.8 Question. *Is the product of two θ-separable spaces θ-separable?*

4.9 Question. *Is the product of two Ω-separable spaces Ω-separable?*

4.10 Question. *Is the product of an Ω-separable space and a separable space Ω-separable?*

4.11 Question. *Is any σ-compact topological group Ω-separable?*

4.12 Question. *Is any Lindelöf-Σ topological group Ω-separable?*

4.13 Question. *Is any Lindelöf-Σ topological group θ-separable?*

4.14 Question. *Let X be an Ω-separable space. Must then the Markov free topological group $F_M(X)$ be Ω-separable?*

4.15 Question. *Is it consistent with ZFC that every hereditarily θ-separable space is separable?*

4.16 Question. *Is it consistent with ZFC that every hereditarily Ω-separable space is separable?*

4.17 Question. *Is there a hereditarily θ-separable space which is not Ω-separable?*

4.18 Question. *Is any hereditarily θ-separable Lindelöf Σ-space hereditarily separable?*

4.19 Question. *Is any hereditarily Ω-separable Lindelöf Σ-space hereditarily separable?*

4.20 Question. *Is there a space X in ZFC on which Ω is undetermined?*

4.21 Question. *Is there a Lindelöf Σ-space X in ZFC on which θ is undetermined?*

4.22 Question. *Is there a Lindelöf Σ-space X in ZFC on which Ω is undetermined?*

4.23 Question. *Is there a compact X in ZFC on which θ is undetermined?*

4.24 Question. *Is there a compact X in ZFC on which Ω is undetermined?*

4.25 Question. *Does $c(X) > \omega$ imply in ZFC that X is Ω-antiseparable?*

4.26 Question. *Is every dyadic compact space Ω-separable?*

4.27 Question. *Is every Corson compact Ω-separable space metrizable in ZFC?*

4.28 Question. *Is every Gul'ko compact Ω-separable space metrizable in ZFC?*

4.29 Question. *Is every Miliutin compact space Ω-separable?*

4.30 Question. *Is every Ω-separable compact X with $t(X) = \omega$ separable in ZFC?*

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Departamento de Matematicas, Universidad Autónoma Metropolitana,
Av. Michoacan y La Purísima, Iztapalapa, A.P.55-532, C.P.09340, Mexico, D.F.

E-mail: vova@xanum.uam.mx
or rolando@redvax1.dgsca.unam.mx to Tkachuk

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