On the extremality of regular extensions of contents and measures

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Abstract. Let $\mathcal{A}$ be an algebra and $\mathcal{K}$ a lattice of subsets of a set X. We show that every content on $\mathcal{A}$ that can be approximated by $\mathcal{K}$ in the sense of Marczewski has an extremal extension to a $\mathcal{K}$-regular content on the algebra generated by $\mathcal{A}$ and $\mathcal{K}$. Under an additional assumption, we can also prove the existence of extremal regular measure extensions.

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1. Introduction

If $\mathcal{A}$, $\mathcal{B}$, are algebras of subsets of some set X with $\mathcal{A} \subset \mathcal{B}$, then Plachky [9] has shown by a Krein-Milman argument that every (finite) content on $\mathcal{A}$ has an extremal extension to a content on $\mathcal{B}$. In [2], this result has been generalized in the following way. If $\mathcal{K}$, $\mathcal{L}$ are lattices of subsets of X with $\mathcal{K} \subset \mathcal{L}$, then every $\mathcal{K}$-regular content on $\alpha(\mathcal{K})$, the algebra generated by $\mathcal{K}$, has an extremal extension to an $\mathcal{L}$-regular content on $\alpha(\mathcal{L})$. It is the aim of this note to give the following further generalization. If $\mathcal{A}$ is an algebra and $\mathcal{K}$ a lattice of subsets of X, then every content on $\mathcal{A}$ which can be approximated by $\mathcal{K}$ in the sense of Marczewski [7] has an extremal extension to a $\mathcal{K}$-regular content on $\alpha(\mathcal{A} \cup \mathcal{K})$. Under an additional assumption, we can also prove the existence of extremal regular measure extensions. Note that extremal measure extensions are considered always under some additional assumptions ([2]) or for special situations (e.g. if the target σ-algebra is generated from a given one by adjunction of a family which either consists of pairwise disjoint sets or is well ordered by inclusion [3], [4], [5]), since, in general, extremal measure extensions do not exist (see [9], [11]).

Now we fix the notation. X will always denote an arbitrary set. Let $\mathcal{C}$ be a subset of $\mathcal{P}(X)$, the power set of X. We write $\alpha(\mathcal{C})$, $\sigma(\mathcal{C})$ for the algebra, σ-algebra generated by $\mathcal{C}$, respectively. Furthermore, $\mathcal{C}_\delta$ denotes the family of all countable intersections of sets from $\mathcal{C}$. $\mathcal{C}$ is said to be <u>semicompact</u> if every countable subfamily of $\mathcal{C}$ having the finite intersection property has nonvoid intersection. $\mathcal{C}$ is called a <u>lattice</u> if $\varnothing \in \mathcal{C}$ and $\mathcal{C}$ is closed under finite unions and finite intersections. For a lattice $\mathcal{C}$, we denote by $\mathcal{F}(\mathcal{C}) := \{F \subset X : F \cap C \in \mathcal{C} \text{ for every } C \in \mathcal{C}\}$ the lattice of so-called "<u>local $\mathcal{C}$-sets</u>". Obviously, $X \in \mathcal{F}(\mathcal{C})$ and $\mathcal{C} \subset \mathcal{F}(\mathcal{C})$; in addition, we have $\mathcal{C} = \mathcal{F}(\mathcal{C})$ iff $X \in \mathcal{C}$.

If $\mathcal{D}$ is another subset of $\mathcal{P}(X)$, then $\mathcal{C}$ is said to be sequentially dominated by $\mathcal{D}$ if whenever $(C_n \in \mathcal{C})_{n\in N}$ and $C_n \downarrow \varnothing$, there exists a sequence $(D_n \in \mathcal{D})_{n\in N}$ such that $D_n \downarrow \varnothing$ and $C_n \subset D_n$ for all $n \in N$. Note that a semicompact family is sequentially dominated by any family $\mathcal{D}$ with $X \in \mathcal{D}$.

By a content (measure) we always understand a $[0,\infty)$-valued, finitely (countably) additive set function defined on an algebra.

