Comment.Math.Univ.Carolin. 36,3 (1995)427–428

An inequality for the coefficients of a cosine polynomial

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Abstract. We prove: If

$$\frac{1}{2} + \sum_{k=1}^{n} a_k(n) \cos(kx) \geq 0 \quad \text{for all} \quad x \in [0, 2\pi),$$

then

$$1 - a_k(n) \geq \frac{1}{2} \frac{k^2}{n^2} \quad \text{for} \quad k = 1, \ldots, n.$$

The constant $1/2$ is the best possible.

Keywords: cosine polynomials, inequalities
Classification: 26D05

In order to determine the saturation classes of optimal and quasi-optimal sequences (for details we refer to [1]), R.A. DeVore [1] proved in 1970 the following interesting integral inequality involving trigonometric polynomials.

Proposition. *Let n and k be integers with $n \geq k \geq 1$. For all non-negative trigonometric polynomials T_n of degree $\leq n$ with $\frac{1}{\pi} \int_{-\pi}^{\pi} T_n(x)\, dx = 1$ we have*

$$\int_{-\pi}^{\pi} (\sin \frac{kx}{2})^2 T_n(x)\, dx \geq \frac{1}{256\pi} \frac{k^2}{n^2}\,.$$

If we only consider cosine polynomials of the form

$$T_n(x) = \frac{1}{2} + \sum_{k=1}^{n} a_k(n) \cos(kx), \tag{1}$$

then the Proposition states that the inequality

$$1 - a_k(n) \geq c_1(k^2/n^2) \tag{2}$$

with $c_1 = 1/(128\pi^2)$ is valid for all non-negative functions T_n and for all $k \in \{1, \ldots, n\}$.

In 1970 E.L. Stark [2] discovered a better bound for $1 - a_k(n)$. He established that (2) holds with $c_2 = \pi^2/36$. This result was improved by Stark in 1976. In [3] he proved the validity of (2) with $c_3 = \pi/9$. In the same paper he mentioned that the "problem of determining the optimal constant … remains open" [3, p. 71]. To the best of my knowledge no solution of this problem has been published until now. It is the aim of this note to show that the best possible constant is $c = 1/2$.

Theorem. *For all non-negative cosine polynomials* (1) *we have*

$$1 - a_k(n) \geq \frac{1}{2}\frac{k^2}{n^2} \qquad (k = 1, \ldots, n). \tag{3}$$

The constant $1/2$ *is the best possible.*

PROOF: As in [3] we define

$$u = u_k(n) = \frac{\pi}{[n/k] + 2} \qquad (1 \leq k \leq n).$$

($[x]$ denotes the greatest integer $\leq x$.) Then we have

$$\frac{\pi}{3}\frac{k}{n} \leq u \leq \frac{\pi}{3}. \tag{4}$$

Since the function $x \mapsto (1 - \cos(x))/x^2$ is strictly decreasing on $(0, \pi]$, we obtain

$$\frac{1 - \cos(u)}{u^2} \geq \frac{1 - \cos(\pi/3)}{(\pi/3)^2}, \tag{5}$$

so that (5) and the left-hand inequality of (4) yield

$$1 - \cos(u) \geq 9u^2/(2\pi^2) \geq k^2/(2n^2). \tag{6}$$

From (6) and

$$|a_k(n)| \leq \cos(u) \tag{7}$$

we conclude

$$1 - a_k(n) \geq 1 - \cos(u) \geq k^2/(2n^2).$$

If we set $T_n(x) = \frac{1}{2} + \frac{1}{2}\cos(nx)$, then the sign of equality holds in (3) for $k = n$, so that the constant $1/2$ cannot be replaced by a greater number. □

Remarks. (1) The proof of the Theorem reveals that the inequality (3) is strict for $k = 1, \ldots, n - 1$.

(2) The "extremely important" [3, p. 71] inequality (7) is due to J. Egerváry and O. Szász; see [4]. Concerning different proofs and extensions of (7) we refer to [3] and the references therein.

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(*Received* November 2, 1994)