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A full descriptive definition of the BV-integral

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Abstract. We present a Cauchy test for the almost derivability of additive functions of bounded BV sets. The test yields a full descriptive definition of a coordinate free Riemann type integral.

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The BV-integral, introduced in [5, Definition 5.1] under the name "variational integral", is a coordinate free generalization of the Lebesgue integral defined on all bounded Caccioppoli sets. Unlike the Lebesgue integral, it integrates partial derivatives of differentiable functions and provides the unrestricted Gauss-Green theorem. The purpose of this note is to present a complete characterization of those additive functions of bounded Caccioppoli sets that are indefinite BV-integrals (Theorem 3.9).

1. Preliminaries

The ambient space of this paper is $\mathbf{R}^m$, where $\mathbf{R}$ is the set of all real numbers and m is a fixed positive integer. The metric in $\mathbf{R}^m$ is induced by the maximum norm, and $U(x,\varepsilon)$ denotes the open ball about $x \in \mathbf{R}^m$ of radius $\varepsilon > 0$. For a set $E \subset \mathbf{R}^m$, we denote by $\operatorname{cl} E$, $\operatorname{int} E$, ∂E, $d(E)$, and $|E|$ the closure, interior, boundary, diameter, and Lebesgue measure of E, respectively. The words "measure", "measurable", and "negligible" as well as the expressions "almost all" and "almost everywhere" always refer to the Lebesgue measure in $\mathbf{R}^m$. The *symmetric difference* of sets A and B is the set $A \triangle B = (A - B) \cup (B - A)$.

Let $E \subset \mathbf{R}^m$. We say an $x \in E$ is, respectively, a *density* or *dispersion* point of E according to whether

$$\liminf_{\varepsilon \to 0+} \frac{|U(x,\varepsilon) \cap E|}{(2\varepsilon)^m} = 1 \qquad \text{or} \qquad \limsup_{\varepsilon \to 0+} \frac{|U(x,\varepsilon) \cap E|}{(2\varepsilon)^m} = 0\,.$$

The set of all nondispersion points of E is called the *essential closure* of E, denoted by $\operatorname{cl}^* E$, and the set of all density points

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- of E is called the *essential interior* of E, denoted by $\mathrm{int}^* E$. The *essential boundary* of E is the set $\partial^* E = \mathrm{cl}^* E - \mathrm{int}^* E$. Clearly, $\mathrm{int}\, E \subset \mathrm{int}^* E \subset \mathrm{cl}^* E \subset \mathrm{cl}\, E$, and so $\partial^* E \subset \partial E$. If E is measurable, the sets $E \triangle \mathrm{cl}^* E$, $E \triangle \mathrm{int}^* E$, and $\partial^* E$ are negligible [7, Chapter IV, Theorem 6.1]. The set E is called *essentially* or *doubly closed* whenever $E = \mathrm{cl}^* E$ or $E = \mathrm{cl}^* E = \mathrm{cl}\, E$, respectively.

The $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ is denoted by $\mathcal{H}$, and a set $T \subset \mathbf{R}^m$ of σ-finite measure $\mathcal{H}$ is called *thin*. Each thin set is negligible but not vice versa. The *perimeter* (in De Giorgi's sense) of a set $A \subset \mathbf{R}^m$ is the number $\|A\| = \mathcal{H}(\partial^* A)$. A bounded set $A \subset \mathbf{R}^m$ with $\|A\| < +\infty$ is called a *Caccioppoli* or BV *set* (BV for *bounded variation* — cf. [2, Section 5.11, Theorem 1]). Each BV set is measurable [6, Corollary 13.2.5], and the family BV of all BV sets is an algebra in $\mathbf{R}^m$. For $E \subset \mathbf{R}^m$, we let $BV_E = \{A \in BV : A \subset E\}$.

The *regularity* of a BV set A is the number

$$r(A) = \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The isoperimetric inequality ([2, Section 5.6, Theorem 2,(i)]) shows that a sequence $\{A_n\}$ of BV sets is regular in the sense of [7, Chapter IV, Section 2] whenever $\inf_n r(A_n) > 0$.

