# Realcompactification of frames

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*Abstract.* We give a construction of Wallman-type realcompactifications of a frame $L$ by considering regular sub $\sigma$-frames the join of which generates $L$. In particular, we show that the largest such regular sub $\sigma$-frame gives rise to the universal realcompactification of $L$.

*Keywords:* frame, $\sigma$-frame, realcompactification

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## 1. Background

We shall be concerned with completely regular frames and regular sub $\sigma$-frames which join generate them. A $\sigma$-frame is a bounded distributive lattice $A$ for which every countable subset $S$ has a join such that

$$a \wedge \bigvee_A S = \bigvee_A \{a \wedge s \mid s \in S\}$$

for each $a \in A$. We denote the top and bottom elements of a $\sigma$-frame $A$ respectively by $1_A$ and $0_A$. $\sigma$-Frame homomorphisms preserve countable joins and finite meets. The resulting category is denoted $\sigma\mathbf{Frm}$. Extending the above notions by allowing arbitrary subsets and arbitrary joins in the definitions leads to the notions of a frame and a frame homomorphism, and the corresponding category **Frm** of frames. For further details on frames and $\sigma$-frames we refer to Johnstone [4] and Banaschewski-Gilmour [1].

Given a bounded distributive lattice $A$, and supposing $a, b \in A$, then $a$ is said to be rather below $b$ (written $a \prec b$) if there exists $s \in A$ such that $a \wedge s = 0_A$ and $b \vee s = 1_A$. We say $a$ is completely below $b$ (written $a \prec\!\prec b$) if there is a family $\{x_i \mid i \in \mathbb{Q} \cap [0,1]\}$ of elements in $A$ satisfying $x_0 = a$, $x_1 = b$ and $i < j$ implies $x_i \prec x_j$. The lattice $A$ is called *normal* if for each pair $a, b$ of elements of $L$ with $a \vee b = 1_A$, there exists $u, v \in A$ such that $a \vee u = 1_A = b \vee v$ and $u \wedge v = 0_A$. In a normal lattice, the two relations $\prec$ and $\prec\!\prec$ coincide.

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Given a frame $L$, an element $a \in L$ is called a completely regular element if $a$ is a join of elements completely below it. We say $L$ is completely regular if each $a \in L$ is completely regular. A $\sigma$-frame $A$ is regular if each $a \in A$ is a countable join of elements rather below it. In [1] it is shown that regular $\sigma$-frames are normal. The full subcategories of **Frm** and $\sigma$**Frm** consisting of completely regular frames and regular $\sigma$-frames are denoted **CrgFrm** and **Reg**$\sigma$**Frm** respectively.

One important notion, particularly in the study of realcompact frames, is that of the cozero part of a frame. Given a completely regular frame $L$, an element $a \in L$ is called a cozero element of $L$ if $a = h(\mathbb{R}\backslash\{0\})$ for some frame homomorphism $\mathcal{O}\mathbb{R} \xrightarrow{h} L$. The sublattice of all cozero elements of $L$ is in fact the largest regular sub $\sigma$-frame of $L$ and is denoted $CozL$. It is immediate from the definition that frame homomorphisms preserve cozero elements. Thus, $Coz$ is a functor from **CrgFrm** to **Reg**$\sigma$**Frm**. The following useful result, as well as other important properties of the cozero-set lattice can be extracted from [4].

**Lemma 1.1.** *Let $L$ be a completely regular frame. Then $a \in CozL$ iff $a = \bigvee_L a_n$, for some sequence $(a_n)$ in $L$ with $a_i \prec\!\prec a_{i+1}$ for all $i \in \mathbb{N}$.*

## 2. Realcompact frames

Madden and Vermeer [5] obtained a localic version of realcompactness by way of the following result:

**Theorem 2.1.** *For a completely regular frame $L$, the following are equivalent:*

(i) *$L$ is Lindelöf.*
(ii) *$L$ is a closed quotient of $\bigoplus_I \mathcal{O}\mathbb{R}$, for some index set $I$.*
(iii) *$L \cong \mathcal{H}CozL$.*

**Remark.** Property (ii) above is suggestive of the well-known characterization of realcompactness. For this reason Schlitt [6] refers to this notion as Stone-realcompactness. However, the frame of open sets of a realcompact topological space need not be Stone-realcompact. Consider an uncountable discrete space $X$ with a non-measurable cardinality. Then $X$ is realcompact, but $\mathcal{O}X$ is not Lindelöf, and therefore not Stone-realcompact. Schlitt formulated a definition of realcompactness, for which a space $X$ is realcompact if $\mathcal{O}X$ is realcompact (which he refers to as Herrlich-realcompactness, or H-realcompactness); and it is this definition which we adopt below.