Consider a lattice $\mathcal{K} \subset \mathcal{P}(X)$ and a content μ on the algebra $\mathcal{A} \subset \mathcal{P}(X)$. Under the assumption $\mathcal{K} \subset \mathcal{A}$, μ is called $\mathcal{K}$-regular if $\mu(A) = \sup\{\mu(K) : K \in \mathcal{K},\ K \subset A\}$ for all $A \in \mathcal{A}$. For the following concept going back to Marczewski [7], we will use the terminology of [8]:
$\mathcal{K}$ is said to μ-approximate $\mathcal{A}$ if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exist sets $B \in \mathcal{A}$ and $K \in \mathcal{K}$ such that $B \subset K \subset A$ and $\mu(A - B) < \varepsilon$ hold. Note that in case $\mathcal{K} \subset \mathcal{A}$, $\mathcal{K}$ μ-approximates $\mathcal{A}$ iff μ is $\mathcal{K}$-regular.

2. The main results

In this section we consider an algebra $\mathcal{A}$ and two lattices $\mathcal{K}$, $\mathcal{L}$ of subsets of X with $\mathcal{K} \subset \mathcal{L}$ as well as a content μ on $\mathcal{A}$ such that $\mathcal{K}$ μ-approximates $\mathcal{A}$.

If $\mathcal{B} \supset \mathcal{A}$ is another algebra, then $\mathrm{ba}(\mu,\mathcal{B})$ denotes the family of all contents on $\mathcal{B}$ that extend μ. In addition, we define $\mathrm{ba}(\mu,\mathcal{B},\mathcal{K}) := \{\nu \in \mathrm{ba}(\mu,\mathcal{B}) : \mathcal{K}$ ν-approximates $\mathcal{B}\}$ and $\mathrm{ca}(\mu,\mathcal{B},\mathcal{K}) := \{\nu \in \mathrm{ba}(\mu,\mathcal{B},\mathcal{K}) : \nu$ is a measure$\}$. Note that $\mathrm{ba}(\mu,\mathcal{B})$, $\mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$ and $\mathrm{ca}(\mu,\mathcal{B},\mathcal{K})$ are convex sets. If D is any of these sets, then $\mathrm{ex}\, D$ denotes the set of extreme points of D.

Lemma 2.1. *Let $\mathcal{B} \supset \mathcal{A}$ be another algebra and $\nu \in \mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$. Then $\nu \in \mathrm{ex}\,\mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$ iff $\nu \in \mathrm{ex}\,\mathrm{ba}(\mu,\mathcal{B})$.*

Proof: Assume $\nu \in \mathrm{ex}\,\mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$ and let $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ with $\nu_1, \nu_2 \in \mathrm{ba}(\mu,\mathcal{B})$. Since $\frac{1}{2}\nu_i \le \nu$ and $\nu \in \mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$ we have $\nu_i \in \mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$ for $i = 1,2$. Thus we infer $\nu_1 = \nu_2$ from the extremality of ν. This proves $\nu \in \mathrm{ex}\,\mathrm{ba}(\mu,\mathcal{B})$. The other part of the claim is obvious. □

Lemma 2.2. *If $Q \in \mathcal{F}(\mathcal{K}) - \mathcal{A}$ and $\mathcal{B} := \alpha(\mathcal{A} \cup \{Q\})$ then $\mathrm{ex}\,\mathrm{ba}(\mu,\mathcal{B},\mathcal{K}) \neq \varnothing$.*

Proof: (1) For every $E \in \mathcal{P}(X)$, we define $\mu^*(E) := \inf\{\mu(A) : E \subset A \in \mathcal{A}\}$ and $\mu_*(E) := \sup\{\mu(A) : E \supset A \in \mathcal{A}\}$. It is well known ([6]) that $\mathcal{B} = \{(A_1 \cap Q) \cup (A_2 - Q) : A_1, A_2 \in \mathcal{A}\}$ and $\nu(B) := \mu^*(B \cap Q) + \mu_*(B - Q)$, $B \in \mathcal{B}$, defines an element ν of $\mathrm{ba}(\mu,\mathcal{B})$.