Let A be a BV set. The set of all $x \in \mathrm{int}^* A$ such that

$$\lim_{\varepsilon \to 0+} \frac{\mathcal{H}[U(x,\varepsilon) \cap \partial^* A]}{(2\varepsilon)^{m-1}} = 0$$

is called the *critical interior* of A, denoted by $\mathrm{int}^c A$. According to [8, Section 4], we have $\mathcal{H}(\mathrm{int}^* A - \mathrm{int}^c A) = 0$; in particular, the set $\mathrm{cl}^* A - \mathrm{int}^c A$ is negligible. The next lemma, proved in [4, Lemma 1.2], gives an important property of the critical interior.

Lemma 1.1. *Let $A \in BV$ and $x \in \mathrm{int}^c A$. Suppose $\{B_n\}$ is a sequence of BV sets such that $x \in \mathrm{cl}^* B_n$ and $r(B_n) > \varepsilon > 0$ for $n = 1, 2, \ldots$. Then $x \in \mathrm{cl}^*(A \cap B_n)$ for $n = 1, 2, \ldots$, and if $\lim d(B_n) = 0$, then $r(A \cap B_n) > \varepsilon^{m+1}$ for all sufficiently large integers $n \geq 1$.*

2. The integral

Unless specified otherwise, by a function we always mean a real-valued function. An *additive function* in a BV set A is a function F defined on BV_A such that

$$F(B \cup C) = F(B) + F(C)$$

for each pair of disjoint sets $B, C \in BV_A$. Such an F is called *continuous* if given $\varepsilon > 0$, we can find an $\eta > 0$ so that $|F(B)| < \varepsilon$ for each $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

A *partition* is a collection $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are disjoint BV sets and $x_i \in \mathrm{cl}^* A_i$ for $i = 1, \ldots, p$. When $E \subset \mathbf{R}^m$ and $\bigcup_{i=1}^p A_i \subset E$ or $\{x_1, \ldots, x_p\} \subset E$, we say P is a partition *in* E or a partition *anchored in* E, respectively. Clearly, each partition in E is anchored in $\mathrm{cl}^* E$. Given an $\varepsilon > 0$ and a nonnegative function δ on E, the partition P is called

(i) *ε-regular* if $r(A_i) > \varepsilon$ for $i = 1, \ldots, p$;

(ii) *δ-fine* if P is anchored in E and $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$.

It is convenient to denote $\bigcup_{i=1}^p A_i$ by $\bigcup P$.

A *gage* in a set $E \subset \mathbf{R}^m$ is a nonnegative function δ defined on $\mathrm{cl}^* E$ whose *null set* $N_\delta = \{x \in \mathrm{cl}^* E : \delta(x) = 0\}$ is thin.

Definition 2.1. Let A be a BV set and let f be a function defined on $\mathrm{cl}^* A$. We say f is BV-*integrable* in A if there is a continuous additive function F in A satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ in A so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

In view of [5, Propositions 7.7 and 7.8], BV-integrability coincides with variational integrability introduced in [5, Definition 5.1]. The function F, uniquely determined by f, is called the *indefinite* BV-*integral* of f in A. If f is Lebesgue integrable in A, it is BV-integrable in A, and the two indefinite integrals coincide [5, Proposition 5.8].

Let A be a BV set and let f be a function defined on $A \cup \mathrm{cl}^* A$. Since the BV-integral extends the Lebesgue integral, neither the BV-integrability nor the BV-integral of f is affected by the values f takes on negligible subsets of $\mathrm{cl}^* A$; in particular, they are not affected by the values of f on $A \triangle \mathrm{cl}^* A$. Thus, in the obvious way, we can and will define the BV-integrability and BV-integral for the extended real-valued functions defined almost everywhere in A.

For additional properties of the BV-integral, including the Gauss-Green theorem, we refer to [5].