**Definition 2.2.** For any frame $L$, an ideal $I \subseteq L$ is $\sigma$-proper if $\bigvee_L S \neq 1_L$ for any countable $S \subseteq I$. $I$ is said to be completely proper if $\bigvee_L I \neq 1_L$.

**Definition 2.3** (Schlitt)**.** A completely regular frame $L$ is realcompact if every $\sigma$-proper maximal completely regular ideal is completely proper.

The definition given above differs from the original definition given by Schlitt, which he chose for the reason of avoiding choice principles. On the assumption of the axiom of choice, the two definitions are equivalent, as pointed out by Schlitt.

Given a bounded distributive lattice $L$, $MaxL$ denotes the topological space consisting of all maximal ideals on $L$ with a base for open sets consisting of the sets $\{I \in MaxL \mid a \notin I\}$, where $a \in L$. We denote by $Max_c L$ the topological space consisting of all maximal completely regular ideals with basic open sets of the form $\{I \in Max_c L \mid a \notin I\}$, where $a \in L$. We also denote by $k_L(a)$ the set $\{b \in L \mid b \prec\!\prec a\}$.

The following lemma is a generalization of a result obtained by Schlitt [6], and allows for our characterization of realcompactness in Proposition 2.5.

**Lemma 2.4.** *For any completely regular frame $L$, $Max_c L \cong MaxCozL$.*

PROOF: Consider the maps

$$\varphi : Max_c L \longrightarrow MaxCozL$$
$$\psi : MaxCozL \longrightarrow Max_c L$$

defined by

$$\varphi(I) = \{a \in CozL \mid k_L(a) \subseteq I\}$$
$$\psi(J) = \{u \in L \mid u \prec\!\prec v, \text{ for some } v \in J\}\,.$$

We show that the maps $\varphi$ and $\psi$ are indeed well-defined. It is clear that $\varphi(I)$ is an ideal in $CozL$. To see that $\varphi(I)$ is maximal, consider $a \in CozL$ with $a \notin \varphi(I)$. Then $k_L(a) \nsubseteq I$, and consequently $k_L(a) \vee I = L$, since $I$ is a maximal completely regular ideal in $L$. Thus, $k_L(a) \vee k_L(s) = L$ and hence $a \vee s = 1_L$, for some $s \in I$. But $s \in \varphi(I)$ since $k_L(s) \subseteq I$, so it follows that $\varphi(I)$ is maximal.

To see that $\psi$ is well-defined, let $J$ be a maximal ideal in $CozL$. Since the relation $\prec\!\prec$ interpolates, it follows that $\psi(J)$ is a completely regular ideal in $L$. Suppose that $K$ is a completely regular ideal properly containing $\psi(J)$. Let $u \in K \backslash \psi(J)$. Then there exists $v \in K$ such that $u \prec\!\prec v$. From Lemma 1.1 there exists $w \in CozL$ such that $u \prec\!\prec w \prec\!\prec v$, ie $w \notin J$. Now, $J$ is maximal so there exists $a \in J$ such that $a \vee w = 1_L$. Since $CozL$ is normal, there exists $t \prec\!\prec a$ with $t \vee w = 1_L$. But then $K$ is not a proper ideal, since $t \in \psi(J) \subseteq K$, and $w \vee t = 1_L$.

It is easily seen that $\varphi$ and $\psi$ are continuous and are inverse to each other. $\square$

**Proposition 2.5.** *A completely regular frame $L$ is realcompact if every $\sigma$-proper maximal ideal in $CozL$ is completely proper.*

**Remark.** From the above Proposition it is clear that Schlitt's definition of realcompactness is conservative, ie a completely regular space $X$ is realcompact iff $\mathcal{O}X$ is realcompact. Also, every Lindelöf frame is realcompact, since a completely regular frame $L$ is Lindelöf iff every $\sigma$-ideal in $CozL$ is completely proper.

The full subcategory of realcompact frames is denoted $\mathbb{R}\mathbf{KFrm}$.

## 3. Realcompactification of frames

**Definition 3.1.** Let $L$ be a completely regular frame. Then $(M, h)$ is a realcompactification of $L$ if $M$ is a realcompact frame, and $M \xrightarrow{h} L$ is a dense surjection.