(2) To prove $\nu \in \mathrm{ba}(\mu,\mathcal{B},\mathcal{K})$ let $B \in \mathcal{B}$ and $\varepsilon > 0$ be given. Then $B = (A_1 \cap Q) \cup (A_2 - Q)$ with some $\mathcal{A}$-sets A_1, A_2. Since $\mu_*(B - Q) = \mu_*(A_2 - Q) = \sup\{\mu(A) : A \in \mathcal{A},\ A \subset A_2 - Q\}$, there is an $\mathcal{A}$-set C satisfying $C \subset A_2 - Q$ and $\mu_*(B-Q) < \mu(C) + \frac{\varepsilon}{4}$. In addition, there exist sets $C_0 \in \mathcal{A}$ and $K_0 \in \mathcal{K}$ such that $C_0 \subset K_0 \subset C$ and $\mu(C) < \mu(C_0) + \frac{\varepsilon}{4}$. This together yields $\mu_*(B-Q) < \mu(C_0) + \frac{\varepsilon}{2}$. Furthermore, one can choose sets $C_1 \in \mathcal{A}$ and $K_1 \in \mathcal{K}$ such that $C_1 \subset K_1 \subset A_1$ and $\mu(A_1 - C_1) < \frac{\varepsilon}{2}$ which implies $\mu^*((A_1 \cap Q) - C_1) \le \mu(A_1 - C_1) < \frac{\varepsilon}{2}$ and hence $\mu^*(A_1 \cap Q) \le \mu^*((A_1 \cap Q) - C_1) + \mu^*(A_1 \cap Q \cap C_1) < \mu^*(C_1 \cap Q) + \frac{\varepsilon}{2}$. Now

$B^* := (C_1 \cap Q) \cup (C_0 - Q) \in \mathcal{B}$, $K^* := (K_1 \cap Q) \cup K_0 \in \mathcal{K}$, $B^* \subset K^* \subset B$ and $\nu(B) = \mu^*(B \cap Q) + \mu_*(B - Q) < \mu^*(A_1 \cap Q) + \mu(C_0) + \frac{\varepsilon}{2} < \mu^*(C_1 \cap Q) + \mu(C_0) + \varepsilon = \mu^*(C_1 \cap Q) + \mu_*(C_0 - Q) + \varepsilon = \nu(B^*) + \varepsilon$. Thus $\nu \in \mathrm{ba}(\mu, \mathcal{B}, \mathcal{K})$.

(3) To prove $\nu \in \mathrm{ex\,ba}(\mu, \mathcal{B}, \mathcal{K})$ it suffices to show $\nu \in \mathrm{ex\,ba}(\mu, \mathcal{B})$. For an arbitrary $\varepsilon > 0$, choose $A \in \mathcal{A}$ such that $Q \subset A$ and $\mu(A) < \mu^*(Q) + \varepsilon$. Then $\nu(A \vartriangle Q) = \nu(A - Q) = \mu_*(A - Q) = \mu(A) - \mu^*(Q) < \varepsilon$. From [9], Theorem 1 and the associated Remark 2, we infer $\nu \in \mathrm{ex\,ba}(\mu, \mathcal{B})$. □

If $\mathcal{B}$ is an algebra satisfying $\mathcal{A} \cup \mathcal{K} \subset \mathcal{B}$, then $\mathrm{ba}(\mu, \mathcal{B}, \mathcal{K})$ is the family of all $\mathcal{K}$-regular contents on $\mathcal{B}$ that extend μ. According to [1, Theorem 3.4], μ can be extended to a $\mathcal{K}$-regular content on $\alpha(\mathcal{A} \cup \mathcal{F}(\mathcal{K}))$. The following basic result shows that even an extremal extension exists.

Theorem 2.3. $\mathrm{ex\,ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{E}), \mathcal{K}) \neq \varnothing$ *for every sublattice* $\mathcal{E}$ *of* $\mathcal{F}(\mathcal{K})$.