3. BV-ACG$_*$ functions

Let $E \subset \mathbf{R}^m$, let F be a function defined on BV_E, and let $x \in \mathrm{cl}^* E$. Set

$$\underline{F}(x) = \inf_{\alpha > 0} \sup_{\delta > 0} \left[\inf \frac{F(B)}{|B|} \right]$$

where the infimum in the brackets is taken over all sets $B \in BV_E$ with $x \in \mathrm{cl}^* B$, $d(B) < \delta$, and $r(B) > \alpha$; furthermore, let $\overline{F}(x) = -\underline{(-F)}(x)$. The extended real-valued functions $x \mapsto \underline{F}(x)$ and $x \mapsto \overline{F}(x)$ defined on $\mathrm{cl}^* E$ are denoted by $\underline{F}$ and $\overline{F}$, respectively. When $\underline{F}(x) = \overline{F}(x)$ is a real number, we denote it by $F'(x)$

and say F is BV-*derivable* at x. By F' we denote the function $x \mapsto F'(x)$ defined on the set of all $x \in \mathrm{cl}^* E$ at which F is BV-derivable.

The following lemma, proved in [4, Lemma 2,3], facilitates applications of Vitali's covering theorem.

Lemma 3.1. *Let F be an additive continuous function in a BV set A. If $x \in \mathrm{int}^c A$, then*

$$\underline{F}(x) = \inf_{\alpha>0} \sup_{\delta>0} \left[\inf \frac{F(A \cap C)}{|A \cap C|} \right]$$

where the infimum in the brackets is taken over all doubly closed BV sets C with $x \in C$, $d(C) < \delta$, and $r(C) > \alpha$. In particular, $\underline{F}(x) \leq \overline{F}(x)$ for each $x \in \mathrm{int}^c A$.

Definition 3.2. Let F be an additive continuous function in a BV set A. We say F is BV-AC$_*$ on a set $E \subset \mathrm{cl}^* A$ if given $\varepsilon > 0$, there is an $\eta > 0$ and a gage δ in A such that

$$\left| F\left(\bigcup P\right) - F\left(\bigcup Q\right) \right| < \varepsilon$$

for all ε-regular δ-fine partitions P and Q in A anchored in E for which

$$\left| \left(\bigcup P\right) \triangle \left(\bigcup Q\right) \right| < \eta\,.$$

If $\mathrm{cl}^* A = \bigcup_{n=1}^{\infty} E_n$ and F is BV-AC$_*$ on each E_n, we say that F is BV-ACG$_*$.

In a more general setting BV-ACG$_*$ functions were introduced in [1]. Our results below parallel some of those obtained in [3], where a concept closely related to the BV-ACG$_*$ functions has been applied to Perron type integrals.

Following [4, Definition 2.5], we say an additive continuous function F in a BV set A is BV-*absolutely continuous* if given a negligible set $N \subset \mathrm{cl}^* A$ and an $\varepsilon > 0$, there is a gage δ in A such that $|F(\bigcup P)| < \varepsilon$ for each ε-regular δ-fine partition P in A anchored in N.

Proposition 3.3. *Each BV-ACG$_*$ function in a BV set A is BV-absolutely continuous.*

Proof: Let F be a BV-ACG$_*$ function in A. With no loss of generality, we may assume that there are disjoint sets $E_1, E_2, \ldots$ such that $\mathrm{cl}^* A = \bigcup_{n=1}^{\infty} E_n$ and F is BV-AC$_*$ on each E_n. Choose a negligible set $N \subset \mathrm{cl}^* A$ and $\varepsilon > 0$, and fix an integer $n \geq 1$. Letting $Q = \emptyset$ in Definition 3.2, find a gage δ_n in A and $\eta_n > 0$ so that $|F(\bigcup P)| < \varepsilon 2^{-n}$ for each ε-regular δ_n-fine partition P in A anchored in E_n with $|\bigcup P| < \eta_n$. There is an open set U_n containing $N \cap E_n$ with $|U_n| < \eta_n$. Making δ_n smaller, we may assume that each δ_n-fine partition P anchored in $N \cap E_n$ is a partition in U_n; in particular, $|\bigcup P| < \eta_n$. Define a gage δ in A by setting $\delta(x) = \delta_n(x)$ if $x \in E_n$ for $n = 1, 2, \dots$. If $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ is an ε-regular δ-fine partition in E anchored in N, we obtain

$$\left|F\Big(\bigcup P\Big)\right| \leq \sum_{n=1}^{\infty}\left|\sum_{x_i \in E_n} F(A_i)\right| < \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon\,,$$

which establishes the BV-absolute continuity of F. □

The following characterizations are called, respectively, the *partial* and *full descriptive definitions* of the BV-integral:

1. Among all continuous additive functions that are derivable almost everywhere in a BV set A, characterize those which are indefinite BV-integrals in A.
2. Among all continuous additive functions in a BV set A (derivable or not), characterize those which are indefinite BV-integrals in A.

A partial descriptive definition was given in [4, Theorem 2.6] employing the concept of BV-absolutely continuous functions. Using the stronger concept of BV-ACG$_*$ functions we shall present a full descriptive definition in Theorem 3.7 below.

Lemma 3.4. *Let F be a continuous additive function in a BV set A, and let*

$$E = \{x \in \mathrm{cl}^* A : \underline{F}(x) < r < s < \overline{F}(x)\}\,.$$

If F is BV-AC$_*$ on E, then E is negligible.

Proof: With no loss of generality we may assume $E \subset \mathrm{int}^c A$. Choose positive numbers ε and η, and let $\varepsilon' = \varepsilon^{m+1}$. If F is BV-AC$_*$ on E, we can find a positive number $\eta' \leq \eta$ and a gage δ in A so that

$$\left|F\Big(\bigcup P\Big) - F\Big(\bigcup Q\Big)\right| < \varepsilon$$

for all ε'-regular δ-fine partitions P and Q in A anchored in E for which

$$\left|\Big(\bigcup P\Big) \triangle \Big(\bigcup Q\Big)\right| < 4\eta'\,.$$

Select an open set U containing E with $|U| < |E| + \eta'$, and let $\mathcal{R}$ and $\mathcal{S}$ be the families of all doubly closed sets $C \subset U$ such that $d(C) < \delta(x_C)$ for an $x_C \in E \cap C$, $r(C) > \varepsilon$, and respectively,

$$F(A \cap C) < r|A \cap C| \qquad \text{and} \qquad F(A \cap C) > s|A \cap C|\,.$$

In view of Lemma 1.1, making δ smaller, we may assume $r(A \cap C) > \varepsilon'$ for each $C \in \mathcal{R} \cup \mathcal{S}$. Clearly, $\mathcal{R}$ and $\mathcal{S}$ are Vitali covers of $E - N_\delta$. Since N_δ is a negligible set (in fact, a thin set), applying Vitali's covering theorem [7, Chapter IV, Theorem 3.1], we obtain collections $\{A_1, \dots, A_p\} \subset \mathcal{R}$ and $\{B_1, \dots, B_q\} \subset \mathcal{S}$, each consisting of disjoint sets and such that

$$\min\left\{\left|\bigcup_{i=1}^{p}(E \cap A_i)\right|, \left|\bigcup_{j=1}^{q}(E \cap B_j)\right|\right\} > |E| - \eta' .$$

From the definitions of $\mathcal{R}$ and $\mathcal{S}$, we see that

$$\begin{aligned} P &= \{(A \cap A_1, x_{A_1}), \dots, (A \cap A_p, x_{A_p})\} , \\ Q &= \{(A \cap B_1, x_{B_1}), \dots, (A \cap B_q, x_{B_q})\} \end{aligned}$$

are ε'-regular δ-fine partitions in A anchored in E. Since $\bigcup P \subset U$ and $|E \cap \bigcup P| > |E| - \eta'$, we have $|E \triangle \bigcup P| < 2\eta'$; by symmetry, also $|E \triangle \bigcup Q| < 2\eta'$. Thus $|(\bigcup P) \triangle (\bigcup Q)| < 4\eta'$, and we obtain

$$\begin{aligned} \varepsilon &> \left|F\left(\bigcup P\right) - F\left(\bigcup Q\right)\right| \geq \sum_{j=1}^{q} F(A \cap B_j) - \sum_{i=1}^{p} F(A \cap A_i) \\ &> s\left|\bigcup Q\right| - r\left|\bigcup P\right| > s(|E| - \eta') - r(|E| + \eta') \\ &= (s - r)|E| - (s + r)\eta' \geq (s - r)|E| - |s + r|\eta . \end{aligned}$$