For any regular $\sigma$-frame $A$, we denote by $\mathcal{H}A$ the frame of all $\sigma$-ideals of $A$. It was shown by Madden and Vermeer [5] that $\mathcal{H}A$ is in fact regular Lindelöf. We construct a realcompactification of a completely regular frame $L$ by forming a suitable quotient of $\mathcal{H}A$, where $A$ is a regular sub $\sigma$-frame of $L$, with the property that each $a \in L$ can be written as a join of elements in $A$, ie $A$ join generates $L$. The technique used here is essentially an adaptation of that used by Schlitt [6] in his construction of the Hewitt realcompactification for frames, and is motivated by the construction of realcompactification of spaces using Alexandroff bases, and the adjunction between Alexandroff spaces and regular $\sigma$-frames obtained by Gilmour [3]. As this construction is akin to that of Wallman for compactifications, we will call the realcompactification obtained the Wallman realcompactification of $L$ with respect to $A$.

Let $L$ be a completely regular frame, and let $A$ be a regular sub $\sigma$-frame join generating $L$. Define $\mathcal{H}A \xrightarrow{h_L} \mathcal{H}A$ by

$$h_L I = \downarrow (\bigvee_L I) \cap \bigcap \{J \in \sigma PMaxA \mid I \subseteq J\},$$

where $\sigma PMaxA$ is the collection of all $\sigma$-proper maximal ideals in $A$.

**Lemma 3.2.** *The map $h_L$, given above, is a nucleus.*

Proof: We firstly show that $h_L$ is well-defined. Let $I \in \mathcal{H}A$. Suppose $S$ is a countable subset of $h_L I$, then $\bigvee_L S \in \downarrow (\bigvee_L I)$. Let $J$ be a $\sigma$-proper maximal ideal containing $I$. Then $J$ is a $\sigma$-ideal, and since $S \subseteq J$, it follows that $\bigvee_L S \subseteq J$. Let $u \in h_L I$, and suppose $v \leq u$. Then $v \in \downarrow (\bigvee_L I)$, since $u \in \downarrow (\bigvee_L I)$. Given any $\sigma$-proper maximal ideal $J \supseteq I$. Then $u \in J$, and hence $v \in J$, ie $v \in h_L I$. Thus $h_L I \in \mathcal{H}A$ and hence $h_L$ is well-defined. We now show that $h_L$ is a nucleus:

(i) It is clear that $I \subseteq h_L I$.

(ii) Since $h_L$ is order-preserving, it follows that $h_L(I \cap K) \subseteq h_L I \cap h_L K$. Now, suppose $u \in h_L I \cap h_L K$. Then, $u \leq \bigvee_L I \wedge \bigvee_L K = \bigvee_L (I \cap K)$. Suppose $J \supseteq I \cap K$ is a $\sigma$-proper maximal ideal in $A$. Since $J$ is maximal, and hence prime, $J \supseteq I$, or $J \supseteq K$. But then $u \in J$ and so $u \in h_L(I \cap K)$, ie $h_L(I \cap K) = h_L I \cap h_L K$.

(iii) Let $u \in h_L^2 I$. Then $u \leq \bigvee_L h_L I \leq \bigvee_L I$. Now, suppose $I \subseteq J$, where $J$ is a $\sigma$-proper maximal ideal in $A$. Then $h_L I \subseteq J$, by definition of $h_L$. Thus, $u \in J$ and hence $u \in h_L I$, ie $h_L^2 I \subseteq h_L I$.

□

Let $(\mathcal{H}A)_{h_L} = \{I \in \mathcal{H}A \mid h_L I = I\}$ be the quotient frame corresponding to the nucleus $h_L$.

**Lemma 3.3.** *Let $L$ be a completely regular frame, and let $A$ be a regular sub $\sigma$-frame join generating $L$. Then $Coz(\mathcal{H}A)_{h_L} \cong A$.*

Proof: We show that the $\sigma$-frame $Coz(\mathcal{H}A)_{h_L}$ is precisely the $\sigma$-frame of principal ideals in $A$. Firstly, suppose $I \in Coz(\mathcal{H}A)_{h_L}$. Then there exists a sequence $(J_n)$ in $(\mathcal{H}A)_{h_L}$ with $J_1 \prec\prec J_2 \prec\prec J_3 \prec\prec \cdots$, and $I = \bigvee_{(\mathcal{H}A)_{h_L}} J_i$. Now, for each $n \in \mathbb{N}, J_n \prec I$. Thus, for each $n \in \mathbb{N}$, there exists $S_n \in (\mathcal{H}A)_{h_L}$ such that $J_n \wedge S_n = \{0_A\}$ and $I \vee S_n = A$. For each $n \in \mathbb{N}$, take $s_n \in S_n$. Then $j_{n_i} \wedge s_n = 0_L$ for each $j_{n_i} \in J_n$, and there exists $k_n \in I$ such that $k_n \vee s_n = 1_L$, ie $j_{n_i} \prec k_n$ for each $j_{n_i} \in J_n$. Let $k = \bigvee_L k_n$. Then $k \in I$, since $I$ is a $\sigma$-ideal. Also, $j_{n_i} \prec k$, for each $j_{n_i} \in J_n$ and each $n \in \mathbb{N}$. But then $J_n \subseteq \downarrow k$ for each $n \in \mathbb{N}$. Hence $\bigvee_{\mathcal{H}A} J_i \subseteq \downarrow k$. Now, $h_L$ is order preserving, and therefore $h_L \bigvee_{\mathcal{H}A} J_i = \bigvee_{(\mathcal{H}A)_{h_L}} J_i \subseteq h_L \downarrow k = \downarrow k$. Since $k \in I$, it follows that $I = \downarrow k$.