Proof: (1) Fix some sublattice $\mathcal{E}$ of $\mathcal{F}(\mathcal{K})$ and define $\Gamma := \{(\mathcal{M}, \varrho) : \mathcal{M}$ is a sublattice of $\mathcal{E}$ and $\varrho \in \mathrm{ex\,ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}), \mathcal{K})\}$. Note that $(\{\varnothing\}, \mu) \in \Gamma$. We order the elements of Γ in the following way: $(\mathcal{M}, \varrho) \leq (\mathcal{M}', \varrho')$ iff $\mathcal{M} \subset \mathcal{M}'$ and ϱ' is an extension of ϱ.

(2) Now we show that Γ is inductively ordered. Consider a chain $(\mathcal{M}_i, \varrho_i)_{i \in I}$ in Γ. Then $\mathcal{M} := \bigcup_{i \in I} \mathcal{M}_i$ is a sublattice of $\mathcal{E}$ and $\alpha(\mathcal{A} \cup \mathcal{M}) = \bigcup_{i \in I} \alpha(\mathcal{A} \cup \mathcal{M}_i)$. For $C \in \alpha(\mathcal{A} \cup \mathcal{M})$, define $\varrho(C) := \varrho_i(C)$ provided that $C \in \alpha(\mathcal{A} \cup \mathcal{M}_i)$. ϱ is a content on $\alpha(\mathcal{A} \cup \mathcal{M})$ that extends every ϱ_i. It is easy to see that $\varrho \in \mathrm{ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}), \mathcal{K})$.

To prove $\varrho \in \mathrm{ex\,ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}), \mathcal{K})$ consider $\tau_1, \tau_2 \in \mathrm{ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}), \mathcal{K})$ with $\varrho = \frac{1}{2}(\tau_1 + \tau_2)$. Fix some $i_0 \in I$ and define $\widehat{\tau}_j := \tau_j \mid \alpha(\mathcal{A} \cup \mathcal{M}_{i_0})$ for $j = 1, 2$. Then $\widehat{\tau}_j \in \mathrm{ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}_{i_0}))$, $j = 1, 2$, and $\varrho_{i_0} = \frac{1}{2}(\widehat{\tau}_1 + \widehat{\tau}_2)$. Since $\varrho_{i_0} \in \mathrm{ex\,ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}_{i_0}), \mathcal{K})$, we infer $\widehat{\tau}_1 = \widehat{\tau}_2$ from 2.1.

Now consider an arbitrary $A \in \alpha(\mathcal{A} \cup \mathcal{M})$. Then $A \in \alpha(\mathcal{A} \cup \mathcal{M}_{i_0})$ for some $i_0 \in I$ and hence $\tau_1(A) = \widehat{\tau}_1(A) = \widehat{\tau}_2(A) = \tau_2(A)$. Thus $\tau_1 = \tau_2$ which proves $\varrho \in \mathrm{ex\,ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{M}), \mathcal{K})$.

Consequently, $(\mathcal{M}_i, \varrho_i) \leq (\mathcal{M}, \varrho) \in \Gamma$ for all $i \in I$. So Γ is inductively ordered.

(3) By Zorn's lemma, there is a maximal element $(\widetilde{\mathcal{M}}, \widetilde{\varrho})$ in Γ. We will show $\widetilde{\mathcal{M}} = \mathcal{E}$ which implies that $\widetilde{\varrho}$ is the desired extremal element of $\mathrm{ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{E}), \mathcal{K})$.

Assume that there is a set $Q \in \mathcal{E} - \widetilde{\mathcal{M}}$. Denoting by $\check{\mathcal{K}}$ the lattice generated by $\widetilde{\mathcal{M}} \cup \{Q\}$, we have $\alpha(\mathcal{A} \cup \check{\mathcal{K}}) = \alpha(\mathcal{B} \cup \{Q\})$ with $\mathcal{B} := \alpha(\mathcal{A} \cup \widetilde{\mathcal{M}})$. It follows $Q \notin \mathcal{B}$. By 2.2, there exists an element $\check{\mu} \in \mathrm{ex\,ba}(\widetilde{\varrho}, \alpha(\mathcal{A} \cup \check{\mathcal{K}}), \mathcal{K})$.