The negligibility of E follows from the arbitrariness of ε and η. □

Lemma 3.5. *Let F be a continuous additive function in a BV set A, and let*

$$E = \{x \in \mathrm{cl}^* A : \underline{F}(x) = +\infty\} .$$

If F is BV-AC$_*$ on E, then E is negligible.

Proof: We may assume $E \subset \mathrm{int}^c A$. Proceeding towards a contradiction, suppose $|E| > 0$. If $0 < \varepsilon < 1/(2m)$ and F is BV-AC$_*$ on E, we can find a gage δ in A and a positive $\eta < |E|/2$ so that

$$\left|F\left(\bigcup P\right) - F\left(\bigcup Q\right)\right| < 1$$

for all ε-regular δ-fine partitions P and Q in A anchored in E for which

$$\left|\left(\bigcup P\right) \triangle \left(\bigcup Q\right)\right| < 4\eta .$$

Let U be an open set containing E with $|U| < |E| + \eta$. Using [5, Lemma 3.4] and Vitali's covering theorem, find disjoint closed cubes $C_1, \dots, C_p$ contained in U such that $|E - \bigcup_{i=1}^p C_i| < \eta$ while $r(A \cap C_i) > \varepsilon$ and $d(C_i) < \delta(x_i)$ for an $x_i \in E \cap C_i$ and $i = 1, \dots, p$. Thus $P = \{(A \cap C_1, x_1), \dots, (A \cap C_p, x_p)\}$ is an ε-regular δ-fine partition in A anchored in E, and $|E \triangle (\bigcup P)| < 2\eta$.

In view of [5, Lemma 3.4] and Lemma 3.1, the family $\mathcal{B}$ of all closed cubes $B \subset U$ such that $r(A \cap B) > \varepsilon$, $d(B) < \delta(y_B)$ for a $y_B \in E \cap B$, and

$$\frac{F(A \cap B)}{|A \cap B|} > \frac{2}{|E|} \left| F\left(\bigcup P\right) + 1 \right| ,$$

is a Vitali cover of E. Using Vitali's covering theorem again, find a disjoint collection $\{B_1, \dots, B_q\} \subset \mathcal{B}$ so that $|E - \bigcup_{j=1}^q B_j| < \eta$. Then $Q = \{(A \cap B_1, y_{B_1}), \dots, (A \cap B_q, y_{B_q})\}$ is an ε-regular δ-fine partition in A anchored in E, and $|E \triangle \bigcup Q| < 2\eta$. Observing that $\sum_{j=1}^q |A \cap B_j| \ge |E|/2$, we obtain

$$\begin{aligned} F\left(\bigcup Q\right) &= \sum_{j=1}^q F(A \cap B_j) \\ &> \frac{2}{|E|} \left| F\left(\bigcup P\right) + 1 \right| \cdot \sum_{j=1}^q |A \cap B_j| \\ &\ge F\left(\bigcup P\right) + 1 . \end{aligned}$$

This is a contradiction, since $|(\bigcup P) \triangle (\bigcup Q)| < 4\eta$. □

Proposition 3.6. *A continuous additive function F in a BV set A that is BV-ACG$_*$ is derivable almost everywhere in* $\mathrm{cl}^* A$.