On the other hand, let $a \in A$, Then, since $A$ is a regular $\sigma$-frame, $a = \bigvee_A S$, where $S = \{a_i \mid i \in \mathbb{N}\}$ and $a_i \prec a$ for each $i \in \mathbb{N}$. Now, the relation $\prec$ interpolates in regular $\sigma$-frames, so $\downarrow a_i \prec\prec \downarrow a$ in $\mathcal{H}A$. But then there is an $I_i \in Coz\mathcal{H}A$ such that $\downarrow a_i \prec\prec I_i \prec\prec \downarrow a$. Hence $\bigvee_{\mathcal{H}A} I_i = \downarrow a$, and thus $\downarrow a \in Coz\mathcal{H}A$ since it is a countable join of cozero elements. □

**Lemma 3.4.** *The frame $(\mathcal{H}A)_{h_L}$ is realcompact.*

Proof: Let $\mathcal{J}$ be a $\sigma$-proper maximal ideal in $Coz(\mathcal{H}A)_{h_L}$. Then $\mathcal{J} = \{\downarrow a \mid a \in J\}$, where $J$ is a $\sigma$-proper maximal ideal in $A$. But then $\bigvee_{(\mathcal{H}A)_{h_L}} \mathcal{J} = h_L(\bigvee_{\mathcal{H}A} \mathcal{J}) = h_L(J) = J$. Thus $\bigvee_{(\mathcal{H}A)_{h_L}} \mathcal{J}$ is a $\sigma$-proper maximal ideal in $A$, so that $\mathcal{J}$ is completely proper in $(\mathcal{H}A)_{h_L}$. □

**Proposition 3.5.** *The map $(\mathcal{H}A)_{h_L} \xrightarrow{j_L} L$ given by join is a dense surjection.*

Proof: Firstly note that for any family $\{I_\lambda \mid \lambda \in \Lambda\} \subseteq (\mathcal{H}A)_{h_L}$, we have:

$$\begin{aligned} j_L \bigvee_{(\mathcal{H}A)_{h_L}} I_\lambda &= \bigvee_L h_L \bigvee_{\mathcal{H}A} I_\lambda \\ &= \bigvee_L \bigvee_{\mathcal{H}A} I_\lambda \\ &= \bigvee_L j_L(I_\lambda)\,. \end{aligned}$$

Obviously, $j_L$ preserves binary meets, so $j_L$ is indeed a frame homomorphism. Now, for each $a \in L$, $h_L(\downarrow a \cap A) = (\downarrow a \cap A)$, and hence $(\downarrow a \cap A) \in (\mathcal{H}A)_{h_L}$. Also $j_L(\downarrow a \cap A) = a$ for each $a \in L$, since $A$ join generates $L$, so $j_L$ is surjective. Suppose $j_L(I) = 0_L$, then $I = \{0_L\}$, ie $j_L$ is dense. □

**Definition 3.6.** Let $L$ be a completely regular frame, and let $A$ be a regular sub $\sigma$-frame of $L$, join generating $L$. Then $((\mathcal{H}A)_{h_L}, j_L)$ is called the Wallman realcompactification of $L$ with respect to $A$. $(\mathcal{H}A)_{h_L}$ is denoted $\upsilon_A L$.