Next we shall prove $\check{\mu} \in \mathrm{ex\,ba}(\mu, \alpha(\mathcal{A} \cup \check{\mathcal{K}}), \mathcal{K})$ which implies $(\check{\mathcal{K}}, \check{\mu}) \in \Gamma$. On the other hand, $(\widetilde{\mathcal{M}}, \widetilde{\varrho}) \leq (\check{\mathcal{K}}, \check{\mu})$ and $\widetilde{\mathcal{M}} \neq \check{\mathcal{K}}$ which, however, is in contrast to the maximality of $(\widetilde{\mathcal{M}}, \widetilde{\varrho})$.

It is obvious that $\check{\mu} \in \mathrm{ba}(\mu, \alpha(\mathcal{A} \cup \check{\mathcal{K}}), \mathcal{K})$. To prove the extremality of $\check{\mu}$, let $\check{\mu} = \frac{1}{2}(\mu_1 + \mu_2)$ with $\mu_1, \mu_2 \in \mathrm{ba}(\mu, \alpha(\mathcal{A} \cup \check{\mathcal{K}}), \mathcal{K})$ and define $\widetilde{\mu}_i := \mu_i \mid \alpha(\mathcal{A} \cup \widetilde{\mathcal{M}})$, $i = 1, 2$. For $B \in \mathcal{B}$, $\widetilde{\varrho}(B) = \check{\mu}(B) = \frac{1}{2}(\widetilde{\mu}_1(B) + \widetilde{\mu}_2(B))$, i.e. $\widetilde{\varrho} = \frac{1}{2}(\widetilde{\mu}_1 + \widetilde{\mu}_2)$.

Since $\widetilde{\varrho} \in \operatorname{ex}\operatorname{ba}(\mu, \alpha(\mathcal{A} \cup \widetilde{\mathcal{M}}))$ by 2.1, we infer $\widetilde{\mu}_1 = \widetilde{\mu}_2 = \widetilde{\varrho}$. Consequently, $\mu_1, \mu_2 \in \operatorname{ba}(\widetilde{\varrho}, \alpha(\mathcal{A} \cup \check{\mathcal{K}}))$. As $\check{\mu} \in \operatorname{ex}\operatorname{ba}(\widetilde{\varrho}, \alpha(\mathcal{A} \cup \check{\mathcal{K}}))$ by 2.1, we obtain $\mu_1 = \mu_2$ proving $\check{\mu} \in \operatorname{ex}\operatorname{ba}(\mu, \alpha(\mathcal{A} \cup \check{\mathcal{K}}), \mathcal{K})$. □

Corollary 2.4. $\operatorname{ex}\operatorname{ba}(\mu, \alpha(\mathcal{A} \cup \mathcal{E}), \mathcal{L}) \neq \varnothing$ *for every sublattice* $\mathcal{E}$ *of* $\mathcal{F}(\mathcal{L})$.

Proof: Since $\mathcal{K} \subset \mathcal{L}$ and $\mathcal{K}$ μ-approximates $\mathcal{A}$, so does $\mathcal{L}$. Thus our claim follows from 2.3 (with $\mathcal{L}$ instead of $\mathcal{K}$). □

In case $\mathcal{A} = \alpha(\mathcal{K})$, the assumption that $\mathcal{K}$ μ-approximates $\mathcal{A}$ is equivalent to $\mathcal{K}$-regularity of μ. Thus we obtain from 2.4

Corollary 2.5 ([2, Theorem 2.3])**.** *Every* $\mathcal{K}$*-regular content on* $\alpha(\mathcal{K})$ *admits an extremal extension to an* $\mathcal{L}$*-regular content on* $\alpha(\mathcal{L})$.

Our next result is concerned with the existence of extremal measure extensions.

Theorem 2.6. *If* μ *is a measure and* $\mathcal{K}$ *is sequentially dominated by* $\mathcal{A}$*, then* $\operatorname{ex}\operatorname{ca}(\mu, \sigma(\mathcal{A} \cup \mathcal{E}), \mathcal{K}_\delta) \neq \varnothing$ *for every sublattice* $\mathcal{E}$ *of* $\mathcal{F}(\mathcal{K}_\delta)$.