PROOF: Let $\mathrm{cl}^* A = \bigcup_{n=1}^\infty E_n$, and let F be BV-AC$_*$ on each E_n. The set of all $x \in \mathrm{cl}^* A$ at which F is not derivable is the union of the sets

$$E_{n,+\infty} = \{x \in E_n : \underline{F}(x) = +\infty\} , \qquad E_{n,-\infty} = \{x \in E_n : \overline{F}(x) = -\infty\} ,$$

and

$$E_{n,r,s} = \{x \in E_n : \underline{F}(x) < r < s < \overline{F}(x)\}$$

where r and s are rational numbers. Lemma 3.5 applied to F and $-F$ shows that the sets $E_{n,\pm\infty}$ are negligible, and the sets $E_{n,r,s}$ are negligible by Lemma 3.4. Since we have only countably many of these sets, the proposition follows.

Corollary 3.7. *Let F be a continuous additive function in a BV set A. If F is BV-ACG$_*$, then F' is BV-integrable in A and F is its indefinite BV-integral.*

PROOF: According to Proposition 3.6, the derivate F' is defined almost everywhere in $\mathrm{cl}^* A$, and by Proposition 3.3, the function F is BV-absolutely continuous. An application of [4, Theorem 2.6] completes the proof. □

Proposition 3.8. *Let f be a BV-integrable function in a BV set A. If F is the indefinite BV-integral of f in A, then F is BV-ACG$_*$.*

Proof: We may assume that f is a real-valued function defined on the whole of $\mathrm{cl}^* A$, and let $E_n = \{x \in \mathrm{cl}^* A : |f(x)| \leq n\}$ for $n = 1, 2, \ldots$. Since $\mathrm{cl}^* A = \bigcup_{n=1}^{\infty} E_n$, it suffices to show that F is BV-AC$_*$ on each E_n. To this end, fix a positive integer n and let $E = E_n$. It follows from [5, Corollary 5.12] that E is measurable and f is Lebesgue integrable in E. Hence, if χ is the characteristic function of E restricted to $\mathrm{cl}^* A$, then $f\chi$ is Lebesgue integrable in A. By [5, Proposition 5.8], the function $f\chi$ is also BV-integrable in A, and we denote by G its indefinite BV-integral in A.

Choose an $\varepsilon > 0$. Using the absolute continuity of the indefinite Lebesgue integral, find an $\eta > 0$ so that $|G(Z)| < \varepsilon$ for each BV set $Z \subset A$ with $|Z| < \eta$. There is a gage δ in A such that

$$\sum_{i=1}^{r} \Big| f(z_i)|C_i| - F(C_i) \Big| < \varepsilon \qquad \text{and} \qquad \sum_{i=1}^{r} \Big| f(z_i)\chi(z_i)|C_i| - G(C_i) \Big| < \varepsilon$$

for each ε-regular δ-fine partition $R = \{(C_1, z_1), \ldots, (C_r, z_r)\}$ in A. If such a partition R is anchored in E, then $\chi(z_i) = 1$ for $i = 1, \ldots, r$, and we have

$$\left| F\Big(\bigcup R\Big) - G\Big(\bigcup R\Big) \right| \leq \sum_{i=1}^{r} |F(C_i) - G(C_i)| < 2\varepsilon\,.$$

Now choose ε-regular δ-fine partitions $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $Q = \{(B_1, y_1), \ldots, (B_q, y_q)\}$ in A anchored in E for which $|(\bigcup P) \triangle (\bigcup Q)| < \eta$. Letting $X = \bigcup P$ and $Y = \bigcup Q$, observe that

$$\begin{aligned} |G(X) - G(Y)| &= |G(X - Y) - G(Y - X)| \\ &\leq |G(X - Y)| + |G(Y - X)| < 2\varepsilon\,; \end{aligned}$$

for $\max\{|X - Y|, |Y - X|\} \leq |X \triangle Y| < \eta$. Thus

$$|F(X) - F(Y)| \leq |F(X) - G(X)| + |F(Y) - G(Y)| + |G(X) - G(Y)| < 6\varepsilon\,,$$

which establishes F is BV-AC$_*$ on E. □

Combining Corollary 3.7 and Proposition 3.8, we obtain the following full descriptive definition of the BV-integral.

Theorem 3.9. *A continuous additive function F in a BV set A is BV-ACG$_*$ if and only if F' exists almost everywhere in A and F is its indefinite BV-integral.*

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