**Lemma 3.7.** *If $L$ is realcompact, then $\upsilon_{CozL}L \cong L$.*

PROOF: It suffices to show that $\upsilon_{CozL}L \xrightarrow{j_L} L$ is codense. Suppose $j_L(I) = 1_L$, ie $I$ is not completely proper. Now, if $I \neq 1_{IdlCozL}$, then since $I \in \upsilon_{CozL}L$, there is a $\sigma$-proper maximal ideal $J \supseteq I$. But then $J$ is completely proper, since $L$ is realcompact. This contradicts the fact that $I$ is not completely proper. Hence $I = 1_{IdlCozL}$. □

**Remark.** The above result cannot be generalized to arbitrary regular sub $\sigma$-frames of $L$. As a counterexample, let $L = \mathcal{P}\mathbb{R}$, the power set of $\mathbb{R}$ and let $A$ be the collection of all countable and cocountable subsets of $\mathbb{R}$. Then $A$ is a regular sub $\sigma$-frame join generating $L$, but $\upsilon_A L \ncong L$. For if $\upsilon_A L \cong L$, then $Coz\upsilon_A L$ and $CozL$ would be isomorphic as $\sigma$- frames. But $Coz\upsilon_A L \cong A$, and $CozL = L$, from which it would follow that $L$ and $A$ are isomorphic as $\sigma$-frames.

**Definition 3.8.** Let $L$ and $M$ be completely regular frames and let $A$ be a regular sub $\sigma$-frame join generating $L$. Then the map $M \xrightarrow{h} L$ is said to be over $A$ if $M$ is generated by a regular sub $\sigma$-frame $B$ such that $h[B] \subseteq A$.

**Lemma 3.9.** *Let $L$ be a completely regular frame, and let $A$ be a regular sub $\sigma$-frame join generating $L$. If $\{K_\lambda \mid \lambda \in \Lambda\}$ is a collection of $\sigma$-ideals in $A$, then $h_L \bigvee_{\mathcal{H}A} h_L K_\lambda = h_L \bigvee_{\mathcal{H}A} K_\lambda$.*

PROOF: So as not to complicate the notation, we shall suppress mentioning the index set $\Lambda$.

$$\begin{aligned}
h_L \bigvee_{\mathcal{H}A} h_L K_\lambda &= h_L \bigvee_{\mathcal{H}A} (\downarrow (\bigvee_L K_\lambda) \cap \bigcap \{K \in \sigma PMaxA \mid K \supseteq K_\lambda\}) \\
&= h_L (\bigvee_{\mathcal{H}A} \downarrow (\bigvee_L K_\lambda) \cap \bigvee_{\mathcal{H}A} \bigcap \{K \in \sigma PMaxA \mid K \supseteq K_\lambda\}) \\
&= h_L J, \text{ say} \\
&= \downarrow (\bigvee_L \bigvee_{\mathcal{H}A} \downarrow (\bigvee_L K_\lambda)) \cap \downarrow (\bigvee_L \bigvee_{\mathcal{H}A} \bigcap \{K \in \sigma PMaxA \mid K \supseteq K_\lambda\}) \cap \\
&\qquad \bigcap \{I \in \sigma PMaxA \mid I \supseteq J\} \\
&= \downarrow (\bigvee_L \bigvee_{\mathcal{H}A} K_\lambda) \cap \bigcap \{I \in \sigma PMaxA \mid I \supseteq J\}.
\end{aligned}$$

Let $I$ be a $\sigma$-proper maximal ideal in $A$ containing $\bigvee_{\mathcal{H}A} K_\lambda$, then $I$ contains $\bigcap \{K \in \sigma PMaxA \mid K \supseteq K_\lambda\}$, for each $\lambda$ from which it follows that $I$ contains $\bigvee_{\mathcal{H}A} \bigcap \{K \in \sigma PMaxA \mid K \supseteq K_\lambda\}$.

Consequently, $I \supseteq \bigvee_{\mathcal{H}A} \downarrow \bigvee_L K_\lambda \cap \bigvee_{\mathcal{H}A} \bigcap \{K \in \sigma PMaxA \mid K \supseteq K_\lambda\} = J$ so that

$$\bigcap \{I \in \sigma PMaxA \mid I \supseteq J\} \subseteq \bigcap \{K \in \sigma PMaxA \mid K \supseteq \bigvee_{\mathcal{H}A} K_\lambda\}$$

from which it follows that

$$\begin{aligned} h_L \bigvee_{\mathcal{H}A} h_L K_\lambda &\subseteq \downarrow (\bigvee_L \bigvee_{\mathcal{H}A} K_\lambda) \cap \bigcap \{K \in \sigma PMaxA \mid K \supseteq \bigvee_{\mathcal{H}A} K_\lambda\} \\ &= h_L \bigvee_{\mathcal{H}A} K_\lambda \,. \end{aligned}$$

For the reverse inclusion, note that $K_\lambda \subseteq h_L K_\lambda$ for each $\lambda$. Since $h_L$ is order-preserving, it follows that $h_L \bigvee_{\mathcal{H}A} K_\lambda \subseteq h_L \bigvee_{\mathcal{H}A} h_L K_\lambda$.

