

A characterization of the existence of solutions to some higher order boundary value problems

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Abstract. The aim of this short note is to present a theorem that characterizes the existence of solutions to a class of higher order boundary value problems. This result completely answers a question previously set by the authors in [Differential Integral Equations **6** (1993), 1119–1123].

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The aim of this short paper is to point out Theorem 1 below, which characterizes the existence of solutions to a class of higher order boundary value problems and, moreover, completely answers a question previously investigated in [2] and [3, Section 3.2]. Related results can also be found in the extensive survey by Agarwal [1, Chapter 9].

Our notation and terminology are standard. In any case, we refer to [2]. The symbol $C([a, b] \times \mathbb{R}^{n+1})$ is used to denote the space of all continuous real-valued functions defined on $[a, b] \times \mathbb{R}^{n+1}$.

Theorem 1. *The following assertions are equivalent:*

- (i) *The length of $[a, b]$ is less than $\pi/2$.*
- (ii) *For every function $f \in C([a, b] \times \mathbb{R}^{n+1})$ and every bounded sequence $\{x_h\} \subseteq \mathbb{R}$, there exists an integer $\nu \geq n + 1$ such that, for any $k \geq \nu$ and any $t_1, t_2, \dots, t_k \in [a, b]$, the problem*

$$\begin{cases} x^{(k)} = f(t, x, x', \dots, x^{(n)}) \\ x^{(i-1)}(t_i) = x_i, \quad i = 1, 2, \dots, k \end{cases}$$

admits at least one solution $u \in C^k([a, b])$.

PROOF: If (i) is true, so does (ii), by Theorem 1.1 of [2]. Example 1 below shows that (ii) \Rightarrow (i). □

Remark 1. The implication (i) \Rightarrow (ii) of Theorem 1 actually holds even if the function f satisfies Carathéodory’s type conditions only (see [2, Theorem 1.1] or [3, Theorem 5]). However, in this case, one achieves generalized solutions.

Remark 2. We emphasize that in assertion (ii) of Theorem 1 no condition on the finite sequence t_1, t_2, \dots, t_k is assumed. Moreover, whenever we specialize the choice of points t_i , assertion (ii) may be true also for $b - a \geq \pi/2$. As an example, this is the case if $t_1 \leq t_2 \leq \dots \leq t_k$; see [1, Theorem 9.2] and [3, Theorem 2].

Example 1. Let $f : [0, \pi/2] \times \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f(t, z) = \sin t + z, \quad (t, z) \in [0, \pi/2] \times \mathbb{R},$$

and let $x_h = 0$ for each $h \in \mathbb{N}$.

Then, for every positive integer ν , the problem

$$\begin{cases} x^{(4\nu)} = f(t, x) \\ x^{(i-1)}(t_i) = x_i, \quad i = 1, 2, \dots, 4\nu, \end{cases}$$

where $t_i = 0$ for i odd and $t_i = \pi/2$ for i even, admits no solutions in $C^{4\nu}([0, \pi/2])$.

Indeed, if $u \in C^{4\nu}([0, \pi/2])$ is a solution to the preceding problem for some $\nu \in \mathbb{N}$, then

$$(1) \quad \int_0^{\pi/2} u^{(4\nu)}(t) \sin t \, dt = \int_0^{\pi/2} u(t) \sin t \, dt + \int_0^{\pi/2} \sin^2 t \, dt$$

and

$$(2) \quad u^{(i-1)}(0) = 0 \quad \text{for } i \text{ odd,} \quad u^{(i-1)}(\pi/2) = 0 \quad \text{for } i \text{ even,} \quad i = 1, 2, \dots, 4\nu.$$

Owing to (2), integrating by parts, one has

$$\int_0^{\pi/2} u^{(4\nu)}(t) \sin t \, dt = \int_0^{\pi/2} u^{(4\nu-2)}(t) \sin t \, dt = \dots = \int_0^{\pi/2} u(t) \sin t \, dt;$$

therefore, identity (1) becomes

$$\int_0^{\pi/2} u(t) \sin t \, dt = \int_0^{\pi/2} u(t) \sin t \, dt + \frac{\pi}{4}.$$

This is clearly absurd.

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