Proof: Fix some sublattice $\mathcal{E}$ of $\mathcal{F}(\mathcal{K}_\delta)$ and define $\mathcal{B} := \alpha(\mathcal{A} \cup \mathcal{E})$. By 2.4, there exists an element $\varrho \in \operatorname{ex}\operatorname{ba}(\mu, \mathcal{B}, \mathcal{K}_\delta)$. To show the countable additivity of ϱ, consider a sequence (B_n) of sets from $\mathcal{B}$ with $B_n \downarrow \varnothing$. For any $\varepsilon > 0$ and $n \in N$, choose $C_n \in \mathcal{B}$ and $K_n \in \mathcal{K}_\delta$ such that $C_n \subset K_n \subset B_n$ and $\varrho(B_n - C_n) < \varepsilon \cdot 2^{-n}$. Then $D_n := \bigcap_{i=1}^n C_i \subset \bigcap_{i=1}^n K_i \subset B_n$ and $\varrho(B_n - D_n) \leq \varrho(\bigcup_{i=1}^n (B_i - C_i)) \leq \sum_{i=1}^n \varrho(B_i - C_i) < \varepsilon$ for $n \in N$. Furthermore, $K_n' := \bigcap_{i=1}^n K_i \in \mathcal{K}_\delta$ and $K_n' \downarrow \varnothing$. Since also $\mathcal{K}_\delta$ is sequentially dominated by $\mathcal{A}$, there is a sequence (A_n) of $\mathcal{A}$-sets satisfying $A_n \downarrow \varnothing$ and $K_n' \subset A_n$ for $n \in N$. This implies $\varrho(B_n) \leq \varrho((B_n - D_n) \cup A_n) \leq \varrho(B_n - D_n) + \varrho(A_n) < \varepsilon + \mu(A_n) < 2\varepsilon$ for all sufficiently large n. Therefore ϱ is a measure.

Denote by $\widetilde{\varrho}$ the unique measure extension of ϱ to $\sigma(\mathcal{B}) = \sigma(\mathcal{A} \cup \mathcal{E})$. Then $\widetilde{\varrho} \in \operatorname{ca}(\mu, \sigma(\mathcal{B}), \mathcal{K}_\delta)$ by [8, (2.10)]. To prove $\widetilde{\varrho} \in \operatorname{ex}\operatorname{ca}(\mu, \sigma(\mathcal{B}), \mathcal{K}_\delta)$ consider $\widetilde{\varrho}_1, \widetilde{\varrho}_2 \in \operatorname{ca}(\mu, \sigma(\mathcal{B}), \mathcal{K}_\delta)$ with $\widetilde{\varrho} = \frac{1}{2}(\widetilde{\varrho}_1 + \widetilde{\varrho}_2)$. Let $\varrho_i := \widetilde{\varrho}_i \mid \mathcal{B}$ for $i = 1, 2$. Then $\varrho = \frac{1}{2}(\varrho_1 + \varrho_2)$. As $\mathcal{K}_\delta$ ϱ-approximates $\mathcal{B}$ and $\frac{1}{2}\varrho_i \leq \varrho$, $\mathcal{K}_\delta$ also ϱ_i-approximates $\mathcal{B}$ which implies $\varrho_i \in \operatorname{ba}(\mu, \mathcal{B}, \mathcal{K}_\delta)$ for $i = 1, 2$. Since $\varrho \in \operatorname{ex}\operatorname{ba}(\mu, \mathcal{B}, \mathcal{K}_\delta)$, we conclude $\varrho_1 = \varrho_2$ and hence $\widetilde{\varrho}_1 = \widetilde{\varrho}_2$. □

Corollary 2.7. *If* $\mathcal{K}$ *is semicompact, then* $\operatorname{ex}\operatorname{ca}(\mu, \sigma(\mathcal{A} \cup \mathcal{E}), \mathcal{K}_\delta) \neq \varnothing$ *for every sublattice* $\mathcal{E}$ *of* $\mathcal{F}(\mathcal{K}_\delta)$.