□

**Proposition 3.10.** *Let $L$ be a completely regular frame and let $A$ be a regular sub $\sigma$-frame join generating $L$. Then the map $\upsilon_A L \xrightarrow{j_L} L$ is the universal realcompactification of $L$ over $A$.*

Proof: Let $M$ be a realcompact frame, and suppose the frame homomorphism $M \xrightarrow{\varphi} L$ is over $A$, with $B$ a regular sub $\sigma$-frame join generating $M$ such that $h[B] \subseteq A$.

$$\begin{tikzcd} M \arrow[r, "\varphi"] & L \\ & \\ \upsilon_B M \arrow[r, "\bar{\varphi}"] & \upsilon_A L \arrow[uu, "j_L"'] \end{tikzcd}$$

Consider the map $\upsilon_B M \xrightarrow{\bar{\varphi}} \upsilon_A L$ defined by $\bar{\varphi}(J) = h_L < \varphi[J] >$, where $< \varphi[J] >$ is the $\sigma$-ideal in $A$ generated by $\varphi[J]$. (In order not to complicate the notation we shall simply denote $< \varphi[J] >$ by $\varphi[J]$.) We now show that $\bar{\varphi}$ is a frame homomorphism.

It is clear that $\bar{\varphi}$ preserves intersection. We show that $\bar{\varphi}$ preserves arbitrary joins. Let $\{J_\lambda \mid \lambda \in \Lambda\}$ be a collection of $\sigma$-ideals in $(\mathcal{H}B)_{h_M}$. Then

$$\begin{aligned} \bar{\varphi}(\bigvee_{(\mathcal{H}B)_{h_M}} J_\lambda) &= \bar{\varphi}(h_M \bigvee_{\mathcal{H}B} J_\lambda) \\ &= h_L \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda] \\ &= \downarrow \bigvee_L \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda] \cap \bigcap \{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\} \\ &= \downarrow (\varphi(\bigvee_M h_M \bigvee_{\mathcal{H}B} J_\lambda)) \cap \bigcap \{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\} \,. \end{aligned}$$

Now, for any $J \in \mathcal{H}B$, $h_M J \subseteq \downarrow (\bigvee_M J)$, and hence $\bigvee_M h_M J \leq \bigvee_M J$. Also, $J \subseteq h_M J$, so that $\bigvee_M J \leq \bigvee_M h_M J$. Thus, $\bigvee_M J = \bigvee_M h_M J$. So we get:

$$\begin{aligned} \bar{\varphi}(\bigvee_{(\mathcal{H}B)_{h_M}} J_\lambda) &= \downarrow (\varphi(\bigvee_M \bigvee_{\mathcal{H}B} J_\lambda)) \cap \bigcap \{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\} \\ &= \downarrow (\bigvee_L \varphi[\bigvee_{\mathcal{H}B} J_\lambda]) \cap \bigcap \{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\} \,. \end{aligned}$$

On the other hand,

$$\begin{aligned}\bigvee_{(\mathcal{H}A)_{h_L}} \bar{\varphi}(J_\lambda) &= h_L \bigvee_{\mathcal{H}A} h_L\varphi[J_\lambda] \\ &= h_L \bigvee_{\mathcal{H}A} \varphi[J_\lambda], \text{ from Lemma 3.9 above} \\ &= \downarrow (\bigvee_L \bigvee_{\mathcal{H}A} \varphi[J_\lambda]) \cap \bigcap\{J \in \sigma PMaxA \mid J \supseteq \varphi[\bigvee_{\mathcal{H}B} J_\lambda]\}\,.\end{aligned}$$

We now show that

$$\{J \in \sigma PMaxA \mid J \supseteq \varphi[\bigvee_{\mathcal{H}B} J_\lambda]\} = \{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\}\,.$$

It is clear that

$$\{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\} \subseteq \{J \in \sigma PMaxA \mid J \supseteq \varphi[\bigvee_{\mathcal{H}B} J_\lambda]\}\,.$$

Conversely, suppose $J$ is a $\sigma$-proper maximal ideal in $A$, containing $\varphi[\bigvee_{\mathcal{H}B} J_\lambda]$. Let $K = \bigvee_{\mathcal{H}B}\{I \in \mathcal{H}B \mid \varphi[I] \subseteq J\}$. Suppose $K'$ is an ideal properly containing $K$. Then $\varphi[K'] = 1_{IdlA}$, ie there exists $k' \in K'$ such that $\varphi(k') = 1_M$. Now, since $k' \in B$, and $B$ is a regular $\sigma$-frame, there exists a set $S = \{k_i \in B \mid i \in \mathbb{N}\}$ with $k_i \prec k'$ for each $i \in \mathbb{N}$, and $k' = \bigvee_M S$. But, since $K$ is $\sigma$-proper, there exists $j \in \mathbb{N}$ such that $k_j \notin K$, ie $\varphi(k_j) \vee p = 1_L$, for some $p \in J$. But then $\varphi(k_j^*) \leq (\varphi(k_j))^* \leq p$, where $k_j^*$ denotes the pseudocomplement of $k_j$ in $M$. Since $k_j \prec k'$, there exists $s \in B$ such that $k_j \wedge s = 0_M$ and $k' \vee s = 1_M$. But then $s \leq k_j^*$, and therefore $\varphi(s) \leq \varphi(k_j^*) \leq p$. Hence, $s \in K'$, from which it follows that $K' = 1_{IdlB}$. Thus, $K$ is a $\sigma$-proper maximal ideal in $B$, and hence $h_M K = K$. Now, $\bigvee_{\mathcal{H}B} J_\lambda \subseteq K$ so that $\varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda] \subseteq \varphi[h_M K] = \varphi[K] \subseteq J$, and hence

$$\{J \in \sigma PMaxA \mid J \supseteq \varphi[h_M \bigvee_{\mathcal{H}B} J_\lambda]\} \subseteq \{J \in \sigma PMaxA \mid J \supseteq \varphi[\bigvee_{\mathcal{H}B} J_\lambda]\}\,.$$

This gives equality, and therefore $\bar{\varphi}(\bigvee_{(\mathcal{H}B)_{h_L}} J_\lambda) = \bigvee_{(\mathcal{H}A)_{h_L}} \bar{\varphi}(J_\lambda)$.

Also, the map $M \xrightarrow{d} \upsilon_B M$ defined by $d(m) = \downarrow m$ is a frame homomorphism. Obviously, $d$ preserves binary meets. To see that $d$ preserves arbitrary joins, consider $\{m_i \mid i \in I\} \subseteq M$, with $m = \bigvee_M m_i$. Then

$$\begin{aligned}\bigvee_{\upsilon_B M} \downarrow m_i &= \downarrow \bigvee_M \bigvee_{\mathcal{H}B} \downarrow m_i \cap \bigcap\{J \in \sigma PMaxB \mid J \supseteq \bigvee_{\mathcal{H}B} \downarrow m_i\} \\ &= \downarrow m\,.\end{aligned}$$

Note that the above equality holds even if $m = 1_M$, in which case $\{J \in \sigma PMaxB \mid J \supseteq \bigvee_{\mathcal{H}B} \downarrow m_i\} = \phi$, since $M$ is realcompact.
Now,

$$\begin{aligned} j_L \cdot \bar{\varphi} \cdot d(m) &= \bigvee_L h_L \varphi[\downarrow m] \\ &= \bigvee_L h_L \downarrow \varphi(m) \\ &= \bigvee_L \downarrow \varphi(m) \\ &= \varphi(m)\,. \end{aligned}$$

Uniqueness of the map $\bar{\varphi} \cdot d$ follows from the fact that $j_L$ is dense and hence monic in the category of frames. □

**Corollary 3.11.** *Let $L$ be a completely regular frame. Then $\upsilon_{CozL} L \xrightarrow{j_L} L$ is the universal realcompactification of $L$.*

Gilmour [3] showed that the Alexandroff bases of Alexandroff spaces are precisely those bases giving rise to Wallman realcompactifications for the underlying topologies. These bases were shown to be precisely the regular sub $\sigma$-frames of the frame of open sets of the underlying topological space.

Let $X$ be a topological space with $A$ an Alexandroff base on $X$. We shall denote by $\upsilon_A X$, the Wallman realcompactification of $X$ with respect to $A$.

Recall that the spectrum $\Sigma L$ of a given frame $L$ is the topological space consisting of all frame homomorphisms $\xi : L \to \mathbf{2}$ with open sets $\Sigma_a = \{\xi \mid \xi(a) = 1\}$. A frame $L$ is called spatial if $\mathcal{O}\Sigma L \cong L$, ie the map $\eta : L \to \mathcal{O}\Sigma L$ given by $\eta(a) = \Sigma_a$ is an isomorphism.

An element $s \in L$ is called *prime* if $s = a \wedge b \Rightarrow s = a$ or $s = b$. It is a well-known fact that $L$ is spatial iff each $a \in L$ is a meet of prime elements.