Proof: The semicompactness of $\mathcal{K}$ implies that both μ is a measure and $\mathcal{K}$ is sequentially dominated by $\mathcal{A}$. Thus the assertion follows from 2.6. □

Under the additional assumption $\mathcal{K} \subset \mathcal{A}$, the previous results can be strengthened in the following way, thus obtaining an "extremal version" of the extension theorem 3.6 of [1].

Theorem 2.8. *Assume* $\mathcal{K} \subset \mathcal{A}$.

(a) *Then* $\operatorname{ex}\operatorname{ba}(\mu, \mathcal{B}, \mathcal{L}) \neq \varnothing$ *for every algebra* $\mathcal{B}$ *satisfying* $\mathcal{A} \cup \mathcal{L} \subset \mathcal{B} \subset \alpha(\mathcal{A} \cup \mathcal{F}(\mathcal{K}) \cup \mathcal{F}(\mathcal{L}))$.

(b) *If, in addition, μ is a measure and $\mathcal{L}$ is sequentially dominated by $\sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta))$, then $\operatorname{ex}\operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta) \neq \varnothing$ for every σ-algebra $\mathcal{B}$ satisfying $\mathcal{A} \cup \mathcal{L} \subset \mathcal{B} \subset \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$.*

Proof: We only prove (b), since the (simpler) proof of (a) can be performed in the same way.

(1) We first consider the special case $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$. Define $\mathcal{C} = \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta))$, and let ν be the $\mathcal{K}_\delta$-regular measure on $\mathcal{C}$ extending μ that has been constructed in the proof of [1, 3.6 (b)]. Since $\mathcal{L}$ is sequentially dominated by $\mathcal{C}$, so is $\mathcal{L}_\delta$. In addition, $\mathcal{K}_\delta \subset \mathcal{L}_\delta$ and $\mathcal{B} = \sigma(\mathcal{C} \cup \mathcal{F}(\mathcal{L}_\delta))$. Thus, by 2.6, there exists an element $\tau \in \operatorname{ex}\operatorname{ca}(\nu, \mathcal{B}, \mathcal{L}_\delta)$. Clearly $\tau \in \operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$. To prove $\tau \in \operatorname{ex}\operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ consider $\tau_1, \tau_2 \in \operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ with $\tau = \frac{1}{2}(\tau_1 + \tau_2)$. Then

$$\nu(C) \leq \tau_i(C) \quad \text{for} \quad C \in \mathcal{C} \quad \text{and} \quad i = 1, 2. \tag{2.1}$$

Assume that (2.1) fails to be true. Then $\nu(C) > \tau_i(C)$ for some $C \in \mathcal{C}$ and some $i \in \{1, 2\}$. Thus we can find a $\mathcal{K}_\delta$-set $\overline{K}$ satisfying $\overline{K} \subset C$ and $\nu(\overline{K}) > \tau_i(C)$. Choosing a sequence (K_n) in $\mathcal{K}$ such that $K_n \downarrow \overline{K}$, we obtain the contradiction $\inf_n \mu(K_n) = \inf_n \nu(K_n) = \nu(\overline{K}) > \tau_i(C) \geq \tau_i(\overline{K}) = \inf_n \tau_i(K_n) = \inf_n \mu(K_n)$. Thus (2.1) holds true.

Since also $\tau_i(X) = \mu(X) = \nu(X)$ for $i = 1, 2$, we infer from (2.1) $\tau_1 \mid \mathcal{C} = \tau_2 \mid \mathcal{C} = \nu$. Thus $\tau_1, \tau_2 \in \operatorname{ca}(\nu, \mathcal{B}, \mathcal{L}_\delta)$ which together with $\tau \in \operatorname{ex}\operatorname{ca}(\nu, \mathcal{B}, \mathcal{L}_\delta)$ implies $\tau_1 = \tau_2$. So $\tau \in \operatorname{ex}\operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$.

(2) Now we consider an arbitrary σ-algebra $\mathcal{B}$ satisfying $\mathcal{A} \cup \mathcal{L} \subset \mathcal{B} \subset \mathcal{E}$ where $\mathcal{E} := \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$. By the special case (1), there exists an element $\varrho \in \operatorname{ex}\operatorname{ca}(\mu, \mathcal{E}, \mathcal{L}_\delta)$. Then $\nu := \varrho \mid \mathcal{B} \in \operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$. To prove $\nu \in \operatorname{ex}\operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ consider $\nu_1, \nu_2 \in \operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ with $\nu = \frac{1}{2}(\nu_1 + \nu_2)$. For every $E \in \mathcal{E}$, $\varrho(E) = \sup\{\varrho(L) : L \in \mathcal{L}_\delta, L \subset E\} = \sup\{\nu(L) : L \in \mathcal{L}_\delta, L \subset E\} = \frac{1}{2}(\sup\{\nu_1(L) : L \in \mathcal{L}_\delta, L \subset E\} + \sup\{\nu_2(L) : L \in \mathcal{L}_\delta, L \subset E\}) \leq \frac{1}{2}(\widetilde{\nu}_1(E) + \widetilde{\nu}_2(E))$ where $\widetilde{\nu}_i$ denotes an arbitrary content on $\mathcal{E}$ that extends ν_i, $i = 1, 2$. It follows $\varrho \leq \frac{1}{2}(\widetilde{\nu}_1 + \widetilde{\nu}_2)$ as well as $\frac{1}{2}(\widetilde{\nu}_1(X) + \widetilde{\nu}_2(X)) = \frac{1}{2}(\nu_1(X) + \nu_2(X)) = \mu(X) = \varrho(X)$ which implies

$$\varrho = \frac{1}{2}(\widetilde{\nu}_1 + \widetilde{\nu}_2). \tag{2.2}$$

From (2.2) we infer both the countable additivity and the $\mathcal{L}_\delta$-regularity of $\widetilde{\nu}_i$, $i = 1, 2$. Therefore $\varrho \in \operatorname{ex}\operatorname{ca}(\mu, \mathcal{E}, \mathcal{L}_\delta)$ and (2.2) imply $\widetilde{\nu}_1 = \widetilde{\nu}_2$ and hence $\nu_1 = \nu_2$. So $\nu \in \operatorname{ex}\operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$. □

An immediate consequence of 2.8 (b) is [2, Theorem 2.4], various applications of which are gathered in Section 3 of [2].

The assumptions of 2.8 (b) are, in particular, satisfied if the lattice $\mathcal{L}$ is semicompact. Thus we obtain

Corollary 2.9. *If $\mathcal{L}$ is semicompact and $\mathcal{K} \subset \mathcal{A}$ holds, then* $\operatorname{ex}\operatorname{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta) \neq \varnothing$ *for every σ-algebra $\mathcal{B}$ satisfying $\mathcal{A} \cup \mathcal{L} \subset \mathcal{B} \subset \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$.*

The following result is an application of 2.9.

Corollary 2.10. *Let $\mathcal{C}$, $\mathcal{D}$ be lattices of subsets of X such that $\mathcal{C} \subset \mathcal{D} \subset \mathcal{F}(\mathcal{C}_\delta)$. If $\mathcal{C}$ is semicompact and $\mathcal{A} \subset \sigma(\mathcal{D})$, then every $\mathcal{C} \cap \mathcal{A}$-regular content on $\mathcal{A}$ admits an extremal extension to a $\mathcal{C}_\delta$-regular measure on $\sigma(\mathcal{D})$.*

Proof: The claim follows with $\mathcal{K} = \mathcal{C} \cap \mathcal{A}$ and $\mathcal{L} = \mathcal{C}$ from 2.9. □

The assumptions of 2.10 are, in particular, satisfied if $\mathcal{C}$, $\mathcal{D}$ are the lattices of compact, respectively closed, subsets of a Hausdorff topological space. Thus one obtains from 2.10 an "extremal version" of Henry's extension theorem (cf. [10, Theorem 16, p. 51]).

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