**Corollary 3.12.** *Let $X$ be a topological space, and let $A$ be an Alexandroff base on $X$. Then $\upsilon_A \mathcal{O}X \cong \mathcal{O}\upsilon_A X$.*

Proof: Let $e : X \hookrightarrow \upsilon_A X$ be the Wallman realcompactification of the space $X$ with respect to the Alexandroff base $A$. Now, $\mathcal{O}\upsilon_A X$ is a realcompact frame, and $\mathcal{O}e$ is over $A$, and thus there is a unique map $\mathcal{O}\upsilon_A X \xrightarrow{\varphi} \upsilon_A \mathcal{O}X$ such that $j_{\mathcal{O}X} \cdot \varphi = \mathcal{O}e$, where the map $\upsilon_A \mathcal{O}X \xrightarrow{j_{\mathcal{O}X}} \mathcal{O}X$ takes an ideal to its union.

We now show that every proper ideal $I \in \upsilon_A \mathcal{O}X$ is a meet of prime ideals. Note that $I = h_{\mathcal{O}X} I = \downarrow \bigcup I \cap \bigcap\{J \in \sigma PMaxA \mid J \supseteq I\}$. If $\downarrow \bigcup I = 1_{\mathcal{O}X}$, then we are done, since maximal ideals are prime. Otherwise, if $\downarrow \bigcup I = a \neq 1_{\mathcal{O}X}$ then $a$ is a meet of prime elements in $\mathcal{O}X$, and $I$ is the meet of the principal ideals generated by these prime elements. Hence $\upsilon_A \mathcal{O}X$ is spatial.

Consider the map $X \xrightarrow{f} \Sigma\upsilon_A \mathcal{O}X$ given by $f(x) = \tilde{x}$, where $\tilde{x}(I) =$ Card $(\bigcup I \cap \{x\})$. Note that $f$ is a continuous function, since $x \in \bigcup I$ iff $\tilde{x}(I) = 1$ iff

$f(x) \in \Sigma_I$. Furthermore, since $v_A\mathcal{O}X$ is spatial $Coz\mathcal{O}\Sigma v_A\mathcal{O}X \cong Cozv_A\mathcal{O}X \cong A$, and hence the cozero-sets of $\Sigma v_A\mathcal{O}X$ are all of the form $\Sigma_{\downarrow a}$, where $a \in A$. Thus $f : (X, A) \to (\Sigma v_A\mathcal{O}X, Coz\Sigma v_A\mathcal{O}X)$ is a coz-map. Gilmour [2] showed that $e : (X, A) \hookrightarrow (v_AX, A)$ is the universal realcompactification of $X$ in the category of Alexandroff spaces. Since $\Sigma v_A\mathcal{O}X$ is a realcompact Alexandroff space, there is a unique map $v_AX \xrightarrow{g} \Sigma v_A\mathcal{O}X$ such that $g \cdot e = f$. Applying the functor $\mathcal{O}$, we obtain:

$$\begin{tikzcd}
\mathcal{O}v_AX \arrow[r, "\mathcal{O}e"] & \mathcal{O}X \\
v_A\mathcal{O}X \arrow[u, "\mathcal{O}g\cdot\eta"'] \arrow[r, "\eta"] & \mathcal{O}\Sigma v_A\mathcal{O}X \arrow[u, "\mathcal{O}f"']
\end{tikzcd}$$

Now, $\mathcal{O}f \cdot \eta(I) = \mathcal{O}f(\Sigma_I) = \bigcup I = j_{\mathcal{O}X}(I)$. Hence $\mathcal{O}g \cdot \eta = \varphi^{-1}$, ie $\mathcal{O}v_AX \cong v_A\mathcal{O}X$. $\square$

## References

[1] Banaschewski B., Gilmour C.R.A., *Stone-Čech compactification and dimension theory for regular σ-frames*, J. London Math. Soc. **39** (1989), 1–8.
[2] Gilmour C.R.A., *Realcompactifications through zero-set spaces*, Quaestiones Math. **6** (1983), 73–95.
[3] ———, *Realcompact spaces and regular σ-frames*, Math. Proc. Camb. Phil. Soc. **96** (1984), 73–79.
[4] Johnstone P.T., *Stone Spaces*, Cambridge Studies in Advanced Math. **3**, Cambridge Univ. Press, 1982.
[5] Madden J., Vermeer J., *Lindelöf locales and realcompactness*, Math. Proc. Camb. Phil. Soc. **99** (1986), 473–480.
[6] G. Schlitt, ℕ-*Compact frames and applications*, Doctoral thesis, McMaster University, 1990.

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